NORMS OF PRODUCTS AND FACTORS OF POLYNOMIALS

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ABSTRACT. We study inequalities connecting a product of uniform norms of polynomials with the norm of their product. Generalizing Gel'fond-Mahler inequality for the unit disk and Kneser-Borwein inequality for the segment $[-1, 1]$, we prove an asymptotically sharp inequality for norms of products of algebraic polynomials over an arbitrary compact set in plane. Applying similar techniques, we produce a related inequality for the norm of a single monic factor of a monic polynomial. The best constants in both inequalities are obtained by potential theoretic methods. We also consider applications of the general results to the cases of a disk and a segment.

1. INTRODUCTION

Let E be a compact set in the complex plane $\mathbb{C}$. Define the uniform (sup) norm on E as follows:

$$\|f\|_E = \sup_{z \in E} |f(z)|.$$

Consider algebraic polynomials $\{p_k(z)\}_{k=1}^m$ and their product

$$p(z) := \prod_{k=1}^{m} p_k(z).$$

We are interested here in polynomial inequalities of the form

$$(1.1) \qquad \prod_{k=1}^{m} \|p_k\|_E \leq C \|p\|_E.$$

One of the first results in this direction is due to Kneser [16], for $E = [-1, 1]$ and $m = 2$ (see also Aumann [1]), who proved that

$$(1.2) \qquad \|p_1\|_{[-1,1]} \|p_2\|_{[-1,1]} \leq K_{\ell,n} \|p_1 p_2\|_{[-1,1]},$$

where

$$(1.3) \qquad K_{\ell,n} := 2^{n-1} \prod_{k=1}^{\ell} \left(1 + \cos \frac{2k-1}{2n} \pi\right) \prod_{k=1}^{n-\ell} \left(1 + \cos \frac{2k-1}{2n} \pi\right),$$

$\deg p_1 = \ell$ and $\deg(p_1 p_2) = n$. Note that (1.2) becomes an equality for the Chebyshev polynomial $t(z) = \cos n \arccos z = p_1(z) p_2(z)$, with a proper choice of the factors $p_1(z)$ and $p_2(z)$. P. B. Borwein [6] gave a new proof of (1.2)-(1.3) and generalized this to the multifactor inequality

$$(1.4) \qquad \prod_{k=1}^{m} \|p_k\|_{[-1,1]} \leq 2^{n-1} \prod_{k=1}^{\left[\frac{n}{2}\right]} \left(1 + \cos \frac{2k-1}{2n} \pi\right)^2 \|p\|_{[-1,1]}.$$

He has also showed that

$$(1.5) \qquad 2^{n-1} \prod_{k=1}^{\left[\frac{n}{2}\right]} \left(1 + \cos \frac{2k-1}{2n} \pi\right)^2 \sim (3.20991\ldots)^n, \text{ as } n \to \infty.$$

Another case of the inequality (1.1) was considered by Gel'fond [13, p. 135] in connection with the theory of transcendental numbers, for $E = \overline{D}$, where $D := \{w : |w| < 1\}$ is the unit disk:

$$(1.6) \qquad \prod_{k=1}^{m} \|p_k\|_{\overline{D}} \leq e^n \|p\|_{\overline{D}}.$$

The latter inequality was improved by Mahler [19], who replaced e by 2:

$$(1.7) \qquad \prod_{k=1}^{m} \|p_k\|_{\overline{D}} \leq 2^n \|p\|_{\overline{D}}.$$

It is easy to see that the base 2 cannot be decreased, if $m = n$ and $n \to \infty$. However, (1.7) has recently been further improved in two directions. D. W. Boyd [8, 9] showed that, by taking in account the number of factors m in (1.7), one has

$$(1.8) \qquad \prod_{k=1}^{m} \|p_k\|_{\overline{D}} \leq (C_m)^n \|p\|_{\overline{D}},$$

where

$$(1.9) \qquad C_m := \exp\left(\frac{m}{\pi} \int_0^{\pi/m} \log\left(2 \cos \frac{t}{2}\right) dt\right)$$

is asymptotically best possible for each fixed m, as $n \to \infty$. Kroó and Pritsker [17] showed that, for any $m \le n$,

$$(1.10) \qquad \prod_{k=1}^{m} \|p_k\|_{\overline{D}} \le 2^{n-1} \|p\|_{\overline{D}},$$

where equality holds in (1.10) for *each* $n \in \mathbb{N}$, with $m = n$ and $p(z) = z^n - 1$. We give an asymptotically sharp inequality for the norm of products of polynomials on arbitrary compact set in Section 2, which generalizes the results of Mahler, Kneser and Borwein. This inequality and other connected to it results were originally obtained in [22].

