# ORTHOGONAL DECOMPOSITION OF THE SPACE OF ALGEBRAIC NUMBERS AND LEHMER'S PROBLEM

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ABSTRACT. Building on work of Dubickas and Smyth regarding the metric Mahler measure and the authors regarding extremal norms associated to the Mahler measure, the authors introduce a new set of norms associated to the Mahler measure of algebraic numbers which allow for an equivalent reformulation of problems like the Lehmer problem and the Schinzel-Zassenhaus conjecture on a single spectrum. We present several new geometric results regarding the space of algebraic numbers modulo torsion using the $L^p$ Weil height introduced by Allcock and Vaaler, including an canonical decomposition of an algebraic number into an orthogonal series with respect to the $L^2$ height.

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## 1. Introduction

Let $K$ be a number field with set of places $M_K$. For each $v \in M_K$ lying over a rational prime $p$, let $\|\cdot\|_v$ be the absolute value on $K$ extending the usual $p$-adic absolute value on $\mathbb{Q}$ if $v$ is finite or the usual archimedean absolute value if $v$ is infinite. Then for $\alpha \in K^\times$, the *absolute logarithmic Weil height* $h$ is given by

$$h(\alpha) = \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log^+ \|\alpha\|_v$$

where $\log^+ t = \max\{0, \log t\}$. As the expression on the right hand side of this equation does not depend on the choice of field $K$ containing $\alpha$, $h$ is a well-defined function mapping $\overline{\mathbb{Q}}^\times \to [0, \infty)$ which vanishes precisely on the roots of unity $\mathrm{Tor}(\overline{\mathbb{Q}}^\times)$. Closely related to the Weil height is the *logarithmic Mahler measure*, given by

$$m(\alpha) = (\deg \alpha) \cdot h(\alpha)$$

where $\deg \alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Though seemingly related to the Weil height in a simple fashion, the Mahler measure is in fact a fair bit more mysterious. Perhaps the most important open question regarding the Mahler measure is *Lehmer's problem*, which asks if there exists an absolute constant $c$ such that

$$m(\alpha) \geq c > 0 \quad \text{for all} \quad \alpha \in \overline{\mathbb{Q}}^\times \setminus \mathrm{Tor}(\overline{\mathbb{Q}}^\times). \tag{1.1}$$

The question of the existence of algebraic numbers with small Mahler measure was first posed in 1933 by D.H. Lehmer [8] and since then the conjectured existence of an absolute lower bound away from zero has come to be known as *Lehmer's conjecture*. The current best known lower bound, due to Dobrowolski [3], is of the form

$$m(\alpha) \gg \left( \frac{\log\log\deg\alpha}{\log\deg\alpha} \right)^3 \quad \text{for all} \quad \alpha \in \overline{\mathbb{Q}}^\times \setminus \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$$

where the implied constant is absolute.

Recently, Allcock and Vaaler [1] observed that the absolute logarithmic Weil height $h : \overline{\mathbb{Q}}^\times \to [0, \infty)$ can in fact be viewed in an equivalent fashion as the $L^1$ norm on a certain measure space $(Y, \lambda)$. The points of $Y$ are the places of $\overline{\mathbb{Q}}$ endowed with a topology which makes $Y$ a totally disconnected locally compact Hausdorff space, and each equivalence class of the algebraic numbers modulo torsion gives rise to a unique locally constant real-valued function on $Y$ with compact support. The purpose of this paper is to construct analogous function space norms in order to study the Mahler measure. Once we have introduced our new norms, we will give a general $L^p$ formulation of the Lehmer conjecture which is equivalent to the classical Lehmer conjecture for $p = 1$ and to the Schinzel-Zassenhaus conjecture [9] for $p = \infty$.

We first briefly recall here the notation of [1], which we will use throughout this paper. To each equivalence class $\alpha$ in $\overline{\mathbb{Q}}^\times / \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$, we can uniquely associate the function $f_\alpha : Y \to \mathbb{R}$ given by

$$f_\alpha(y) = \log \|\alpha\|_y.$$

(We will often drop the subscript $\alpha$ when convenient.) We denote the space of functions given by algebraic numbers modulo torsion by $\mathcal{F}$. If $\alpha \in K$, then the function $f_\alpha(y)$ is constant on the sets $Y(K, v) = \{y \in Y : y | v\}$ for $v \in M_K$ and takes the value $\log \|\alpha\|_v$. The measure $\lambda$ is constructed so that it assigns measure $[K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$ to the set $Y(K, v)$, so that if $\alpha \in K^\times$ for some number field $K$, we have

$$\|f_\alpha\|_1 = \int_Y |f_\alpha(y)|\, d\lambda(y) = \sum_{v \in M_K} |\log \|\alpha\|_v| \, \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} = 2\, h(\alpha).$$

The product formula takes the form $\int_Y f_\alpha \, d\lambda = 0$. We also have a well-defined inner product on $\mathcal{F}$ given by

$$\langle f, g \rangle = \int_Y f(y) g(y) \, d\lambda(y)$$

which satisfies $\|f\|_2 = \langle f, f\rangle^{1/2}$. The geometry of the space $\mathcal{F}$ will play a significant role in our study.

The study of the Mahler measure on the vector space of algebraic numbers modulo torsion $\mathcal{F}$ presents several difficulties absent for the Weil height, the first of which is that $m$, unlike $h$, is not well-defined modulo torsion. Recent attempts to find topologically better-behaved objects related to the Mahler measure include the introduction of the *metric Mahler measure*, a well-defined metric on $\mathcal{F}$ by Dubickas and Smyth [5] and the introduction of the *ultrametric Mahler measure* by the first author and Samuels [7]. Both metrics induce the discrete topology if (and only if) Lehmer's conjecture is true. Later, the authors introduced vector space norms associated to the Mahler measure [6] which satisfied an extremal condition akin to those in the papers of Dubickas and Smyth and of the first author and Samuels. The norms introduced in this paper do not satisfy the same extremal condition, however, they allow the introduction of much advantageous geometry which allows for stronger results.

In order to construct our norms related to the Mahler measure, we first construct an orthogonal decomposition of the space $\mathcal{F}$ of algebraic numbers modulo torsion. We fix our algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and let $\mathcal{K}$ denote the set of finite extensions of $\mathbb{Q}$. We let $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group, and let $\mathcal{K}^G = \{K \in \mathcal{K} : \sigma K = K \text{ for all } \sigma \in G\}$. Let $V_K$ denote the $\mathbb{Q}$-vector space span of the functions given by

$$V_K = \mathrm{span}_{\mathbb{Q}} \langle \{ f_\alpha : \alpha \in K^\times / \operatorname{Tor}(K^\times) \} \rangle.$$

We first prove the following result, which gives the orthogonal decomposition by Galois field:

**Theorem 1.** *There exist projection operators $T_K : \mathcal{F} \to \mathcal{F}$ for each $K \in \mathcal{K}^G$ such that $T_K(\mathcal{F}) \subset V_K$, $T_K(\mathcal{F}) \perp T_L(\mathcal{F})$ for all $K \neq L \in \mathcal{K}^G$ with respect to the inner product on $\mathcal{F}$, and*

$$\mathcal{F} = \bigoplus_{K \in \mathcal{K}^G} T_K(\mathcal{F}).$$

The notation $\mathcal{F} = \bigoplus_{K \in \mathcal{K}^G} T_K(\mathcal{F})$ is a direct sum in the usual $\mathbb{Q}$-vector space sense, specifically, that every element of the $\mathbb{Q}$-vector space $\mathcal{F}$ is uniquely expressible as a finite sum of elements from the $\mathbb{Q}$-vector spaces $T_K(\mathcal{F})$ as $K$ ranges over the set $\mathcal{K}^G$. The term projection means an idempotent linear operator which is continuous (here, with respect to the $L^p$ norm for any $1 \leq p \leq \infty$). It is of note in this theorem that our projections are defined on the underlying $\mathbb{Q}$-vector space $\mathcal{F}$ (as well as extending by continuity to each of the closures).

