BOUNDARY SINGULARITIES OF FABER AND FOURIER SERIES

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Abstract: Given an analytic Jordan curve Γ, with interior G and exterior Ω, and given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ satisfying $\limsup_{n\to\infty} |a_n|^{1/n} = 1$, we consider here three series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z),$$

where the polynomials $P_n(z)$ are chosen to be (i) the Faber polynomials associated with G, (ii) the polynomials orthogonal over the area of G, and (iii) the polynomials orthogonal over the contour Γ. Here, we study the nature of the singularities of $f(z)$ in $\Omega \cup \Gamma$, for these three types of series, based solely on the coefficients $\{a_n\}_{n=0}^{\infty}$.

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1. Introduction

Let Γ be an analytic Jordan curve with interior G and exterior Ω. As is well known, the Riemann mapping theorem gives the existence of a unique conformal mapping Φ of Ω onto $D' := \{w \in \overline{\mathbf{C}} : |w| > 1\}$, normalized by the conditions that

$$\Phi(\infty) = \infty \text{ and } \lim_{z\to\infty} \frac{\Phi(z)}{z} = \frac{1}{C} > 0,$$

where C denotes the *capacity* of Γ. Let $\Psi := \Phi^{[-1]}$ denote the conformal inverse of Φ, so that Ψ maps D' conformally onto Ω, with

$$\Psi(\infty) = \infty \text{ and } \lim_{w\to\infty} \frac{\Psi(w)}{w} = C > 0.$$

Because Γ is an analytic Jordan curve, there exists a smallest r_0, with $0 \le r_0 < 1$, such that Ψ can be extended, as a conformal mapping, to $D_{r_0} := \{w \in \mathbf{C} : |w| > r_0\}$. (We assume that $r_0 > 0$, as the case $r_0 = 0$ is essentially trivial.) For each ρ with $\rho \ge r_0$, we define the *level curves* of Φ by $\Gamma_\rho := \{z \in \mathbf{C} : |\Phi(z)| = \rho\}$, so that $\Gamma_1 = \Gamma$. For each Γ_ρ, we define $G_\rho := \mathit{Int}\ \Gamma_\rho$ and $\Omega_\rho := \mathit{Ext}\ \Gamma_\rho$, and it follows that Φ can be similarly extended, as a conformal mapping, from Ω_{r_0} to D'_{r_0}.

Given a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ satisfying

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1, \tag{1.1}$$

we consider a series of the form

$$f(z) := \sum_{n=0}^{\infty} a_n P_n(z), \tag{1.2}$$

where $P_n(z)$ is a polynomial of degree at most n $(n = 0, 1, 2, \ldots)$. If the sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$ is chosen appropriately, then the series (1.2) converges locally uniformly in G (i.e., it converges uniformly on any compact subset of G), making $f(z)$ an analytic function in G. We shall deal in this paper with the following three special sequences of polynomials: Faber polynomials associated with $\overline{G}$, polynomials orthogonal over the area of G, and polynomials orthogonal over the contour Γ. Correspondingly, a Faber series and two cases of Fourier series are then obtained in (1.2). Although all three series converge only in G and diverge for any $z \in \Omega$ (see Lemma 4.2), the function f defined in (1.2) may have an analytic continuation in a larger region containing G. Our interest here is in the question: "What can be deduced about the singularities (in $\Omega \cup \Gamma$) and the analytic continuation properties of f in (1.2), solely from the sequence of coefficients $\{a_n\}_{n=0}^{\infty}$ in (1.1)?". This has been the subject of numerous classical studies for the special case of Taylor series. We investigate to what extent it is possible to transfer these classical results on Taylor series to the Faber and Fourier series in (1.2). In what follows, we always assume that (1.1) is valid.