A closely related problem is to estimate the norm of a single factor via the norm of the whole polynomial. Clearly, we have to normalize the problem by assuming that $p(z)$ is a monic polynomial of degree n, with a monic factor $q(z)$, so that

$$p(z) = q(z)\, r(z).$$

In the case of the unit disk, Boyd [8] proved an asymptotically sharp inequality

$$(1.11) \qquad \|q\|_{\overline{D}} \le \beta^n \|p\|_{\overline{D}},$$

with

$$(1.12) \qquad \beta := \exp\left(\frac{1}{\pi} \int_0^{2\pi/3} \log\left(2\cos\frac{t}{2} \right) dt \right).$$

This inequality improved upon a series of results by Mignotte [21], Granville [15] and Glesser [14].

Further progress was made by Borwein in [6], for the segment $[-a, a]$, $a > 0$ (see Theorems 2 and 5 there or see Section 5.3 in [7]). In particular, Borwein proved that if $\deg q = m$ then

$$(1.13) \qquad |q(-a)| \le \|p\|_{[-a,a]} a^{m-n} 2^{n-1} \prod_{k=1}^{m} \left(1 + \cos\frac{2k-1}{2n}\pi \right),$$

where the bound is attained for a monic Chebyshev polynomial of degree n on $[-a, a]$ and a factor q. He also showed that, for $E = [-2, 2]$, the constant

in the above inequality satisfies

$$\limsup_{n\to\infty} \left(2^{m-1} \prod_{k=1}^{m} \left(1 + \cos \frac{2k-1}{2n}\pi \right) \right)^{1/n}$$

$$\leq \lim_{n\to\infty} \left(2^{[2n/3]-1} \prod_{k=1}^{[2n/3]} \left(1 + \cos \frac{2k-1}{2n}\pi \right) \right)^{1/n}$$

$$= \exp \left(\int_0^{2/3} \log \left(2 + 2\cos \pi x \right) dx \right) = 1.9081\ldots,$$

which hints that

$$(1.14) \qquad C_{[-2,2]} = \exp \left(\int_0^{2/3} \log \left(2 + 2\cos \pi x \right) dx \right) = 1.9081\ldots.$$

In Section 3, we find an asymptotically sharp inequality of this type for a rather arbitrary compact set E. The general result is then applied to the cases of a disk and a line segment, so that we recover (1.11)-(1.12) and confirm (1.14). Also see [23] for these results.

Considered problems have applications in transcendence theory (see [13]) and in designing algorithms for factoring polynomials (see [10] and [18]). We confine ourselves to studying the sup norms for polynomials of one variable only. A survey of the results involving other norms (e.g., Bombieri norms) can be found in [10]. These inequalities are also of considerable interest for polynomials in several variables, where very little is known about sharp constants (cf. [2], [3], [5] and [20]).

2. Products of Polynomials in Uniform Norms

Inequalities (1.2)-(1.10) clearly indicate that the constant C in (1.1) grows exponentially fast with n, with the base for the exponential depending on the set E. A natural general problem arising here is to find the *smallest* constant $M_E > 0$, such that

$$(2.1) \qquad \prod_{k=1}^{m} \|p_k\|_E \leq M_E^n \|p\|_E$$

for arbitrary algebraic polynomials $\{p_k(z)\}_{k=1}^{m}$ with complex coefficients, where $p(z) = \prod_{k=1}^{m} p_k(z)$ and $n = \deg p$. The solution of this problem is based on the logarithmic potential theory (cf. [26] and [25]). Let $\mathrm{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\mathrm{cap}(E) > 0$, denote the *equilibrium measure* of E by μ_E. We remark that μ_E is a positive

unit Borel measure supported on E (see [26, p. 55]). Define

$$(2.2) \qquad d_E(z) := \max_{t \in E} |z - t|, \qquad z \in \mathbb{C},$$

which is clearly a positive and continuous function on $\mathbb{C}$.

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a compact set, $\mathrm{cap}(E) > 0$. Then the best constant M_E in (2.1) is given by*

$$(2.3) \qquad M_E = \frac{\exp\left(\int \log d_E(z) d\mu_E(z)\right)}{\mathrm{cap}(E)}.$$

One can see from (2.1) or (2.3) that M_E is invariant with respect to the rigid motions and dilations of the set E in the plane.

Note that the restriction $\mathrm{cap}(E) > 0$ excludes only very *thin* sets from our consideration (see [26, pp. 63-66]), e.g., finite sets in the plane. On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [26, p. 56]). In particular, if E is a continuum, i.e., a connected set, then we obtain a simple universal bound for M_E.