In particular, it follows from Theorem 1 that the projection operators $T_K$ are orthogonal projections with respect to the inner product on $\mathcal{F}$, and thus, in the completion with respect to the $L^2$ norm, this gives a Hilbert space decomposition in the usual sense of a Hilbert space direct sum (in which each element of the Hilbert space has a unique expansion as a series of vectors, one from each summand). A decomposition by Galois field alone, however, does not give enough information about the degree of a specific number in order to bound the Mahler measure of the number (and further, as we will see in Remark 2.21, a canonical decomposition along the entire collection of number fields is not possible because of linear dependence between conjugate fields). We therefore define the vector subspace

$$V^{(n)} = \sum_{\substack{K\in\mathcal{K} \\ [K:\mathbb{Q}]\leq n}} V_K$$

(where the sum indicates a usual sum of $\mathbb{Q}$-vector spaces) and determine the following decomposition:

**Theorem 2.** *There exist projections $T^{(n)} : \mathcal{F} \to \mathcal{F}$ for each $n \in \mathbb{N}$ such that $T^{(n)}(\mathcal{F}) \subset V^{(n)}$, $T^{(m)}(\mathcal{F}) \perp T^{(n)}(\mathcal{F})$ for all $m \neq n$, and*

$$\mathcal{F} = \bigoplus_{n=1}^{\infty} T^{(n)}(\mathcal{F}).$$

These decompositions are independent of each other in the following sense:

**Theorem 3.** *The projections $T_K$ and $T^{(n)}$ commute with each other for each $K \in \mathcal{K}^G$ and $n \in \mathbb{N}$.*

In particular, as a result of commutativity, we can form projections $T_K^{(n)} = T_K T^{(n)}$ and so we have an orthogonal decomposition

$$\mathcal{F} = \bigoplus_{n=1}^{\infty} \bigoplus_{K\in\mathcal{K}^G} T_K^{(n)}(\mathcal{F}).$$

Again, when we pass to the completion in the $L^2$ norm, the projections extend by continuity and the above decomposition extends to the respective closures and the direct sum becomes a direct sum in the usual Hilbert space sense.

As a simple example of how natural this orthogonal decomposition is, we note that the 2-height of $\alpha = 2 + \sqrt{2}$ can be decomposed as:

$$\|f_{2+\sqrt{2}}\|_2^2 = \|f_{\sqrt{2}}\|_2^2 + \|f_{1+\sqrt{2}}\|_2^2$$

as the numbers $\sqrt{2}$ and $1 + \sqrt{2}$ will be seen to be orthogonal to each other. We refer the reader to Example 2.20 below for more details.

This geometric structure within the algebraic numbers allows us to define linear operators, for all $L^p$ norms with $1 \leq p \leq \infty$, which capture the contribution of the degree to the Mahler measure in such a way that we can define our Mahler norms. Specifically, we define the operator

$$\begin{aligned} M : \mathcal{F} &\to \mathcal{F} \\ f &\mapsto \sum_{n=1}^{\infty} n\, T^{(n)} f. \end{aligned}$$

The sum is finite for each $f \in \mathcal{F}$. $M$ is a well-defined, unbounded (in any $L^p$ norm, $1 \le p \le \infty$), invertible linear map defined on the incomplete vector space $\mathcal{F}$. We define the *Mahler p-norm* on $\mathcal{F}$ for $1 \le p \le \infty$ to be

$$\|f\|_{m,p} = \|Mf\|_p$$

where $\|\cdot\|_p$ denotes the usual $L^p$ norm on the incomplete vector space $\mathcal{F}$. The Mahler $p$-norm is, in fact, a well-defined vector space norm on $\mathcal{F}$, and hence the completion $\mathcal{F}_{m,p}$ with respect to $\|\cdot\|_{m,p}$ is a Banach space.

In order to see that these norms form a suitable generalization of the Mahler measure of algebraic numbers, we will show that the Lehmer conjecture can be reformulated in terms of these norms. First, let us address what form the Lehmer conjecture takes inside $\mathcal{F}$. For any $\alpha \in \overline{\mathbb{Q}}^\times$, let $h_p(\alpha) = \|f_\alpha\|_p$. (Recall that $h_1(\alpha) = 2\,h(\alpha)$.)

**Conjecture 1** ($L^p$ Lehmer conjectures)**.** *For $1 \le p \le \infty$, there exists an absolute constant $c_p$ such that the $L^p$ Mahler measure satisfies the following equation:*

$$(*_p) \qquad m_p(\alpha) = (\deg\alpha)\cdot h_p(\alpha) \ge c_p > 0 \quad \textit{for all} \quad \alpha \in \overline{\mathbb{Q}}^\times \setminus \mathrm{Tor}(\overline{\mathbb{Q}}^\times).$$

From the fact that $h_1(\alpha) = 2h(\alpha)$ it is clear that when $p = 1$ this statement is equivalent to the Lehmer conjecture. For $p = \infty$, we will show in Proposition 4.6 below that the statement is equivalent to the Schinzel-Zassenhaus conjecture.

In order to translate the Lehmer conjecture into a bound on function space norms which, unlike the metric Mahler measure, cannot possibly be discrete, it is necessary to reduce the Lehmer problem to a sufficiently small set of numbers which we can expect to be bounded away from zero in norm. This requires the introduction in Section 3 of two classes of algebraic numbers modulo torsion in $\mathcal{F}$, the *representable* elements $\mathcal{R}$ and the *projection irreducible elements* $\mathcal{P}$. Let $\mathcal{U} \subset \mathcal{F}$ denote the subspace of algebraic units. We prove the following theorem:

**Theorem 4.** *For each $1 \le p \le \infty$, equation $(*_p)$ holds if and only if*

$$(**_p) \qquad \|f\|_{m,p} \ge c_p > 0 \quad \textit{for all} \quad 0 \ne f \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$$

*where $\mathcal{R}$ denotes the set of representable elements, $\mathcal{P}$ the set of projection irreducible elements, and $\mathcal{U}$ the subspace of algebraic units. Further, for $1 \le p \le q \le \infty$, if $(**_p)$ holds then $(**_q)$ holds as well.*

The last statement of the theorem, which is proven by reducing to a place of measure 1 and applying the usual inequality for the $L^p$ and $L^q$ norms on a probability space, generalizes the well-known fact that Lehmer's conjecture implies the conjecture of Schinzel-Zassenhaus.

Let $\mathcal{U}_{m,p}$ denote the Banach space which is the completion of the vector space $\mathcal{U}$ of units with respect to the Mahler $p$-norm $\|\cdot\|_{m,p}$. The set $\mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$ has another useful property which we will prove, namely, that the additive subgroup it generates contains a subgroup $\Gamma = \Gamma_p$,

$$\Gamma \le \langle \mathcal{R} \cap \mathcal{P} \cap \mathcal{U} \rangle,$$

which is also a set of equivalence for the Lehmer conjecture, that is, we will show that the $L^p$ Lehmer conjecture $(*_p)$ is equivalent to the condition that $\Gamma$ be a discrete subgroup in $\mathcal{U}_{m,p}$. Specifically, we prove:

**Theorem 5.** *Equation $(*_p)$ holds if and only if the additive subgroup $\Gamma \subset \mathcal{U}_{m,p}$ is closed.*

This leads us to a new conjecture, equivalent to $(*_p)$ for each $1 \leq p \leq \infty$:

**Conjecture 2.** *The additive subgroup $\Gamma \subset \mathcal{U}_{m,p}$ is closed for each $1 \leq p \leq \infty$.*

Lastly, the presence of orthogonal decompositions raises a particular interest in the study of the $L^2$ norm. In this case, the norm associated to the Mahler measure has a particularly simple form which is in sympathy with the geometry of $L^2$.