It is well known that the classical Pringsheim theorem (cf. Titchmarsh [11, p. 214]) asserts that if (1.1) is valid and if $a_n \ge 0$ $(n = 0, 1, 2, \ldots)$, then the function defined by the

Maclaurin expansion

$$\sum_{n=0}^{\infty} a_n w^n,$$

which is analytic in $|w| < 1$, necessarily has a singularity at $w = 1$. If $\{F_n(z)\}_{n=0}^{\infty}$ denotes the Faber polynomials associated with the analytic Jordan curve Γ, it was recently shown in Hasson and Walsh [6] that if (1.1) is valid and if $a_n \geq 0$ $(n = 0, 1, 2, \ldots)$, then the function defined by the Faber series

$$\sum_{n=0}^{\infty} a_n F_n(z),$$

which is analytic in G, necessarily has a singularity in the point $\Psi(1)$ of Γ. This paper has motivated our investigations here, and many extensions of the results of [6] will be given here, not only to Faber series, but to the two associated Fourier series associated with G and Γ.

2. Faber Series

Let the Laurent expansion for $\Phi(z)$, at $z = \infty$, be given by

$$\Phi(z) = \frac{z}{C} + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots. \tag{2.1}$$

¿From (2.1), the associated Faber polynomials $\{F_n(z)\}_{n=0}^{\infty}$ are then defined by the identity (cf. Curtiss [3, p. 578])

$$\Phi^n(z) =: F_n(z) + h_{n,1}(z), \tag{2.2}$$

where $h_{n,1}(z)$ is a Laurent expansion consisting only of negative powers of z, i.e., $F_n(z)$ is the polynomial part of $\Phi^n(z)$. (Note from (2.1) that the coefficient of z^n in $F_n(z)$ is $\frac{1}{C^n}$, and is hence positive for all $n \geq 0$.) Then, with a sequence $\{a_n\}_{n=0}^{\infty}$ satisfying (1.1), we define its associated Faber series as

$$f_1(z) := \sum_{n=0}^{\infty} a_n F_n(z). \tag{2.3}$$

As will be shown below in Lemma 4.2, the series in (2.3) is convergent, locally uniformly in G, and this series defines an analytic function in G. With $f_1(z)$ also denoting any analytic continuation of $f_1(z)$ of (2.3), it follows that any singularities of $f_1(z)$ must reside in $\Omega \cup \Gamma$.

The following statements of theorems give the complete analogues of classical results, for Taylor series, to the case of Faber series associated with Γ. (Their proofs are given in Section 4.)

THEOREM 2.1. *Given* $\{a_n\}_{n=0}^{\infty}$ *satisfying* (1.1), *suppose that* $a_n \geq 0 \quad (n = 0, 1, 2, \dots)$. *Then,* $\Psi(1) \in \Gamma$ *is a singularity of* f_1 *of* (2.3).

THEOREM 2.2. *Given* $\{a_n\}_{n=0}^{\infty}$ *satisfying* (1.1), *assume further that the following limit exists:*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = e^{i\theta}, \text{ for some real } \theta \text{ with } 0 \leq \theta < 2\pi. \tag{2.4}$$

Then, $\Psi(e^{i\theta}) \in \Gamma$ *is a singularity of* f_1 *of* (2.3).

THEOREM 2.3. *Given* $\{a_n\}_{n=0}^{\infty}$ *satisfying* (1.1), *let the determinant* $D_n^{(s)}$ *be defined, for any nonnegative integers* s *and* n, *by*

$$D_n^{(s)} := \det \begin{bmatrix} a_n & a_{n+1} \cdots & a_{n+s} \\ a_{n+1} & a_{n+2} \ \cdots & a_{n+s+1} \\ \vdots & & \vdots \\ a_{n+s} & a_{n+s+1} \ \cdots & a_{n+2s} \end{bmatrix}. \tag{2.5}$$