Corollary 2.2. *Let $E \subset \mathbb{C}$ be a bounded continuum (not a single point). Then we have*

$$(2.4) \qquad M_E \leq \frac{\mathrm{diam}(E)}{\mathrm{cap}(E)} \leq 4,$$

where $\mathrm{diam}(E)$ is the Euclidean diameter of the set E.

For the unit disk $D = \{w : |w| < 1\}$, we have that $\mathrm{cap}(\overline{D}) = 1$ [26, p. 84] and that

$$(2.5) \qquad \mu_{\overline{D}} = \frac{1}{2\pi} d\theta,$$

where $d\theta$ is the arclength on ∂D. Thus Theorem 2.1 yields
(2.6)
$$M_{\overline{D}} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log d_{\overline{D}}(e^{i\theta}) \, d\theta\right) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log 2 \, d\theta\right) = 2,$$

so that we immediately obtain Mahler's inequality (1.7).

If $E = [-1, 1]$ then $\mathrm{cap}([-1, 1]) = 1/2$ and

$$(2.7) \qquad \mu_{[-1,1]} = \frac{dx}{\pi \sqrt{1 - x^2}}, \qquad x \in [-1, 1],$$

which is the Chebyshev (or arcsin) distribution (see [26, p. 84]). Using Theorem 2.1, we obtain

$$
\begin{aligned}
M_{[-1,1]} \quad &= \quad 2\exp\left(\frac{1}{\pi}\int_{-1}^{1}\frac{\log d_{[-1,1]}(x)}{\sqrt{1-x^2}}dx\right) = 2\exp\left(\frac{2}{\pi}\int_{0}^{1}\frac{\log(1+x)}{\sqrt{1-x^2}}dx\right) \\
(2.8) \quad &= \quad 2\exp\left(\frac{2}{\pi}\int_{0}^{\pi/2}\log(1+\sin t)dt\right) \approx 3.2099123.
\end{aligned}
$$

This gives the asymptotic version of Borwein's inequality (1.4)-(1.5).

It appears that the upper bound 4 in Corollary 2.2 is not the best possible. One might conjecture that the sharp universal bounds are as follows

$$
(2.9) \qquad\qquad 2 = M_{\overline{D}} \le M_E \le M_{[-1,1]} \approx 3.2099123,
$$

for any bounded non-degenerate continuum E.

It is of interest to determine the nature of the extremal polynomials for (2.1). We characterized the asymptotically extremal polynomials for (2.1), i.e., those polynomials, for which (2.1) becomes an asymptotic equality as $n \to \infty$, by their asymptotic zero distributions. The precise statements of these results can be found in Theorems 2.3-2.5 and Corollaries 3.1-3.3 of [22].

3. Uniform Norm of a Single Factor

In the same way as in Section 2, we naturally arrive at the problem to find the best (the smallest) constant C_E, such that

$$
(3.1) \qquad\qquad \|q\|_E \le C_E^n \|p\|_E, \quad \deg p = n,
$$

is valid for *any* monic polynomial $p(z)$ and *any* monic factor $q(z)$. Our solution of this problem is based on similar ideas, involving the logarithmic capacity and the equilibrium measure of E.

Theorem 3.1. *Let $E \subset \mathbb{C}$ be a compact set, $\mathrm{cap}(E) > 0$. Then the best constant C_E in (3.1) is given by*

$$
(3.2) \qquad C_E = \frac{\displaystyle\max_{u \in \partial E}\exp\left(\int_{|z-u|\ge 1}\log|z-u|d\mu_E(z)\right)}{\mathrm{cap}(E)}.
$$

Furthermore, if E is regular then

$$
(3.3) \qquad C_E = \max_{u \in \partial E}\exp\left(-\int_{|z-u|\le 1}\log|z-u|d\mu_E(z)\right).
$$

The above notion of regularity is to be understood in the sense of exterior Dirichlet problem (cf. [26, p. 7]).

One can readily see from (3.1) or (3.2) that the best constant C_E is invariant under the rigid motions of the set E in the plane. Therefore we consider applications of Theorem 3.1 to the family of disks $D_r := \{z : |z| < r\}$, which are centered at the origin, and to the family of segments $[-a, a]$, $a > 0$.