**Theorem 6.** *The Mahler 2-norm satisfies*

$$\|f\|_{m,2}^2 = \sum_{n=1}^{\infty} n^2 \, \|T^{(n)}(f)\|_2^2 = \sum_{K \in \mathcal{K}^G} \sum_{n=1}^{\infty} n^2 \, \|T_K^{(n)}(f)\|_2^2.$$

*Further, the Mahler 2-norm arises from the inner product*

$$\langle f, g\rangle_m = \langle Mf, Mg\rangle = \sum_{n=1}^{\infty} n^2 \, \langle T^{(n)} f, T^{(n)} g\rangle = \sum_{K \in \mathcal{K}^G} \sum_{n=1}^{\infty} n^2 \, \langle T_K^{(n)} f, T_K^{(n)} g\rangle$$

*where $\langle f, g\rangle = \int_Y fg \, d\lambda$ denotes the usual inner product in $L^2(Y)$, and therefore the completion $\mathcal{F}_{m,2}$ of $\mathcal{F}$ with respect to the Mahler 2-norm is a Hilbert space.*

The structure of our paper is as follows. In Section 2 we introduce the basic operators and subspaces of our study, namely, those arising naturally from number fields and Galois automorphisms. The proofs of Theorems 1, 2 and 3 regarding the orthogonal decompositions of the space $\mathcal{F}$ with respect to Galois field and degree will then be carried out in sections 2.4, 2.5, and 2.6. In Section 3 we prove our results regarding the reduction of the classical Lehmer problem and introduce the relevant classes of algebraic numbers which are essential to our theorems. Finally in Section 4 we introduce the Mahler $p$-norms and prove the remaining results.

## 2. Orthogonal Decompositions

In this section we will develop the machinery to prove our main decomposition theorems. First, however, we must introduce several auxiliary constructions and results which will be needed later. We will start by introducing the basic isometries of our space associated to Galois automorphisms in Section 2.1, then exploring the relationships between the subspaces associated to number fields in Section 2.2 and their associated projection maps in Section 2.3. We will then prove a general decomposition for vector spaces in Section 2.4 and apply this to obtain Theorem 1, and finally in Sections 2.4 and 2.5 we will prove Theorems 2 and 3 respectively.

2.1. **Galois isometries.** Let $\mathcal{F}_p$ denote the completion of $\mathcal{F}$ with respect to the $L^p$ norm. By [1, Theorems 1-3],

$$\mathcal{F}_p = \begin{cases} \{f \in L^1(Y, \lambda) : \int_Y f \, d\lambda = 0\} & \text{if } p = 1 \\ L^p(Y, \lambda) & \text{if } 1 < p < \infty \\ C_0(Y, \lambda) & \text{if } p = \infty. \end{cases}$$

We begin by introducing our first class of operators, the isometries arising from Galois automorphisms. Let us recall how the Galois group acts on the places of an arbitrary Galois extension $K$. Suppose $\alpha \in K$, $v \in M_K$ is a place of $K$, and $\sigma \in G$. We define $\sigma v$ to be the place of $K$ given by $\|\alpha\|_{\sigma v} = \|\sigma^{-1}\alpha\|_v$, or in other words, $\|\sigma\alpha\|_v = \|\alpha\|_{\sigma^{-1} v}$.

**Lemma 2.1.** *Each $\sigma \in G$ is a measure-preserving homeomorphism of the measure space $(Y, \lambda)$.*

*Proof.* That the map $\sigma : Y \to Y$ is a well-defined bijection follows from the fact that $G$ gives a well-defined group action. Continuity of $\sigma$ and $\sigma^{-1}$ follow from [1, Lemma 3]. It remains to show that $\sigma$ is measure-preserving, but this follows immediately from [1, (4.6)]. □

In accordance with the action on places, we define for $\sigma \in G$ the operator

$$L_\sigma : \mathcal{F}_p \to \mathcal{F}_p$$

given by

$$(L_\sigma f)(y) = f(\sigma^{-1} y).$$

Thus for $f_\alpha \in \mathcal{F}$, we have $L_\sigma f_\alpha = f_{\sigma\alpha}$, and in particular $L_\sigma(\mathcal{F}) \subseteq \mathcal{F}$ for all $\sigma \in G$. Further, by our definition of the action on places, we have $L_\sigma L_\tau = L_{\sigma\tau}$.

Let $\mathcal{B}(\mathcal{F}_p)$ denote the bounded linear maps from $\mathcal{F}_p$ to itself, and let $\mathcal{I}(\mathcal{F}_p) \subset \mathcal{B}(\mathcal{F}_p)$ denote the subgroup of isometries of $\mathcal{F}_p$. By the construction of $\lambda$, each $\sigma \in G$ is a measure-preserving topological homeomorphism of the space of places $Y$, so it follows immediately that $L_\sigma$ is an isometry for all $1 \leq p \leq \infty$, that is, $\|L_\sigma f\|_p = \|f\|_p$ for all $\sigma \in G$. Thus we have a natural map

$$\begin{aligned} \rho : G &\to \mathcal{I}(\mathcal{F}_p) \\ \sigma &\mapsto L_\sigma \end{aligned}$$

where $(L_\sigma f)(y) = f(\sigma^{-1} y)$. We will show that $\rho$ gives an injective infinite dimensional representation of the absolute Galois group (which is unitary in the case of $L^2$), and further, that the map $\rho$ is continuous if $G$ is endowed with its natural profinite topology and $\mathcal{I}$ is endowed with the strong operator topology inherited from $\mathcal{B}(\mathcal{F}_p)$. (Recall that the strong operator topology, which is weaker than the norm topology, is the weakest topology such that the evaluation maps $A \mapsto \|Af\|_p$ are continuous for every $f \in L^p$.)

**Proposition 2.2.** *The map $\rho : G \to \mathcal{I}$ is injective, and it is continuous if $\mathcal{I}$ is endowed with the strong operator topology and $G$ has the usual profinite topology.*

*Proof.* First we will observe that the image $\rho(G)$ is discrete in the norm topology, so that $\rho$ is injective. To see this, fix $\sigma \neq \tau \in G$, so that there exists some finite Galois extension $K$ and an element $\alpha \in K^\times$ such that $\sigma\alpha \neq \tau\alpha$. By [4, Theorem 3], we can find a rational integer $n$ such that $\beta = n + \alpha$ is torsion-free, that is, if $\beta/\beta' \neq 1$ then $\beta/\beta' \notin \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$ for any conjugate $\beta'$ of $\beta$, and in particular, the conjugates of $\beta$ give rise to distinct functions in $\mathcal{F}$. Thus $L_\sigma f_\beta \neq L_\tau f_\beta$, so in particular, there exists some place $v$ of $K$ such that $\sigma(Y(K, v)) \neq \tau(Y(K, v))$ and are therefore disjoint sets. Choose a Galois extension $L/K$ with distinct places $w_1, w_2 | v$. Since $L/K$ is Galois, the local degrees agree and so $\lambda(Y(L, w_1)) = \lambda(Y(L, w_2))$ by [1, Theorem 5]. Define

$$f(y) = \begin{cases} 1 & \text{if } y \in Y(L, w_1) \\ -1 & \text{if } y \in Y(L, w_2) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f \in \mathcal{F}_p$ for all $1 \leq p \leq \infty$ and $L_\sigma f$ and $L_\tau f$ have disjoint support. Thus,

$$\|(L_\sigma - L_\tau) f\|_p = \left(\|L_\sigma f\|_p^p + \|L_\tau f\|_p^p\right)^{1/p} = 2^{1/p} \|f\|_p$$

(where we let $2^{1/p} = 1$ when $p = \infty$). But this implies that $1 \le 2^{1/p} \le \|L_\sigma - L_\tau\|$ for all $\sigma \neq \tau \in G$, and thus the image $\rho(G)$ is discrete in the norm topology of $\mathcal{I}$, and $\rho$ is injective.