A necessary and sufficient condition that f_1 *of* (2.3) *has exactly* p *poles on* Γ *(counted according to multiplicities), and no other singularities on* Γ, *is that*

$$\limsup_{n\to\infty} |D_n^{(s)}|^{1/n} = 1, \text{ for } s = 0, 1, \dots, p-1, \tag{2.6}$$

and

$$\limsup_{n\to\infty} |D_n^{(p)}|^{1/n} < 1. \tag{2.7}$$

THEOREM 2.4. *Given* $\{a_n\}_{n=0}^{\infty}$ *satisfying* (1.1), *assume that there is a function* g, *defined on* N_0, *such that* $g(n) = a_n$ $(n = 0, 1, 2, \dots)$. *Then,* $f_1(z)$ *of* (2.3) *has the point* $\Psi(1)$ *as its only singular point in* $\overline{\mathbf{C}}$ *if and only if* g *is an entire function such that for any* $\epsilon > 0$, *there exists an* $R > 0$, *such that*

$$|g(z)| \leq e^{\epsilon|z|}, \ |z| > R. \tag{2.8}$$

Moreover, $\Psi(1)$ *is a pole of order* p *if and only if* g *is a polynomial of degree* $p-1$.

We remark that (2.8) implies that g is an entire function of order 1 and type 0 (cf. Boas [2, p. 8]).

As an example illustrating *all* of the above theorems, let Γ be an analytic Jordan curve, and consider the particular sequence of numbers $\{\tilde{a}_n := n+1\}_{n=0}^{\infty}$, which satisfies (1.1). In this case, $g(w) := w+1$, a polynomial of exact degree 1, is such that $g(n) = \tilde{a}_n$ for all $n \geq 0$, and, as $\sum_{n=0}^{\infty} \tilde{a}_n w^n = (1-w)^{-2}$, the only singularity of this function in $\overline{\mathbf{C}}$ is a pole of order 2 at $w = 1$. As a consequence of Proposition 4.3 in Section 4, the only singularity of the associated Faber series $\sum_{n=0}^{\infty} \tilde{a}_n F_n(z)$ is a pole of order 2 at $\Psi(1)$. (In this case (cf. (2.5)), $D_n^{(0)} = n+1, D_n^{(1)} = -1$, and $D_n^{(2)} = 0$.)

THEOREM 2.5. *Let* $\{n_k\}_{k=0}^{\infty}$ *be an increasing sequence of nonnegative integers satisfying*

$$\lim_{k\to\infty} \frac{k}{n_k} = 0. \tag{2.9}$$

If $\{a_n\}_{n=0}^{\infty}$ *is any sequence satisfying* (1.1) *such that*

$$a_m = 0 \text{ for any } m \notin \{n_k\}_{k=0}^{\infty}, \tag{2.10}$$

then Γ *is a natural boundary of* f_1 *of* (2.3).

Having stated our main results, we remark that our Theorems 2.1 and 2.5 duplicate the main results of Hasson and Walsh [6, Theorems 1 and 3.1]. We also remark that our Theorem 2.5 has appeared earlier in 1988 in Adepoju [1].

It will become clear from the proofs in Section 4 that most of the known results on Taylor series can be translated directly to related results on Faber series. For this reason, the list of theorems in this section is purposely not complete.

3. Fourier Series

This section deals with two set of orthogonal polynomials, $\{K_n(z)\}_{n=0}^{\infty}$ and $\{Q_n(z)\}_{n=0}^{\infty}$, and the corresponding Fourier series

$$f_2(z) := \sum_{n=0}^{\infty} a_n K_n(z), \tag{3.1}$$

and

$$f_3(z) := \sum_{n=0}^{\infty} a_n Q_n(z). \tag{3.2}$$

The unique set of polynomials $\{K_n(z)\}_{n=0}^{\infty}$ is determined by the orthonormalization of the polynomials $\{z^j\}_{j=0}^{\infty}$ with respect to the area measure in G, i.e.,

$$\iint_G K_n(z)\overline{K_m(z)}dxdy = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{3.3}$$

The other unique set of polynomials $\{Q_n(z)\}_{n=0}^{\infty}$ is determined by the orthonormalization of the polynomials $\{z^j\}_{j=0}^{\infty}$ over the contour Γ, i.e.,

$$\int_\Gamma Q_n(z)\overline{Q_m(z)}|dz| = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{3.4}$$

In both cases, we require, as usual, that the coefficient of z^n in $K_n(z)$ or $Q_n(z)$ be positive for every $n \geq 0$.