Corollary 3.2. *Let D_r be a disk of radius r. Then the best constant $C_{\overline{D}_r}$, for $E = \overline{D}_r$, is given by*

$$(3.4) \quad C_{\overline{D}_r} = \begin{cases} \dfrac{1}{r}, & 0 < r \leq 1/2, \\[3mm] \dfrac{1}{r} \exp\left(\dfrac{1}{\pi} \displaystyle\int_0^{\pi - 2\arcsin\frac{1}{2r}} \log\left(2r\cos\dfrac{x}{2} \right) dx \right), & r > 1/2. \end{cases}$$

Note that (1.11)-(1.12) immediately follow from (3.4) for $r = 1$. The graph of $C_{\overline{D}_r}$ is in Figure 1.
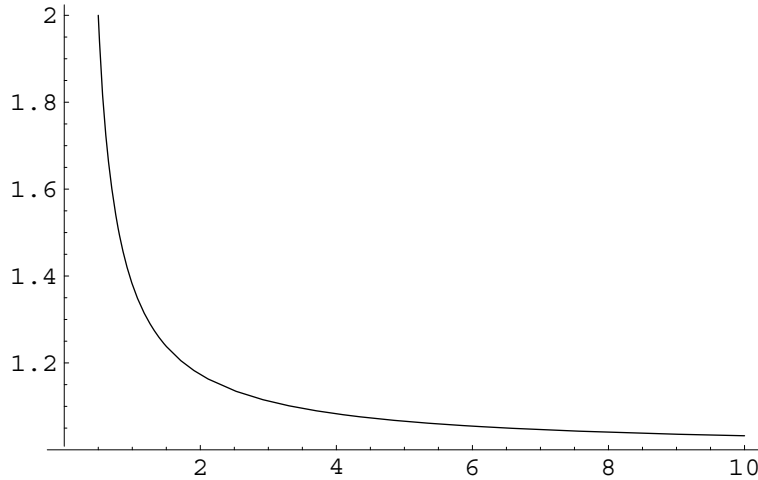

FIGURE 1. $C_{\overline{D}_r}$ as a function of r.

Corollary 3.3. *If $E = [-a, a]$, $a > 0$, then*

(3.5) $$C_{[-a,a]} = \begin{cases} \dfrac{2}{a}, & 0 < a \leq 1/2, \\[3ex] \dfrac{2}{a} \exp\left(\displaystyle\int_{1-a}^{a} \frac{\log(t+a)}{\pi\sqrt{a^2 - t^2}}\, dt \right), & a > 1/2. \end{cases}$$

Observe that (3.5), with $a = 2$, implies (1.14) by the change of variable $t = 2\cos\pi x$. We include the graph of $C_{[-a,a]}$ in Figure 2.
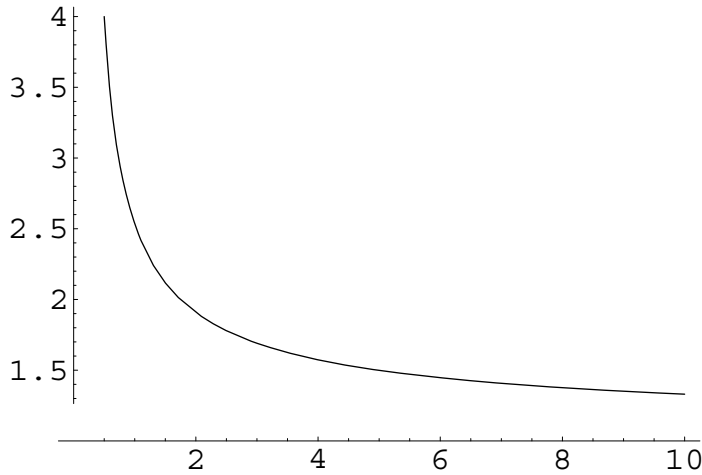

FIGURE 2. $C_{[-a,a]}$ as a function of a.

We now state two general consequences of Theorem 3.1. They explain some interesting features of C_E, which the reader may have noticed in Corollaries 3.2 and 3.3. Recall that the Euclidean diameter of E is defined by

$$\mathrm{diam}(E) := \max_{z,\zeta \in E} |z - \zeta|.$$

Corollary 3.4. *Suppose that* $\mathrm{cap}(E) > 0$. *If* $\mathrm{diam}(E) \leq 1$ *then*

(3.6) $$C_E = \frac{1}{\mathrm{cap}(E)}.$$

It is well known that $\mathrm{cap}(D_r) = r$ and $\mathrm{cap}([-a, a]) = a/2$ (see [25, p. 135]), which clarifies the first lines of (3.4) and (3.5) by (3.6).

The next Corollary shows how the constant C_E behaves under dilations of the set E. Let αE be the dilation of E with a factor $\alpha > 0$.

Corollary 3.5. *If E is regular then*

$$(3.7) \qquad \lim_{\alpha \to +\infty} C_{\alpha E} = 1.$$

Thus Figures 1 and 2 clearly illustrate (3.7).

We conclude this section with two remarks.