Let us now prove continuity. Recall that a basis for the strong operator topology on $\mathcal{I}$ is given by sets of the form

$$U = \{A \in \mathcal{I} : \|(A - B)f_i\| < \epsilon \text{ for all } 1 \le i \le k\}$$

where $B \in \mathcal{I}$, $f_1, \ldots, f_k$ is a finite set of functions in $\mathcal{F}_p$, and $\epsilon > 0$. Fix such an open set $U$ for a given $B = L_\sigma$ for some $\sigma \in G$. Approximate each $f_i$ by an element $g_i \in \mathcal{F}$ such that $\|f_i - g_i\|_p < \epsilon/2^{1/p}$. Let $V_K$ be a subspace of $\mathcal{F}$ containing $g_1, \ldots, g_k$. Let

$$N = \{\tau \in G : \sigma|_K = \tau|_K\}.$$

Then $N$ is an open subset of $G$ in the profinite topology. We claim that $\rho(N) \subseteq U$, and thus that $\rho$ is continuous. To see this, observe that for $\tau \in N$,

$$\begin{aligned}\|(L_\tau - L_\sigma)f_i\|_p &\le \|(L_\tau - L_\sigma)g_i\|_p + \|(L_\tau - L_\sigma)(f_i - g_i)\|_p \\ &< \|(L_\tau - L_\sigma)g_i\|_p + 2^{1/p} \cdot \epsilon/2^{1/p} = \epsilon\end{aligned}$$

where $\|(L_\tau - L_\sigma)g_i\|_p = 0$ because $g_i \in V_K$, and thus is locally constant on the sets $Y(K, v)$ for $v$ a place of $K$, and $\tau \in N$ implies that $\sigma$ and $\tau$ agree on $K$, so $L_\tau g_i = L_\sigma g_i$. $\square$

2.2. **Subspaces associated to number fields.** We will now prove some lemmas regarding the relationship between the spaces $V_K$ and the Galois group. As in the introduction, let us define

$$\mathcal{K} = \{K/\mathbb{Q} : [K : \mathbb{Q}] < \infty\} \quad \text{and} \quad \mathcal{K}^G = \{K \in \mathcal{K} : \sigma K = K \ \forall \sigma \in G\}.$$

As we shall have occasion to use them, let us recall the combinatorial properties of the sets $\mathcal{K}$ and $\mathcal{K}^G$ partially ordered by inclusion. Recall that $\mathcal{K}$ and $\mathcal{K}^G$ are *lattices*, that is, partially ordered sets for which any two elements have a unique greatest lower bound, called the *meet*, and a least upper bound, called the *join*. Specfically, for any two fields $K, L$, the meet $K \wedge L$ is given by $K \cap L$ and the join $K \vee L$ is given by $KL$. If $K, L$ are Galois then both the meet (the intersection) and the join (the compositum) are Galois as well, thus $\mathcal{K}^G$ is a lattice as well. Both lattices have a minimal element, namely $\mathbb{Q}$, and are *locally finite*, that is, between any two fixed elements we have a finite number of intermediate elements.

For each $K \in \mathcal{K}$, let

$$V_K = \operatorname{span}_{\mathbb{Q}}\langle\{f_\alpha : \alpha \in K^\times/\operatorname{Tor}(K^\times)\}\rangle. \tag{2.1}$$

Then $V_K$ is the subspace of $\mathcal{F}$ spanned by the functions arising from numbers of $K$. Suppose we fix a class of an algebraic number modulo torsion $f \in \mathcal{F}$. Then the set

$$\{K \in \mathcal{K} : f \in V_K\}$$

forms a sublattice of $\mathcal{K}$, and by the finiteness properties of $\mathcal{K}$ this set must contain a unique minimal element.

**Definition 2.3.** For any $f \in \mathcal{F}$, the *minimal field* is defined to be the minimal element of the set $\{K \in \mathcal{K} : f \in V_K\}$. We denote the minimal field of $f$ by $K_f$.

**Lemma 2.4.** *For any $f \in \mathcal{F}$, we have* $\operatorname{Stab}_G(f) = \operatorname{Gal}(\overline{\mathbb{Q}}/K_f) \le G$.

*Notation* 2.5. By $\mathrm{Stab}_G(f)$ we mean the $\sigma \in G$ such that $L_\sigma f = f$. As this tacit identification is convenient we shall use it throughout without further comment.

*Proof.* Let $f = f_\alpha$. Then clearly $\mathrm{Gal}(\overline{\mathbb{Q}}/K_f) \leq \mathrm{Stab}_G(f)$, as $\alpha^\ell \in K_f$ for some $\ell \in \mathbb{N}$ by definition of $V_{K_f}$. To see the reverse containment, merely observe that $K_f = \mathbb{Q}(\alpha^\ell)$ for some $\ell \in \mathbb{N}$, as otherwise, there would be a proper subfield of $K_f$ which contains a power of $\alpha$, contradicting the definition of $K_f$. □

*Remark* 2.6. The minimal such exponent $\ell$ used above can in fact be uniquely associated to $f \in \mathcal{F}$ and this will be vital to the concept of representability developed in Section 3 below.

**Lemma 2.7.** *For a given $f \in \mathcal{F}$, we have $f \in V_K$ if and only if $L_\sigma f = f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.*

*Proof.* Necessity is obvious. To see that the condition is sufficient, observe that by definition of $K_f$, we have $f \in V_K$ if and only $K_f \subseteq K$, which is equivalent to $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \leq \mathrm{Gal}(\overline{\mathbb{Q}}/K_f)$ under the Galois correspondence. But by the above lemma, $\mathrm{Gal}(\overline{\mathbb{Q}}/K_f) = \mathrm{Stab}_G(f)$. □

**Proposition 2.8.** *If $E, F \in \mathcal{K}$, then we have $E = F$ if and only if $V_E = V_F$.*

*Proof.* Suppose $E \neq F$ but $V_E = V_F$. Let $E = \mathbb{Q}(\alpha)$. By [4, Theorem 3] we can find a rational integer $n$ such that $\beta = n + \alpha$ is torsion-free, that is, if $\beta/\beta' \neq 1$ then $\beta/\beta' \notin \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$ for any conjugate $\beta'$ of $\beta$, and in particular, the conjugates of $\beta$ give rise to distinct functions in $\mathcal{F}$. Observe therefore that $E = \mathbb{Q}(\beta)$ and $\mathrm{Stab}_G(f_\beta) = \mathrm{Gal}(\overline{\mathbb{Q}}/E)$. By the above if $f_\beta \in V_F$ then we must have $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \leq \mathrm{Gal}(\overline{\mathbb{Q}}/E)$, or $E \subseteq F$. Repeating the same argument for a generator of $F$, we find that $F \subseteq E$ so $E = F$, a contradiction. The reverse implication is obvious. □

*Remark* 2.9. The above proposition is no longer true if we restrict our attention to the space of units $\mathcal{U} \subset \mathcal{F}$. This follows from the well known fact that CM extensions (totally imaginary quadratic extensions of totally real fields) have the same unit group modulo torsion as their base fields, the simplest example being $\mathbb{Q}(i)/\mathbb{Q}$.

2.3. **Orthogonal projections associated to number fields.** For $K \in \mathcal{K}$, define the map $P_K : \mathcal{F} \to V_K$ via

$$(P_K f)(y) = \int_{H_K} (L_\sigma f)(y) d\nu(\sigma)$$

where $H_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $\nu$ is the normalized (measure 1) Haar measure of $H_K$. (Observe that, like $G$, $H_K$ is profinite and thus compact and possesses a Haar measure.) Let us prove that the map is well-defined. Since $f \in \mathcal{F}$, it has a finite Galois orbit and thus a finite orbit under $H_K$. Let us partition $H_K$ into the $k = [H_K : \mathrm{Stab}_{H_K}(f)]$ cosets of equal measure by the translation invariance of the Haar measure. Denote these cosets by $\mathrm{Stab}_{H_K}(f)\sigma_1, \ldots, \mathrm{Stab}_{H_K}(f)\sigma_k$. Then

$$P_K(f) = \frac{1}{k}\left(L_{\sigma_1} f + \cdots + L_{\sigma_k} f\right).$$

But each $L_{\sigma_i} f \in \mathcal{F}$ since $\mathcal{F}$ is closed under the action of the Galois isometries. Thus if $f = f_\alpha$, we have $L_{\sigma_i} f = f_{\sigma_i \alpha}$. Since $\mathcal{F}$ is a vector space, $P_K(f) \in \mathcal{F}$ as well. Further, it is stable under the action of $H_K$, and thus, by Lemma 2.7, we have $P_K(f) \in V_K$. The map $P_K$ is in fact nothing more than the familiar algebraic norm down to $K$, subject to an appropriate normalization, that is, if $f_\beta = P_K f_\alpha$, then we have

$$\beta \equiv \left(\mathrm{N}_K^{K(\alpha)}\,\alpha\right)^{1/[K(\alpha):K]} \mod \operatorname{Tor}(\overline{\mathbb{Q}}^\times). \tag{2.2}$$

(We note in passing that the norm map $N_K^{K(\alpha)} : K(\alpha)^\times \to K^\times$ descends to a well-defined map modulo torsion.)