We refer the reader to Lemma 4.2 below for the local uniform convergence of (3.1) and (3.2) in G. If the domain G is the unit disk $D := \{z \in \mathbf{C} : |z| < 1\}$, then both series (3.1) and (3.2) coincide with a Maclaurin series. In general, they define two *different* analytic functions f_2 and f_3 in G.

As the statements of the results for both Fourier series are the same in most cases, we do not give them separately.

<u>THEOREM 3.1.</u> *Given $\{a_n\}_{n=0}^{\infty}$ satisfying (1.1), suppose that $a_n \geq 0$ $(n = 0, 1, 2, \ldots)$. Then, $\Psi(1) \in \Gamma$ is a singularity of both functions f_2 and f_3 of (3.1) and (3.2).*

<u>THEOREM 3.2.</u> *Given $\{a_n\}_{n=0}^{\infty}$ satisfying (1.1), assume further that the following limit exists:*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = e^{i\theta}, \text{ for some real } \theta \text{ with } 0 \leq \theta < 2\pi. \tag{3.5}$$

Then, $\Psi(e^{i\theta}) \in \Gamma$ is a singularity of both functions f_2 and f_3 of (3.1) and (3.2).

<u>THEOREM 3.3a.</u> *Let $D_n^{(s)}$ be defined as in (2.5). A necessary and sufficient condition that f_3 of (3.2) has exactly p poles (counted according to multiplicity) and no other singularities on Γ, is that (2.6) and (2.7) are satisfied.*

<u>THEOREM 3.3b.</u> *Given $\{a_n\}_{n=0}^{\infty}$ satisfying (1.1), set $b_n := a_n\sqrt{n+1}$ $(n = 0, 1, 2, \ldots)$ and, for any nonnegative integer s and n, set*

$$\tilde{D}_n^{(s)} := \det \begin{bmatrix} b_n & b_{n+1} & \ldots & b_{n+s} \\ b_{n+1} & b_{n+2} & \ldots & b_{n+s+1} \\ \vdots & & & \vdots \\ b_{n+s} & b_{n+s+1} & \ldots & b_{n+2s} \end{bmatrix}. \tag{3.6}$$

A necessary and sufficient conition that f_2 of (3.1) has exactly p poles (counted according to multiplicities) and no other singularities on Γ, is that

$$\limsup_{n\to\infty} |\tilde{D}_n^{(s)}|^{1/n} = 1, \text{ for } s = 0, 1, \ldots, p-1, \tag{3.7}$$

and

$$\limsup_{n\to\infty} |\tilde{D}_n^{(p)}|^{1/n} < 1. \tag{3.8}$$

<u>THEOREM 3.4.</u> *Given* $\{a_n\}_{n=0}^{\infty}$ *satisfying* (1.1), *assume that* $g(n) = a_n$ $(n = 0, 1, 2, \ldots)$, *where* g *is an entire function satisfying* (2.8). *Then,* $\Psi(1)$ *is the only singular point for both* f_2 *and* f_3, *of* (3.1) *and* (3.2), *in* $G_{1/r_0} =$ *Int* Γ_{1/r_0}. *Moreover,* $\Psi(1)$ *is a pole of order* p *if* g *is a polynomial of degree* $p - 1$.

<u>THEOREM 3.5.</u> *Let* $\{a_n\}_{n=0}^{\infty}$ *satisfy* (1.1) *and the conditions of* (2.9) *and* (2.10). *Then,* Γ *is a natural boundary for both* f_2 *and* f_3, *of* (3.1) *and* (3.2).

One can see that Theorems 3.1, 3.2 and 3.5 are identical with the corresponding results on Faber series. However, the results of Theorems 3.3 and 3.4 are slightly weaker than those of Theorems 2.3 and 2.4.

4. Proofs

The proofs of our statements in Section 2 and 3 make use of asymptotic properties of the Faber polynomials and the orthonormal polynomials, associated with the analytic Jordan curve Γ. These are given in the following lemma, where we recall that r_0, with $0 < r_0 < 1$, is the smallest number such that Φ can be extended to a conformal mapping from Ω_{r_0} to D'_{r_0}.