Remark 3.6. One can deduce inequalities of the type (3.1), for various L_p norms, from Theorem 3.1, by using relations between L_p and L_∞ norms of polynomials on E (see, e.g., [24]).

Remark 3.7. Note that the inequalities considered in this section hold for any monic factor $q(z)$ of a monic polynomial $p(z)$, i.e., they hold for the *largest* factor in the terminology of [11]. However, if we are granted the existence of the factoring $p(z) = q(z)r(z)$, then the norm of the *smallest* factor (cf. [12]) can be estimated from (2.1) as follows:

$$(3.8) \qquad \|r\|_E \le M_E^{n/2} \|p\|_E^{1/2}, \quad \deg p = n,$$

which may be better than (3.1) in some cases.

4. Proofs

The following lemma is a generalization of Lemma 2 in [9]. We refer to [22] for its proof.

Lemma 4.1. *Let $E \subset \mathbb{C}$ be a compact set (not a single point) and let*

$$d_E(z) := \max_{t \in E} |z - t|, \quad z \in \mathbb{C}.$$

Then $\log d_E(z)$ is a subharmonic function in $\mathbb{C}$ and

$$(4.1) \qquad \log d_E(z) = \int \log |z - t| d\sigma_E(t), \quad z \in \mathbb{C},$$

where σ_E is a positive unit Borel measure in $\mathbb{C}$ with unbounded support, i.e.,

$$(4.2) \qquad \sigma_E(\mathbb{C}) = 1 \quad and \quad \infty \in \operatorname{supp} \sigma_E.$$

Lemma 4.2. (Bernstein-Walsh) *Let $E \subset \mathbb{C}$ be a compact set, $\operatorname{cap}(E) > 0$, with the unbounded component of $\overline{\mathbb{C}} \setminus E$ denoted by Ω. Then, for any polynomial $p(z)$ of degree n, we have*

$$(4.3) \qquad |p(z)| \le \|p\|_E \, e^{n g_\Omega(z, \infty)}, \quad z \in \mathbb{C},$$

where $g_\Omega(z, \infty)$ is the Green function of Ω, with pole at ∞, satisfying

$$(4.4) \qquad g_\Omega(z, \infty) = \log \frac{1}{\operatorname{cap}(E)} + \int \log |z - t| d\mu_E(t), \quad z \in \mathbb{C}.$$

This is a well known result about the upper bound (4.3) for the growth of $p(z)$ off the set E (see [25, p. 156], for example). The representation (4.4) for $g_\Omega(z, \infty)$ is also classical (cf. Theorem III.37 in [26, p. 82]).

Consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^n$ for a compact set $E \subset \mathbb{C}$ (cf. [25, p. 152]). Let

$$(4.5) \qquad F_n(z) := \prod_{k=1}^n (z - a_{k,n})$$

be the Fekete polynomial of degree n, and define the normalized counting measures in Fekete points by

$$(4.6) \qquad \tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{a_{k,n}}, \quad n \in \mathbb{N},$$

where $\delta_{a_{k,n}}$ is a unit point-mass at $a_{k,n}$.

Lemma 4.3. *For a compact set $E \subset \mathbb{C}$, $\mathrm{cap}(E) > 0$, we have that*

$$(4.7) \qquad \lim_{n \to \infty} \|F_n\|_E^{1/n} = \mathrm{cap}(E)$$

and

$$(4.8) \qquad \tau_n \overset{*}{\to} \mu_E, \quad as\ n \to \infty.$$

Equation (4.7) is standard (see Theorems 5.5.4 and 5.5.2 in [25, pp. 153-155]), while (4.8) follows from (4.7) (see Ex. 5 on page 159 of [25]).

Proof of Theorem 2.1. First we show that the best constant M_E in (2.1) is at most the right hand side of (2.3). Clearly, it is sufficient to prove an inequality of the type (2.1) for monic polynomials only. Thus, we assume that $p_k(z), 1 \leq k \leq m$, are all monic, so that $p(z)$ is monic too. Let $\{z_{k,n}\}_{k=1}^n$ be the zeros of $p(z)$ and let ν_n be the normalized zero counting measure for $p(z)$. Then, we use (4.1), Fubini's theorem and Lemma 4.2 in the following estimate:

$$\frac{1}{n} \log \frac{\prod_{k=1}^m \|p_k\|_E}{\|p\|_E}$$

$$\leq \frac{1}{n} \log \frac{\prod_{k=1}^n \|z - z_{k,n}\|_E}{\|p\|_E} = \log \frac{1}{\|p\|_E^{1/n}} + \int \log d_E(z) d\nu_n(z)$$

$$= \log \frac{1}{\|p\|_E^{1/n}} + \int \int \log|z - t| d\nu_n(z) d\sigma_E(t) = \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} d\sigma_E(t)$$

$$\leq \int g_\Omega(t, \infty) d\sigma_E(t) = \log \frac{1}{\mathrm{cap}(E)} + \int \int \log|z - t| d\sigma_E(t) d\mu_E(z)$$

$$= \log \frac{1}{\mathrm{cap}(E)} + \int \log d_E(z) d\mu_E(z).$$

This gives that

$$(4.9) \qquad M_E \leq \frac{\exp(\int \log d_E(z) d\mu_E(z))}{\operatorname{cap}(E)}.$$