The following alternative formulation will also be helpful:

**Lemma 2.10.** *Let $K \in \mathcal{K}$ and let $M_K$ denote the places of $K$. For each $v \in M_K$, let $\chi_v(y)$ be the characteristic function of the set $Y(K,v)$. Then*

$$P_K f(y) = \sum_{v\in M_K} \left( \frac{1}{\lambda(Y(K,v))} \int_{Y(K,v)} f(z)\,d\lambda(z) \right) \chi_v(y).$$

In other words, $P_K$ is essentially the conditional expectation with respect to the Borel $\sigma$-algebra generated by the collection $\{Y(K,v) : v \in M_K\}$. Of course, $Y$ has infinite measure so this is not a conditional expectation in the usual sense from probability theory, although it shares many of the same properties. If we restrict to the space of units, that is, functions supported on the measure one space $Y(\mathbb{Q},\infty)$, then the restriction of $P_K$ to this space is indeed a conditional expectation.

*Proof.* Fix a value $y \in Y$. Then there exists a unique $v \in M_K$ such that $y \in Y(K,v)$ since $Y = \bigcup_{v\in M_K} Y(K,v)$ is a disjoint union. The claim will be proven if we can show that for this value of $y$,

$$P_K f(y) = \frac{1}{\lambda(Y(K,v))} \int_{Y(K,v)} f(z)\,d\lambda(z).$$

Now,

$$P_K f(y) = \int_{H_K} f(\sigma^{-1}y)\,d\nu(\sigma)$$

where $H_K, \nu$ are as above. By the construction of $\lambda$ (see (4.1) and surrounding remarks in [1]), for any $y \in Y(K,v)$,

$$\frac{1}{\lambda(Y(K,v))} \int_{Y(K,v)} f(z)\,d\lambda(z) = \int_{H_K} f(\sigma^{-1}y)\,d\nu(\sigma)$$

(where we need the normalization factor $1/\lambda(Y(K,v))$ since (4.1) assumes $\lambda(Y(K,v)) = 1$) and so the proof is complete. $\square$

**Proposition 2.11.** *Let $K \subset \overline{\mathbb{Q}}$ be a field of arbitrary degree. Then $P_K$ is a projection onto $V_K$ of norm one with respect to the $L^p$ norms for $1 \le p \le \infty$.*

*Proof.* We first prove that $P_K^2 = P_K$. Let $H = H_K$ as above and $\nu$ the normalized Haar measure on $H$. Suppose that $\tau \in H$. Observe that

$$P_K(f)(\tau^{-1}y) = \int_H f(\sigma^{-1}\tau^{-1}y)d\nu(\sigma) = \int_{\tau H} f(\sigma^{-1}y)d\nu(\sigma) = P_K(f)(y)$$

since $\tau H = H$ for $\tau \in H$. Thus,

$$({P_K}^2 f)(y) = \int_H P_K f(\sigma^{-1}y)d\nu(\sigma) = \int_H P_K f(y)d\nu(\sigma) = P_K f(y),$$

or more succinctly, ${P_K}^2 = P_K$. Since linearity is clear we will now prove that the operator norm of $P_K$, denoted $\|P_K\|$, is equal to 1 with respect to the $L^p$ norm in order to conclude that $P_K$ is a projection. If $p = \infty$, this is immediate, so let us assume that $1 \leq p < \infty$. Let $f \in L^p(Y)$. Then first observe that since $\nu(H) = 1$, Jensen's inequality implies

$$\int_H |f(\sigma^{-1}y)|\, d\nu(\sigma) \leq \left(\int_H |f(\sigma^{-1}y)|^p\, d\nu(\sigma)\right)^{1/p}.$$

Now let us consider the $L^p$ norm of $P_K f$:

$$\begin{aligned}
\|P_K f\|_p = \left(\int_Y |P_K(f)(y)|^p d\lambda(y)\right)^{1/p} &= \left(\int_Y \left|\int_H f(\sigma^{-1}y) d\nu(\sigma)\right|^p d\lambda(y)\right)^{1/p} \\
\leq \left(\int_Y \int_H \left|f(\sigma^{-1}y)\right|^p d\nu(\sigma) d\lambda(y)\right)^{1/p} &= \left(\int_H \int_Y \left|f(\sigma^{-1}y)\right|^p d\lambda(y) d\mu(\sigma)\right)^{1/p} \\
= \left(\int_H \|L_\sigma f\|_p^p d\mu(\sigma)\right)^{1/p} &= \left(\int_H \|f\|_p^p d\mu(\sigma)\right)^{1/p} = \|f\|_p.
\end{aligned}$$

where we have made use of the fact that $L_\sigma$ is an isometry, and the application of Fubini's theorem is justified by the integrability of $|f|^p$. This proves that $\|P_K\| \leq 1$, and to see that the operator norm is not in fact less than 1, observe that the subspace $V_{\mathbb{Q}}$ is fixed for every $P_K$. $\square$

As a corollary, if we extend $P_K$ by continuity to the completion $\mathcal{F}_p$ of $\mathcal{F}$ under the $L^p$ norm, we obtain:

**Corollary 2.12.** *The subspace $\overline{V_K} \subset \mathcal{F}_p$ is complemented in $\mathcal{F}_p$ for all $1 \leq p \leq \infty$.*

As $\mathcal{F}_2 = L^2(Y, \lambda)$ is a Hilbert space, more is in fact true:

**Proposition 2.13.** *For each $K \in \mathcal{K}$, $P_K$ is the orthogonal projection onto the subspace $\overline{V_K} \subset L^2(Y)$.*

Specifically, this means that $\|f\|_2^2 = \|P_K f\|_2^2 + \|(I - P_K)f\|_2^2$, where $I$ is the identity operator.

*Proof.* It suffices to observe that $P_K$ is idempotent and has operator norm $\|P_K\| = 1$ with respect to the $L^2$ norm, and any such projection in a real Hilbert space is orthogonal [12, Theorem III.3]. $\square$

We now explore the relationship between the Galois isometries and the projection operators $P_K$ for $K \in \mathcal{K}$.

**Lemma 2.14.** *For any field $K \subset \overline{\mathbb{Q}}$ of arbitrary degree and any $\sigma \in G$,*

$$L_\sigma P_K = P_{\sigma K}\, L_\sigma.$$

*Equivalently, $P_K\, L_\sigma = L_\sigma P_{\sigma^{-1}K}$.*

*Proof.* We prove the first form, the second obviously being equivalent. By definition of $P_K$, letting $H = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $\nu$ be the normalized Haar measure on $H$ such that $\nu(H) = 1$,

$$\begin{aligned}(L_\sigma P_K f)(y) = (P_K f)(\sigma^{-1}y) &= \int_H f(\tau^{-1}\sigma^{-1}y)\, d\nu(\tau)\\ &= \int_H f(\sigma^{-1}(\sigma\tau\sigma^{-1})^{-1}y)\, d\nu(\tau)\\ &= \int_{\sigma H\sigma^{-1}} (L_\sigma f)(\tau^{-1}y)\, d\nu(\tau) = P_{\sigma K}(L_\sigma f)(y). \quad \square\end{aligned}$$

We will be particularly interested in the case where the projections $P_K, P_L$ commute with each other (and thus $P_K P_L$ is a projection to the intersection of their ranges). To that end, we recall the following results from [6]:

**Lemma 2.15.** *Let $K, L \subset \overline{\mathbb{Q}}$ be fields of arbitrary degree. Then $V_K \cap V_L = V_{K\cap L}$.*

*Proof.* See [6, Lemma 2.11]. $\square$

**Lemma 2.16.** *Suppose $K \in \mathcal{K}$ and $L \in \mathcal{K}^G$. Then $P_K$ and $P_L$ commute, that is,*

$$P_K P_L = P_{K\cap L} = P_L P_K.$$

*In particular, the family of operators $\{P_K : K \in \mathcal{K}^G\}$ is commuting.*

*Proof.* See [6, Lemma 2.12]. $\square$

2.4. **Main decomposition theorem.** We will now prove a very general decomposition theorem, which we will then apply to $\mathcal{F}$ in the next two sections in order to obtain the proof of Theorems 1 and 2, which state that we can orthogonally decompose the space $\mathcal{F}$ of algebraic numbers modulo torsion by their Galois field and by their degree.