<u>LEMMA 4.1.</u> *The following asymptotic formulas are valid for any* $z \in \Omega_{r_0}$*:*

$$F_n(z) = \Phi^n(z) - h_{n,1}(z), \tag{4.1}$$

$$K_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi'(z)\Phi^n(z)(1 + h_{n,2}(z)), \tag{4.2}$$

$$Q_n(z) = \sqrt{\frac{\Phi'(z)}{2\pi}}\Phi^n(z)(1 + h_{n,3}(z)), \tag{4.3}$$

where $h_{n,i}(z)$ *is analytic in* Ω_{r_0} *and* $h_{n,i}(\infty) = 0$ $(i = 1, 2, 3; n = 0, 1, \ldots)$. *Moreover, the following estimates hold locally uniformly, as* $n \to \infty$, *in the domains indicated:*

$$|h_{n,1}(z)| = O(\rho^n),\ \ z \in \Gamma_\rho,\ \ \rho > r_0, \tag{4.4}$$

$$|h_{n,2}(z)| = \begin{cases} O(n^{-1/2})\frac{r_0^n}{\rho^n},\ \ z \in \Gamma_\rho,\ \ r_0 < \rho < 1, \\ O(n^{1/2})r_0^n,\ \ z \in \overline{\Omega}, \end{cases} \tag{4.5}$$

and

$$|h_{n,3}(z)| = \begin{cases} O(n^{1/2})\frac{r_0^n}{\rho^n}, & z \in \Gamma_\rho,\ r_0 < \rho < 1, \\ O(n^{1/2}) r_0^n, & z \in \overline{\Omega}. \end{cases} \tag{4.6}$$

Proof. First, (4.1) is just a restatement of (2.2). To derive (4.4), consider any $z \in \Gamma_\rho$ with $\rho > r_0$. For any $\epsilon > 0$ with $\rho > r_0 + \epsilon$, we have from (4.1) that

$$\frac{1}{2\pi i} \int_{\Gamma_{r_0+\epsilon}} \frac{F_n(t)dt}{t-z} + \frac{1}{2\pi i} \int_{\Gamma_{r_0+\epsilon}} \frac{h_{n,1}(t)}{t-z} = \frac{1}{2\pi i} \int_{\Gamma_{r_0+\epsilon}} \frac{\Phi^n(t)dt}{t-z}.$$

But as $z \notin \overline{G}_{r_0+\epsilon}$ and $z \in \Omega_{r_0+\epsilon}$, the Cauchy integral theorem gives that the first integral above vanishes, while the second integral reduces to $h_{n,1}(z)$, i.e.,

$$h_{n,1}(z) = \frac{1}{2\pi i} \int_{\Gamma_{r_0+\epsilon}} \frac{\Phi^n(t)dt}{t-z}.$$

Consequently, as $t \in \Gamma_{r_0+\epsilon}$ implies $|\Phi(t)| = r_0 + \epsilon$, we have, on taking moduli, in the above display, that

$$|h_{n,1}(z)| \le \frac{(r_0+\epsilon)^n \cdot \ell_{r_0+\epsilon}}{2\pi dist(z; \Gamma_{r_0+\epsilon})}, \qquad \text{where } \ell_{r_0+\epsilon} \text{ denotes the length of } \Gamma_{r_0+\epsilon}.$$

As this upper bound is $O(\rho^n)$, as $n \to \infty$, this gives (4.4). We refer to Gaier [4, p. 12] for (4.2) and (4.5), and to Smirnov and Lebedev [7, p. 338] for (4.3) and (4.6). (We warn the reader that the normalizations used in [7] for the contour integrals are *different* from our normalizations.) ■

LEMMA 4.2. *Under the assumption* (1.1)*, each of the series* (2.3)*,* (3.1) *and* (3.2) *converges, locally uniformly, in G to an analytic and single-valued function, and diverges for any $z \in \Omega$.*