To show that equality holds in (4.9), we consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^n$ for E and the Fekete polynomials $F_n(z)$, $n \in \mathbb{N}$. Observe that

$$\|z - a_{k,n}\|_E = d_E(a_{k,n}), \quad 1 \leq k \leq n, \quad n \in \mathbb{N}.$$

Since $\operatorname{cap}(E) \neq 0$, the set E consists of more than one point and, therefore, $d_E(z)$ is a *strictly positive* continuous function in $\mathbb{C}$. Consequently, $\log d_E(z)$ is also continuous in $\mathbb{C}$, and we obtain by (4.8) of Lemma 4.3 that

$$(4.10) \qquad \lim_{n \to \infty} \left(\prod_{k=1}^n \|z - a_{k,n}\|_E \right)^{1/n} = \lim_{n \to \infty} \exp\left(\frac{1}{n} \sum_{k=1}^n \log d_E(a_{k,n}) \right)$$

$$= \exp\left(\lim_{n \to \infty} \int \log d_E(z) d\tau_n(z) \right) = \exp\left(\int \log d_E(z) d\mu_E(z) \right).$$

Finally, we have from the above and (4.7) that

$$M_E \geq \lim_{n \to \infty} \frac{(\prod_{k=1}^n \|z - a_{k,n}\|_E)^{1/n}}{\|F_n\|_E^{1/n}} = \frac{\exp(\int \log d_E(z) d\mu_E(z))}{\operatorname{cap}(E)}.$$

$\square$

Proof of Corollary 2.2. Since E is a bounded continuum, we obtain from Theorem 5.3.2(a) of [25, p. 138] that

$$\operatorname{cap}(E) \geq \frac{\operatorname{diam}(E)}{4}.$$

Thus, the Corollary follows by combining this estimate with the obvious inequality

$$d_E(z) \leq \operatorname{diam}(E), \quad z \in E,$$

and by using that $\mu_E(\mathbb{C}) = 1$, $\operatorname{supp}\mu_E \subset E$.

$\square$

Proof of Theorem 3.1. The proof of this result is quite similar to that of Theorem 2.1 (also see [8]). For $u \in \mathbb{C}$, consider a function

$$\rho_u(z) := \max(|z - u|, 1), \quad z \in \mathbb{C}.$$

One can immediately see that $\log \rho_u(z)$ is a subharmonic function in $z \in \mathbb{C}$, which has the following integral representation (see [25, p. 29]):

$$(4.11) \qquad \log \rho_u(z) = \int \log |z - t| \, d\lambda_u(t), \quad z \in \mathbb{C},$$

where $d\lambda_u(u + e^{i\theta}) = d\theta/(2\pi)$ is the normalized angular measure on $|t - u| = 1$.

Let $u \in \partial E$ be such that

$$\|q\|_E = |q(u)|.$$

If z_k, $k = 1, \ldots, n$, are the zeros of $p(z)$, arranged so that the first m zeros belong to $q(z)$, then

$$\log \|q\|_E \quad = \quad \sum_{k=1}^{m} \log |u - z_k| \leq \sum_{k=1}^{m} \log \rho_u(z_k) \leq \sum_{k=1}^{n} \log \rho_u(z_k)$$

$$(4.12) \qquad = \quad \sum_{k=1}^{n} \int \log |z_k - t| \, d\lambda_u(t) = \int \log |p(t)| \, d\lambda_u(t),$$

by (4.11).