**Theorem 7.** *Let $V$ be a vector space over $\mathbb{Q}$ with an inner product $\langle \cdot, \cdot \rangle$ and suppose we have a family of subspaces $V_i \subset V$ together with projections $P_i$ indexed by a partially ordered set $I$ such that:*

1. *The index set $I$ has a unique minimal element, denoted $0 \in I$, and $I$ is locally finite, that is, any interval $[i, j] = \{k \in I : i \leq k \leq j\}$ is of finite cardinality.*
2. *Any pair of elements $i, j \in I$ has a unique greatest lower bound, called the meet of $i$ and $j$, and denoted $i \wedge j$. (Such a poset $I$ is called a meet-semilattice.)*
3. *$V_i \subseteq V_j$ if $i \leq j \in I$.*
4. *The projection map $P_i : V \to V_i$ is orthogonal with respect to the inner product of $V$ for all $i \in I$.*
5. *For $i, j \in I$, $P_i P_j = P_j P_i = P_{i\wedge j}$, where $i \wedge j$ is the meet of $i$ and $j$.*
6. *$V = \sum_{i\in I} V_i$ (the sum is in the usual $\mathbb{Q}$-vector space sense).*

*Then there exist mutually orthogonal projections $T_i \leq P_i$ (that is, satisfying $T_i(V) \subseteq V_i$) which form an orthogonal decomposition of $V$:*

$$V = \bigoplus_{i\in I} T_i(V), \quad \textit{and} \quad T_i(V) \perp T_j(V) \textit{ for all } i \neq j \in I.$$

(The notation $V = \bigoplus_{i\in I} T_i(V)$ indicates a direct sum in the $\mathbb{Q}$-vector space sense, that is, that each vector $v \in V$ has a unique expression as a finite sum of vectors, one from each summand.)

We call $T_i$ the *essential projection* associated to the space $V_i$, as it gives the subspace of $V_i$ which is unique to $V_i$ and no other subspace $V_j$ in the given family.

*Remark* 2.17. Theorem 7 can be stated and proven almost identically if $V$ is a real Hilbert space rather than an incomplete vector space over $\mathbb{Q}$, the only changes being that condition (6) is replaced with the condition that the closure of $\sum_{i\in I} V_i$ is $V$, the direct sum is then understood in the usual Hilbert space sense, and the expansion of each $f$ into $\sum_{i\in I} T_i f$ is to be understood as a unique series expansion rather than a finite sum. The construction of the $T_i$ operators and the orthogonality are proven in exactly the same manner, and indeed, we will make use of the fact that if we complete $V$, the decomposition extends by continuity to the completion in the usual Hilbert space sense. The theorem as stated here and as applied to $\mathcal{F}$ is in fact a strictly stronger result than the statement it implies for the decomposition of $L^2(Y)$ as not only must such projections and such a decomposition exist, but this decomposition must also respect the underlying $\mathbb{Q}$-vector space $\mathcal{F}$ of algebraic numbers modulo torsion.

Let us begin by recalling the background necessary to define our $T_i$ projections. Since $I$ is locally finite, it is a basic theorem in combinatorics that there exists a Möbius function $\mu : I \times I \to \mathbb{Z}$, defined inductively by the requirements that $\mu(i,i) = 1$ for all $i \in I$, $\mu(i,j) = 0$ for all $i \not\leq j \in I$, and $\sum_{i\leq j\leq k} \mu(i,j) = 0$ for all $i < k \in I$ (the sums are finite by the assumption that $I$ is locally finite). Since our set $I$ has a minimal element 0 and is locally finite, we can sum over $i \leq j$ as well. The most basic result concerning the Möbius function is *Möbius inversion*, which (in one of the several possible formulations) tells us that given two functions $f, g$ on $I$,

$$f(j) = \sum_{i\leq j} g(i) \quad \text{if and only if} \quad g(j) = \sum_{i\leq j} \mu(i,j)\, f(i).$$

In order that our $T_i$ capture the unique contribution of each subfield $V_i$, we would like our $T_i$ projections to satisfy the condition that:

$$P_j = \sum_{i\leq j} T_i. \tag{2.3}$$

Möbius inversion leads us to define the $T_i$ operators via the equation:

$$T_j = \sum_{i\leq j} \mu(i,j) P_i. \tag{2.4}$$

Since each of the above sums is finite and $\mu$ takes values in $\mathbb{Z}$, we see that $T_j : V \to V_j$ is well-defined. We will prove that the $T_j$ operators form the desired family of projections.

**Lemma 2.18.** *Let the projections $P_i$ for $i \in I$ satisfy the conditions of Theorem 7 and let $T_i$ be defined as above. Then for all $i, j \in I$, $P_i T_j = T_j P_i$, and*

$$P_j T_i = \begin{cases} T_i & \text{if } i \leq j \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The first claim follows immediately from equation (2.4) and condition (5) of the theorem statement. To prove the second claim, we proceed by induction.

Observe that the statement is trivial for $T_0 = P_0$. Now given $j \in I$, suppose the theorem is true for all $i < j$. Observe that from (2.3) we get

$$T_i = P_i - \sum_{k<i} T_k. \tag{2.5}$$

Then, if $i < j$, we have

$$P_j T_i = P_j P_i - \sum_{k<i} P_j T_k = P_i - \sum_{k<i} T_k = T_i,$$

applying the induction hypothesis at the second equality.

Now suppose $i \not< j$, so that $i \wedge j \neq i$. Then

$$\begin{aligned} P_j T_i = P_j P_i - \sum_{k<i} P_j T_k &= P_{i\wedge j} - \sum_{k \leq i\wedge j} P_j T_k - \sum_{\substack{k<i \\ k \not\leq i\wedge j}} P_j T_k \\ &= P_{i\wedge j} - \sum_{k \leq i \wedge j} T_k - 0 = P_{i\wedge j} - P_{i\wedge j} = 0 \end{aligned}$$

by two applications of the induction hypothesis at the third equality. □

**Lemma 2.19.** *Let the $T_i$ be as above and let $i \neq j$ for $i, j \in I$. Then $T_i T_j = T_j T_i = 0$.*

*Proof.* Suppose that $i \wedge j < j$. By Lemma 2.18, $T_i = T_i P_i$ and $T_j = P_j T_j$. Thus,

$$T_i T_j = (T_i P_i)(P_j T_j) = T_i (P_i P_j) T_j = T_i P_{i\wedge j} T_j = 0$$

since $i \neq j$ implies that $i \wedge j < i$ or $i \wedge j < j$, so either $T_i P_{i\wedge j} = 0$ or $P_{i\wedge j} T_j = 0$ by Lemma 2.18. □

We are now ready to prove the theorem statement.

*Proof of Theorem 7.* Let the operators $T_i$ for $i \in I$ be constructed as above. Let us first show that each $T_i$ is a projection, a linear operator of bounded norm such that ${T_i}^2 = T_i$. The fact the $T_i$ is a continuous linear operator of bounded norm follows from the same fact for the $P_i$ operators, since each $T_i$ is a finite linear combination of $P_i$ projections.

Let us now show that $T_i$ is idempotent. The base case $T_0 = P_0$ is trivial. Assume the lemma is true for all $i < j$. Using equation (2.5), we have

$$\begin{aligned} {T_j}^2 = \left( P_j - \sum_{i<j} T_i \right)^2 &= {P_j}^2 - \sum_{i<j} P_j T_i - \sum_{i<j} T_i P_j + \left( \sum_{i<j} T_i \right)^2 \\ &= P_j - \sum_{i<j} T_i - \sum_{i<j} T_i + \sum_{i<j} T_i = P_j - \sum_{i<j} T_i = T_j \end{aligned}$$

where we have used Lemmas 2.18 and 2.19 to simplify the middle and last terms.