Proof. It follows from Lemma 4.1 that the polynomials $F_n(z)$, $K_n(z)$ and $Q_n(z)$ are geometrically small in G and geometrically large in Ω. Indeed, if ρ is fixed with $\rho > r_0$, we can again choose $\epsilon > 0$ with $\rho > r_0 + \epsilon$. Since $h_{n,1}(z)$ is analytic in Ω_{r_0} and since $\overline{\Omega}_\rho \subset \overline{\Omega}_{r_0+\epsilon}$, the maximum principle and (4.4) give that

$$\max_{z \in \Gamma_\rho} |h_{n,1}(z)| \le \max_{z \in \Gamma_{r_0+\epsilon}} |h_{n,1}(z)| = O\left((r_0+\epsilon)^n\right).$$

On the other hand, for any $z \in \Gamma_\rho$ (where $|\Phi^n(z)| = \rho^n$), it follows from (4.1) and the previous display that

$$\rho^n - O\left((r_0+\epsilon)^n\right) \le |F_n(z)| \le \rho^n + O\left((r_0+\epsilon)^n\right)$$

for any $z \in \Gamma_\rho$. Similar estimates also hold for $K_n(z)$ and $Q_n(z)$, giving

$$\lim_{n\to\infty} |F_n(z)|^{1/n} = \lim_{n\to\infty} |K_n(z)|^{1/n} = \lim_{n\to\infty} |Q_n(z)|^{1/n} = \rho, \tag{4.7}$$

where all limits hold uniformly in $z \in \Gamma_\rho$ (for fixed ρ) for any $\rho > r_0$. Thus, with the assumption of (1.1, we have that the series

$$\sum_{n=0}^{\infty} a_n F_n(z) \tag{4.8}$$

converges locally uniformly in G, to a single-valued analytic function. The same also holds, from (4.7), for the Fourier series (3.1) and (3.2).

The divergence in Ω, of the series (4.8) and the associated Fourier series (3.1) and (3.2), then follows from (1.1) and (4.7). ■

This brings us to

PROPOSITION 4.3. *Let* (1.1) *be satisfied, and consider the associated Taylor series*

$$u_1(w) := \sum_{n=0}^{\infty} a_n w^n, \tag{4.9}$$

which is convergent in D. *Then, the complex number* w_0, *with* $|w_0| \geq 1$, *is a singularity of* $u_1(w)$ *if and only if* $z_0 := \Psi(w_0)$ *is a singularity of* $f_1(z)$, *as given by* (2.3). *Furthermore,* w_0 *is a pole of the order* p *for* $u_1(w)$ *if and only if* z_0 *is a pole of the order* p *for* $f_1(z)$.

Proof. Using the definition of (4.1), we obtain

$$\sum_{n=0}^{\infty} a_n F_n(z) = \sum_{n=0}^{\infty} a_n \Phi^n(z) - \sum_{n=0}^{\infty} a_n h_{n,1}(z), \ z \in G. \tag{4.10}$$

Observe that

$$H_1(z) := -\sum_{n=0}^{\infty} a_n h_{n,1}(z) \tag{4.11}$$

is analytic in Ω_{r_0}, with $H_1(\infty) = 0$, because the series on the right of (4.11) converges locally uniformly in Ω_{r_0} from (1.1) and from (4.4), with $1 > \rho > r_0$. Writing $z = \Psi(w)$, we have, with (2.3) and (4.9), that (4.10) can be expressed in the following form:

$$f_1(\Psi(w)) - u_1(w) = H_1(\Psi(w)), \quad r_0 < |w| < 1. \tag{4.12}$$

Now, $u_1(w)$ is regular in $|w| < 1$ and $f_1(\Psi(w))$ is regular in $r_0 < |w| < 1$, but $H_1(\Psi(w))$ is regular in $|w| > r_0$. By means of analytic continuation, it follows that $f_1(\Psi(w))$ and $u_1(w)$ have the exact same singularities and the exact same points of regularity in $|w| \geq 1$. Thus,

w_0, with $|w_0| \geq 1$, is a singularity of $u_1(w)$ if and only if w_0 is also a singularity of $f_1(\Psi(w_0))$ and consequently, if and only if $z_0 = \Psi(w_0)$ is a singularity of $f_1(z)$. In particular, it similarly follows that w_0 is a pole of order p of $\mu_1(w)$ if and only if $z_0 = \Psi(w_0)$ is a pole of order p of $f_1(z)$. ■

We remark that the first part of Proposition 4.3 appears in [12, Theorem 3.1], in a slightly more general form, i.e., a generalized Faber series replaces that to $f_1(z)$ in (2.3). The latter part of Proposition 4.3, which covers the pole characterization of such singularities, is not discussed in [12].