It follows from (4.11)-(4.12), Lemma 4.2 and Fubini's theorem that

$$\frac{1}{n} \log \frac{\|q\|_E}{\|p\|_E} \quad \leq \quad \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} \, d\lambda_u(t) \leq \int g_\Omega(t, \infty) \, d\lambda_u(t)$$

$$= \quad \log \frac{1}{\operatorname{cap}(E)} + \int \int \log |z - t| \, d\lambda_u(t) d\mu_E(z)$$

$$= \quad \log \frac{1}{\operatorname{cap}(E)} + \int \log \rho_u(z) \, d\mu_E(z).$$

Using the definition of $\rho_u(z)$, we obtain from the above estimate that

$$\|q\|_E \quad \leq \quad \left(\frac{\max\limits_{u \in \partial E} \exp \left(\int \log \rho_u(z) \, d\mu_E(z) \right)}{\operatorname{cap}(E)} \right)^n \|p\|_E$$

$$= \quad \left(\frac{\max\limits_{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z - u| \, d\mu_E(z) \right)}{\operatorname{cap}(E)} \right)^n \|p\|_E.$$

Hence

$$(4.13) \qquad C_E \leq \frac{\max\limits_{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z - u| \, d\mu_E(z) \right)}{\operatorname{cap}(E)}.$$

In order to prove the inequality opposite to (4.13), we consider the n-th Fekete points $\{a_{k,n}\}_{k=1}^{n}$ for the set E and the Fekete polynomials $F_n(z)$, $n \in \mathbb{N}$. Let $u \in \partial E$ be a point, where the maximum of the right hand side of (4.13) is attained. Define the factor $q_n(z)$ for $F_n(z)$, with zeros

being the n-th Fekete points satisfying $|a_{k,n} - u| \geq 1$. Then we have by (4.8) that

$$
\begin{aligned}
\lim_{n\to\infty} \|q_n\|_E^{1/n} &\geq \lim_{n\to\infty} |q_n(u)|^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{n} \sum_{|a_{k,n}-u|\geq 1} \log|u - a_{k,n}|\right) \\
&= \exp\left(\lim_{n\to\infty} \int_{|z-u|\geq 1} \log|u - z|\, d\tau_n(z)\right) \\
&= \exp\left(\int_{|z-u|\geq 1} \log|u - z|\, d\mu_E(z)\right).
\end{aligned}
$$

Combining the above inequality with (4.7) and the definition of C_E, we obtain that

$$
C_E \geq \lim_{n\to\infty} \frac{\|q_n\|_E^{1/n}}{\|F_n\|_E^{1/n}} \geq \frac{\exp\left(\int_{|z-u|\geq 1} \log|z - u|\, d\mu_E(z)\right)}{\mathrm{cap}(E)}.
$$

This shows that (3.2) holds true. Moreover, if $u \in \partial E$ is a regular point for Ω, then we obtain by Theorem III.36 of [26, p. 82]) and (4.4) that

$$
\log \frac{1}{\mathrm{cap}(E)} + \int \log|u - t|\, d\mu_E(t) = g_\Omega(u, \infty) = 0.
$$

Hence

$$
\log \frac{1}{\mathrm{cap}(E)} + \int_{|z-u|\geq 1} \log|u - t|\, d\mu_E(t) = -\int_{|z-u|\leq 1} \log|u - t|\, d\mu_E(t),
$$

which implies (3.3) by (3.2). $\qquad\square$

Proof of Corollary 3.2. It is well known [26, p. 84] that $\mathrm{cap}(\overline{D_r}) = r$ and $d\mu_{\overline{D_r}}(re^{i\theta}) = d\theta/(2\pi)$, where $d\theta$ is the angular measure on ∂D_r. If $r \in (0, 1/2]$ then the numerator of (3.2) is equal to 1, so that

$$
C_{\overline{D_r}} = \frac{1}{r}, \quad 0 < r \leq 1/2.
$$

Assume that $r > 1/2$. We set $z = re^{i\theta}$ and let $u_0 = re^{i\theta_0}$ be a point where the maximum in (3.2) is attained. On writing

$$
|z - u_0| = 2r \left|\sin \frac{\theta - \theta_0}{2}\right|,
$$

we obtain that

$$
\begin{aligned}
C_{\overline{D_r}} &= \frac{1}{r} \exp\left(\frac{1}{2\pi} \int_{\theta_0 + 2\arcsin\frac{1}{2r}}^{2\pi + \theta_0 - 2\arcsin\frac{1}{2r}} \log\left| 2r\sin\frac{\theta - \theta_0}{2} \right| d\theta \right) \\
&= \frac{1}{r} \exp\left(\frac{1}{2\pi} \int_{2\arcsin\frac{1}{2r} - \pi}^{\pi - 2\arcsin\frac{1}{2r}} \log\left(2r\cos\frac{x}{2} \right) dx \right) \\
&= \frac{1}{r} \exp\left(\frac{1}{\pi} \int_0^{\pi - 2\arcsin\frac{1}{2r}} \log\left(2r\cos\frac{x}{2} \right) dx \right),
\end{aligned}
$$

by the change of variable $\theta - \theta_0 = \pi - x$. $\qquad\qquad\square$

Proof of Corollary 3.3. Recall that $\mathrm{cap}([-a,a]) = a/2$ (see [26, p. 84]) and

$$
d\mu_{[-a,a]}(t) = \frac{dt}{\pi\sqrt{a^2 - t^2}}, \quad t \in [-a,a].
$$

It follows from (3.2) that

$$
(4.14) \qquad C_{[-a,a]} = \frac{2}{a} \exp\left(\max_{u \in [-a,a]} \int_{[-a,a]\setminus(u-1,u+1)} \frac{\log|t - u|}{\pi\sqrt{a^2 - t^2}} dt \right).
$$