Now, let us show that the $T_i$ decompose $V$. To see this, observe that each element $f \in V$ by condition (6) lies in some $V_{i_1} + \ldots + V_{i_n}$. Let $I' = \bigcup_{m=1}^{n} [0, i_m] \subset I$, and then observe that $\sum_{k \in I'} T_k$ is the projection onto $V_{i_1} + \ldots + V_{i_n}$ and $I'$ is finite by construction, so $f = \sum_{k\in I'} T_k f$. In fact, observe that we can write $f = \sum_{k\in I} T_k f$ as a formally infinite sum, and all terms except those satisfying $k \leq i$ are zero by Lemma 2.18. Thus we can write

$$V = \bigoplus_{i \in I} T_i(V).$$

That the $T_i$ are orthogonal projections now follows from the fact that a continuous operator is an orthogonal projection if and only if it is idempotent and self-adjoint [12, Theorem III.2], for, since the $P_i$ are assumed to be orthogonal, they are self-adjoint and thus the $T_i$ operators are self-adjoint as well as an integral linear combination of the $P_i$ operators, and we have demonstrated that the $T_i$ are continuous and idempotent. □

2.5. **Decomposition by Galois field and proof of Theorem 1.** We will now apply Theorem 7 to $\mathcal{F}$ and its collection of subspaces $\{V_K : K \in \mathcal{K}^G\}$ with their associated projections. (Recall that $\mathcal{K}^G$ is simply the set of finite Galois extensions of $\mathbb{Q}$.)

*Proof of Theorem 1.* As remarked above, it is well known that both $\mathcal{K}$ and $\mathcal{K}^G$ satisfy all of the axioms of a lattice, that is, for any two fields $K, L$, there is a unique meet $K \wedge L$ given by $K \cap L$ and a unique *join* $K \vee L$ given by $KL$. If $K, L$ are Galois then both the meet (the intersection) and the join (the compositum) are Galois as well, thus $\mathcal{K}^G$ is a lattice as well. Further, both $\mathcal{K}$ and $\mathcal{K}^G$ are locally finite posets and possess a minimal element, namely, $\mathbb{Q}$.

Our decomposition will be along $\mathcal{K}^G$ and the associated family of subspaces $V_K$ with their canonical projections $P_K$. Since $\mathcal{K}^G$ is a locally finite lattice, conditions (1) and (2) of Theorem 7 are satisfied. Clearly the subspaces $V_K$ for $K \in \mathcal{K}^G$ satisfy the containment condition (3). By Proposition 2.13, the projections are orthogonal and satisfy condition (4). By Lemma 2.16, the maps $\{P_K : K \in \mathcal{K}^G\}$ form a commuting family and satisfy condition (5). Lastly, since any $f = f_\alpha$ belongs to $V_{K_f} \subset V_K$ where $K \in \mathcal{K}^G$ is the Galois closure of the minimal field $K_f$, we find that condition (6) is satisfied as well. Thus Theorem 7 gives us an orthgonal decomposition

$$\mathcal{F} = \bigoplus_{K \in \mathcal{K}^G} T_K(\mathcal{F}) \tag{2.6}$$

and the relationship between the $P_K$ and $T_K$ operators is given by:

$$P_K = \sum_{\substack{F \in \mathcal{K}^G \\ F \subseteq K}} T_F, \quad \text{and} \quad T_K = \sum_{\substack{F \in \mathcal{K}^G \\ F \subseteq K}} \mu(F, K) P_F \tag{2.7}$$

where $\mu : \mathcal{K}^G \times \mathcal{K}^G \to \mathbb{Z}$ is the Möbius function associated to $\mathcal{K}^G$. □

If $K$ is the Galois closure of the minimal field $K_f$ where $f = f_\alpha$, then $P_K(f) = f$, and so (2.7) gives us a unique representation modulo torsion of the algebraic number $\alpha$ which we call the *M-factorization* of $\alpha$, or the *M-expansion* of $f_\alpha$ in functional notation.

**Example 2.20.** *Let $\alpha = 2 + \sqrt{2}$ and let $f = f_\alpha$. Then $K_f = \mathbb{Q}(\sqrt{2})$. Since $K \in \mathcal{K}^G$, $[K : \mathbb{Q}] = 2$ and it is easy to see that the interval $[\mathbb{Q}, K] = \{\mathbb{Q}, K\} \subset \mathcal{K}^G$, and so $\mu(\mathbb{Q}, K) = -1$, and thus*

$$T_K = P_K - P_{\mathbb{Q}}, \quad T_{\mathbb{Q}} = P_{\mathbb{Q}}.$$

*Thus*

$$T_K(f_\alpha) = f_{1+\sqrt{2}}, \quad T_{\mathbb{Q}}(f_\alpha) = f_{\sqrt{2}},$$

*and the $M$-factorization of $\alpha$ has the form $2+\sqrt{2} = \sqrt{2}\cdot(1+\sqrt{2})$, or in functional notation,*

$$f_{2+\sqrt{2}} = f_{\sqrt{2}} + f_{1+\sqrt{2}}, \quad \textit{and} \quad f_{\sqrt{2}} \perp f_{1+\sqrt{2}}.$$

*Remark* 2.21. We end this section with a remark on why we decompose along $\mathcal{K}^G$ but not $\mathcal{K}$. It is not difficult to see that the $P_K$ projections for $K \in \mathcal{K}$ do not form a commuting family. To see this, suppose $\alpha$ is a cubic algebraic unit with conjugates $\beta, \gamma$ and discriminant $\Delta$ not a square. Then we have the following fields:

$$\begin{tikzcd}
 & & \mathbb{Q}(\alpha,\beta,\gamma) \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[ddr, dash] & \\
\mathbb{Q}(\alpha) \arrow[ddrr, dash] & \mathbb{Q}(\beta) \arrow[ddr, dash] & \mathbb{Q}(\gamma) \arrow[dd, dash] & \\
 & & & \mathbb{Q}(\sqrt{\Delta}) \arrow[dl, dash] \\
 & & \mathbb{Q} &
\end{tikzcd}$$

But the projections associated to the fields $\mathbb{Q}(\alpha)$ and its conjugates do not commute. Specifically, we may compute:

$$P_{\mathbb{Q}(\beta)} f_\alpha = -\frac{1}{2} f_\beta, \quad \text{and} \quad P_{\mathbb{Q}(\alpha)} f_\beta = -\frac{1}{2} f_\alpha$$

which shows that $P_{\mathbb{Q}(\alpha)} P_{\mathbb{Q}(\beta)} \neq P_{\mathbb{Q}(\beta)} P_{\mathbb{Q}(\alpha)}$. This noncommutativity is present precisely because there is a linear dependence among the vector space $V_{\mathbb{Q}(\alpha)}$ and its conjugates, e.g., $f_\alpha + f_\beta + f_\gamma = 0$ (since we assumed $\alpha$ was an algebraic unit). In particular, it is not hard to check that

$$V_{\mathbb{Q}(\alpha)} + V_{\mathbb{Q}(\beta)} = V_{\mathbb{Q}(\alpha)} + V_{\mathbb{Q}(\beta)} + V_{\mathbb{Q}(\gamma)}.$$

Clearly such a dependence would make it impossible to associate a unique component $T_K$ to each of the three fields. However, the commutavity of the $P_K$ for $K \in \mathcal{K}^G$ implies that there is no such barrier to decomposition amongst the Galois fields.

2.6. **Decomposition by degree and proof of Theorems 2 and 3.** In order to associate a notion of degree to a subspace in a meaningful fashion so that we can define our Mahler $p$-norms we will determine a second decomposition of $\mathcal{F}$. Let us define the function $\delta : \mathcal{F} \to \mathbb{N}$ by

$$\delta(f) = \#\{L_\sigma f : \sigma \in G\} = [G : \mathrm{Stab}_G(f)] \tag{2.8}$$

to be the size of the orbit of $f$ under the action of the Galois isometries. Observe that by Lemma 2.4, we have $\mathrm{Stab}_G(f) = \mathrm{Gal}(\overline{\mathbb{Q}}/K_f)$ where $K_f$ is the minimal field of $f$, and so we also have

$$\delta(f) = [K_f : \mathbb{Q}]. \tag{2.9}$$

Let

$$V^{(n)} = \sum_{\substack{K \in \mathcal{K} \\ [K:\mathbb{Q}] \leq n}} V_K \tag{2.10}$$

be the vector space spanned by all elements of whose orbit in $\mathcal{F}$ under $G$ is of size at most $n$. Let $P^{(n)}$ denote the unique orthogonal projection of the Hilbert space $L^2(Y)$ onto the closure $\overline{V^{(n)}}$ of the $\mathbb{Q}$-vector space $V^{(n)}$ inside $L^2(Y)$. We wish to show that the restriction of this orthogonal projection defined on the Hilbert space

$L^2(Y)$ preserves the $\mathbb{Q}$-vector space $\mathcal{F}$ of equivalence classes of algebraic numbers modulo torsion, that is, that $P^{(n)}(\mathcal{F}) \subset \mathcal{F}$, so that the map

$$P^{(n)} : \mathcal{F} \to V^{(n)}$$

is a well-defined. Once this has been demonstrated, we we can apply Theorem 7 above to obtain projections $T^{(n)} : \mathcal{F} \to V^{(n)}$ which will give us the orthogonal decomposition of $\mathcal{F}$ into a subspace spanned by elements whose orbit under $G$ is of order at most $n$. We begin by first showing that the projections $P^{(n)}$ and $P_K$ for $n \in \mathbb{N}$ and $K \in \mathcal{K}^G$ commute.