Proposition 4.3 is the desired bridge that allows us to translate the results on Taylor series in Theorems 2.1-2.5. Indeed, Theorem 2.1 is an immediate consequence of Pringsheim's theorem (cf. Titchmarsh [11, p. 214]). Correspondingly, Theorem 2.2 follows from Fabry's ratio theorem (cf. Dienes [5, p. 377]). Theorem 2.3 follows from Hadamard's theorem on polar singularities (cf. Dienes [5, p. 333]). Theorem 2.4 follows from the result due to Faber (cf. Dienes [5, pp. 337-339]). Theorem 2.5 is a consequence of Fabry's gap theorem (cf. Dienes [5, p. 376]). This then completes the proofs of the theorems of Section 2.

We now proceed to the proofs of the results on Fourier series.

PROPOSITION 4.4. *Suppose that* (1.1) *is valid, so that* $u_1(w)$ *and*

$$u_2(w) := \sum_{n=0}^{\infty} a_n \sqrt{n+1}\, w^n \tag{4.13}$$

converge locally uniformly in D. *Then, the complex number* w_0, *with* $1 \leq |w_0| < 1/r_0$, *is a singularity of* $u_1(w)$ *if and only if* $z_0 = \Psi(w_0)$ *is a singularity of* f_3 *as given by* (3.2). *Correspondingly,* w_0, *with* $1 \leq |w_0| < 1/r_0$, *is a singular point for* $u_2(w)$ *if and only if* $z_0 = \Psi(w_0)$ *is a singular point for* f_2 *as given in* (3.1). *Furthermore,* w_0, *with* $1 \leq |w_0| < 1/r_0$, *is a pole of order* p *of* $u_1(w)$ *or* $u_2(w)$ *if and only if* $z_0 = \Psi(w_0)$ *is a pole of order* p *of* f_3 *or* f_2, *respectively.*

Proof. We proceed as in the proof of Proposition 4.3, using the asymptotic relations of Lemma 4.1. ¿From the definitions in (4.2) and (4.3), consider the following formal series

$$\sum_{n=0}^{\infty} a_n Q_n(z) = \sqrt{\frac{\Phi'(z)}{2\pi}} \left(\sum_{n=0}^{\infty} a_n \Phi^n(z) + \sum_{n=0}^{\infty} a_n \Phi^n(z) h_{n,3}(z) \right), \quad r_0 < |\Phi(z)| < 1, \tag{4.14}$$

and

$$\sum_{n=0}^{\infty} a_n K_n(z) = \frac{\Phi'(z)}{\sqrt{\pi}} \left(\sum_{n=0}^{\infty} a_n \sqrt{n+1}\, \Phi^n(z) + \sum_{n=0}^{\infty} a_n \sqrt{n+1}\, \Phi^n(z) h_{n,2}(z) \right), \tag{4.15}$$

$$r_0 < |\Phi(z)| < 1.$$

With (4.7), the two series of the left in (4.14) and (4.15) define analytic functions in G, while each first series on the right in (4.14) and (4.15) similarly defines an analytic function in $r_0 < |\Phi(z)| < 1$. On considering (1.1) together with the estimates of (4.5) and (4.6), we conclude that each of the final series of (4.14) and (4.15), namely,