If $a \in (0, 1/2]$ then the integral in (4.14) obviously vanishes, so that $C_{[-a,a]} = 2/a$. For $a > 1/2$, let

$$
(4.15) \qquad f(u) := \int_{[-a,a]\setminus(u-1,u+1)} \frac{\log|t - u|}{\pi\sqrt{a^2 - t^2}} dt.
$$

One can easily see from (4.15) that

$$
f'(u) = \int_{u+1}^{a} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} < 0, \quad u \in [-a, 1 - a],
$$

and

$$
f'(u) = \int_{-a}^{u-1} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} > 0, \quad u \in [a - 1, a].
$$

However, if $u \in (1 - a, a - 1)$ then

$$
f'(u) = \int_{u+1}^{a} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} + \int_{-a}^{u-1} \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}}.
$$

It is not difficult to verify directly that

$$
\int \frac{dt}{\pi(u - t)\sqrt{a^2 - t^2}} = \frac{1}{\pi\sqrt{a^2 - u^2}} \log\left| \frac{a^2 - ut + \sqrt{a^2 - t^2}\sqrt{a^2 - u^2}}{t - u} \right| + C,
$$

which implies that

$$f'(u) = \frac{1}{\pi\sqrt{a^2 - u^2}} \log\left(\frac{a^2 - u^2 + u + \sqrt{a^2 - (u-1)^2}\,\sqrt{a^2 - u^2}}{a^2 - u^2 - u + \sqrt{a^2 - (u+1)^2}\,\sqrt{a^2 - u^2}}\right),$$

for $u \in (1 - a, a - 1)$. Hence

$$f'(u) < 0, \ u \in (1 - a, 0), \quad \text{and} \quad f'(u) > 0, \ u \in (0, a - 1).$$

Collecting all facts, we obtain that the maximum for $f(u)$ on $[-a, a]$ is attained at the endpoints $u = a$ and $u = -a$, and it is equal to

$$\max_{u\in[-a,a]} f(u) = \int_{1-a}^{a} \frac{\log(t + a)}{\pi\sqrt{a^2 - t^2}}\, dt.$$

Thus (3.5) follows from (4.14) and the above equation. $\qquad\square$

Proof of Corollary 3.4. Note that the numerator of (3.2) is equal to 1, because $|z - u| \le 1$, $\forall\, z \in E$, $\forall\, u \in \partial E$. Thus (3.6) follows immediately. $\quad\square$

Proof of Corollary 3.5. Observe that $C_E \ge 1$ for any $E \in \mathbb{C}$, so that $C_{\alpha E} \ge 1$. Since E is regular, we use the representation for C_E in (3.3). Let $T : E \to \alpha E$ be the dilation mapping. Then $|Tz - Tu| = \alpha|z - u|$, $z, u \in E$, and $d\mu_{\alpha E}(Tz) = d\mu_E(z)$. This gives that

$$
\begin{aligned}
C_{\alpha E} &= \max_{Tu\in\partial(\alpha E)} \exp\left(-\int_{|Tz-Tu|\le 1} \log|Tz - Tu|\, d\mu_{\alpha E}(Tz)\right) \\
&= \max_{u\in\partial E} \exp\left(-\int_{|z-u|\le 1/\alpha} \log(\alpha|z - u|)\, d\mu_E(z)\right) \\
&= \max_{u\in\partial E} \exp\left(-\mu_E(\overline{D_{1/\alpha}(u)})\log\alpha - \int_{|z-u|\le 1/\alpha} \log|z - u|\, d\mu_E(z)\right) \\
&< \max_{u\in\partial E} \exp\left(-\int_{|z-u|\le 1/\alpha} \log|z - u|\, d\mu_E(z)\right),
\end{aligned}
$$

where $\alpha \ge 1$. Using the absolute continuity of the integral, we have that

$$\lim_{\alpha\to+\infty} \int_{|z-u|\le 1/\alpha} \log|z - u|\, d\mu_E(z) = 0,$$

which implies (3.7). $\qquad\square$

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