$$H_2(z) := \sum_{n=0}^{\infty} a_n \sqrt{n+1}\Phi^n(z) h_{n,2}(z) \tag{4.16}$$

and

$$H_3(z) := \sum_{n=0}^{\infty} a_n \Phi^n(z) h_{n,3}(z), \tag{4.17}$$

are both analytic in the annular region $\{z \in \mathbf{C} : r_0 < |\Phi(z)| < 1/r_0\}$. Since both $\Phi'(z)$ and $\sqrt{\Phi'(z)}$ are analytic in Ω_{r_0}, the statement of Proposition 4.4 then follows from (4.14) and (4.15) with the substitution $w = \Phi(z)$. For the proof of the final statement in Proposition 4.4, suppose that w_0, with $1 \le |w_0| < 1/r_0$, is a pole of order p of $u_1(w)$. On considering the Laurent series expansion in a neighborhood of w_0, one sees that f_2 and f_3 necessarily also have poles of order p at w_0, and conversely. ∎

Theorems 3.1-3.5 follow from the classical results, for the Taylor series of (4.9) and (4.13), in the exact same fashion as in the proofs of Theorems 2.1-2.5.

5. Further remarks and generalizations

We remark that our main results can be proved in the more general case of *weighted* Faber polynomials (cf. Smirnov and Lebedev [7]) and orthogonal polynomials with respect to a *weighted* inner product defined by the area or contour integral [7], if the weight function is analytic in Ω_{r_0}. Since the statements of the results remain the same and since the proofs change only slightly, we omit these considerations for the sake of brevity.

Another generalization, which is worth mentioning, is to consider the analogue of Faber series for multiply connected domains of ([7, p. 145]). We give only a sample of the results which can be obtained in this direction. It is related to the doubly connected case, i.e., we consider the Faber-Laurent series, (cf. Tietz [10]). Let A be an annular region bounded by two analytic Jordan curves Γ_e and Γ_i, where Γ_i is interior to Γ_e. We denote the exterior of Γ_e by Ω (so that $\infty \in \Omega$) and the interior of Γ_i by G, where we assume that $0 \in G$. Consider the conformal mapping $\phi : G \to D' = \{w : |w| > 1\}$ normalized by $\phi(0) = \infty$

and $\lim_{z\to 0} z\phi(z) = \gamma > 0$, together with its inverse $\psi := \phi^{[-1]}$. Then, in a neighborhood of the origin, $\phi(z)$ has an expansion of the form

$$\phi(z) = \frac{\gamma}{z} + \gamma_0 + \gamma_1 z + \cdots + \gamma_k z^k + \cdots .$$

This yields

$$[\phi(z)]^n = L_n\left(\frac{1}{z}\right) + E_n(z), \quad n = 1, 2, \ldots, \tag{5.1}$$

where $L_n(1/z)$ is a polynomial of degree n in $1/z$ and $E_n(z)$ contains only nonnegative powers of z of $[\phi(z)]^n$. Let the classical Faber polynomials $\{F_n(z)\}_{n=0}^{\infty}$ be defined by (2.2), with the help of conformal mapping $\Phi : \Omega \to D'$ (see Sections 1-2). Then, the union of the two sequences $\{L_n(1/z)\}_{n=1}^{\infty}$ and $\{F_n(z)\}_{n=0}^{\infty}$, called the Faber-Laurent system, forms the basis for the definition of the Faber-Laurent series. If a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ satisfies (1.1) and a sequence of complex numbers $\{b_n\}_{n=0}^{\infty}$ satisfies

$$\limsup_{n\to\infty} |b_n|^{1/n} = 1, \tag{5.2}$$

then the *Faber-Laurent series*, defined by

$$f_4(z) := \sum_{n=0}^{\infty} a_n F_n(z) + \sum_{n=1}^{\infty} b_n L_n(1/z), \tag{5.3}$$

converges locally uniformly in the annular region A. This can be deduced by the methods similar to those used in the proof of Lemma 4.2 (see [10] and [7] for the additional information).

THEOREM 5.1. *Given the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ of complex numbers which respectively satisfy* (1.1) *and* (5.2), *suppose that $a_n \geq 0$ and $b_n \geq 0$ $(n = 0, 1, 2, \ldots)$. Then, $\Psi(1) \in \Gamma_e$ and $\psi(1) \in \Gamma_i$ are singularities of f_4 in* (5.3).

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