**Lemma 2.22.** *If $K \in \mathcal{K}^G$, then $\delta(P_K f) \leq \delta(f)$ for all $f \in \mathcal{F}$.*

*Proof.* Let $F = K_f$. Since $K \in \mathcal{K}^G$, we have by Lemma 2.16 that $P_K f = P_K(P_F f) = P_{K\cap F} f$. Thus, $P_K f \in V_{K\cap F}$, and so by (2.9) above, we have $\delta(P_K f) \leq [K \cap F : \mathbb{Q}] \leq [F : \mathbb{Q}] = \delta(f)$. □

**Proposition 2.23.** *Let $n \in \mathbb{N}$ and $K \in \mathcal{K}^G$. Then the orthogonal projections $P^{(n)} : L^2(Y) \to \overline{V^{(n)}}$ and $P_K : L^2(Y) \to \overline{V_K}$ commute (where the closures are taken in $L^2$), and thus $T_K$ and $P^{(n)}$ commute as well.*

*Proof.* Since $\delta(P_K f) \leq \delta(f)$ for all $f \in \mathcal{F}$ by Lemma 2.22 above, we have $P_K(V^{(n)}) \subset V^{(n)}$, and thus by continuity $P_K(\overline{V^{(n)}}) \subset \overline{V^{(n)}}$, so $P_K(\overline{V^{(n)}}) \subset \overline{V^{(n)}} \cap \overline{V_K}$ and $P_K P^{(n)}$ is a projection. Therefore they commute. The last part of the claim now follows from the definition of $T_K$ in (2.4). □

Let $W_K = T_K(\mathcal{F}) \subset V_K$ for $K \in \mathcal{K}^G$. By the above proposition, we see that if we can show that $P^{(n)}(W_K) \subseteq W_K$, then we will have the desired result since

$$P^{(n)}(\mathcal{F}) = \bigoplus_{K \in \mathcal{K}^G} P^{(n)}(W_K)$$

by the commutativity of $P^{(n)}$ and $T_K$. Since we will prove this by reducing to finite dimensional $S$-unit subspaces, let us first prove an easy lemma regarding finite dimensional vector spaces over $\mathbb{Q}$.

**Lemma 2.24.** *Suppose we have a finite dimensional vector space $A$ over $\mathbb{Q}$, and suppose that*

$$A = V_1 \oplus V_1' = V_2 \oplus V_2' = \cdots = V_n \oplus V_n'$$

*for some subspaces $V_i, V_i'$, $1 \leq i \leq n$. Then*

$$A = (V_1 + \cdots + V_n) \oplus (V_1' \cap \cdots \cap V_n').$$

*Proof.* It suffices to prove the lemma in the case $n = 2$ as the remaining cases follow by induction, so suppose $A = V_1 \oplus V_1' = V_2 \oplus V_2'$. It is an easy exercise that

$$\dim_{\mathbb{Q}} V_1 + \dim_{\mathbb{Q}} V_2 = \dim_{\mathbb{Q}}(V_1 + V_2) + \dim_{\mathbb{Q}}(V_1 \cap V_2),$$

and likewise,

$$\dim_{\mathbb{Q}} V_1' + \dim_{\mathbb{Q}} V_2' = \dim_{\mathbb{Q}}(V_1' + V_2') + \dim_{\mathbb{Q}}(V_1' \cap V_2').$$

Now,

$$\begin{aligned} (2.11) \quad 2\dim_{\mathbb{Q}} A &= \dim_{\mathbb{Q}} V_1 + \dim_{\mathbb{Q}} V_1' + \dim_{\mathbb{Q}} V_2 + \dim_{\mathbb{Q}} V_2' \\ &= \dim_{\mathbb{Q}}(V_1 + V_2) + \dim_{\mathbb{Q}}(V_1 \cap V_2) + \dim_{\mathbb{Q}}(V_1' + V_2') + \dim_{\mathbb{Q}}(V_1' \cap V_2'). \end{aligned}$$

Now, $(V_1 + V_2) \oplus (V_1' \cap V_2') \subseteq A$ and $(V_1' + V_2') \oplus (V_1 \cap V_2) \subseteq A$, so

$$\begin{aligned} b &= \dim_{\mathbb{Q}}(V_1 + V_2) + \dim_{\mathbb{Q}}(V_1' \cap V_2') \le \dim_{\mathbb{Q}} A \\ c &= \dim_{\mathbb{Q}}(V_1' + V_2') + \dim_{\mathbb{Q}}(V_1 \cap V_2) \le \dim_{\mathbb{Q}} A. \end{aligned}$$

By (2.11), we have $b + c = 2\dim_{\mathbb{Q}} A$, therefore, we must have $b = c = \dim_{\mathbb{Q}} A$, and in particular $b = \dim_{\mathbb{Q}} A$ proves the claim. ☐

**Proposition 2.25.** *With $W_K = T_K(\mathcal{F})$ as above, $P^{(n)}(W_K) \subseteq W_K$ for every $n \in \mathbb{N}$ and $K \in \mathcal{K}^G$, and thus $P^{(n)}(\mathcal{F}) \subset \mathcal{F}$.*

*Proof.* Let $f \in W_K$, and let $S \subset M_{\mathbb{Q}}$ be a finite set of rational primes, containing the infinite prime, such that

$$\operatorname{supp}_Y(f) \subset \bigcup_{p\in S} Y(\mathbb{Q}, p).$$

Let $V_{K,S} \subset V_K$ denote the subspace spanned by the $S$-units of $K$. By Dirichlet's $S$-unit theorem, $V_{K,S}$ is finite dimensional over $\mathbb{Q}$. Let $W_{K,S} = T_K(V_{K,S})$. Notice that $W_{K,S} \subset V_{K,S}$ since each $P_F$ projection will preserve the support of $f$ over each set $Y(\mathbb{Q}, p)$ for $p \in M_{\mathbb{Q}}$ by Lemma 2.10.

For all fields $F \in \mathcal{K}$ such that $F \subset K$, let

$$Z_{F,S} = P_F(W_{K,S}) \quad \text{and} \quad Z'_{F,S} = Q_F(W_{K,S}),$$

where $Q_F = I - P_F$ is the complementary orthogonal projection. Observe that for each such $F$, we have

$$W_{K,S} = Z_{F,S} \oplus Z'_{F,S}.$$

Then by Lemma 2.24, we have

$$W_{K,S} = \Bigg( \sum_{\substack{F\subseteq K \\ [F:\mathbb{Q}]\le n}} Z_{F,S} \Bigg) \oplus \Bigg( \bigcap_{\substack{F\subseteq K \\ [F:\mathbb{Q}]\le n}} Z'_{F,S} \Bigg). \tag{2.12}$$

This gives us a decomposition $f = f_n + f_n'$ where

$$f_n \in \sum_{\substack{F\subseteq K \\ [F:\mathbb{Q}]\le n}} Z_{F,S} = V^{(n)} \cap W_{K,S},$$

and

$$f_n' \in \bigcap_{\substack{F\subseteq K \\ [F:\mathbb{Q}]\le n}} Z'_{F,S} = (V^{(n)})^{\perp} \cap W_{K,S},$$

But then $f_n \in V^{(n)}$ and $f_n' \in (V^{(n)})^{\perp}$, so by the uniqueness of the orthogonal decomposition, we must in fact have $f_n = P^{(n)} f$ and $f_n' = Q^{(n)} f = (I - P^{(n)})f$. Since this proof works for any $f \in \mathcal{F}$, we have established the desired claim. ☐

Now we observe that the subspaces $V^{(n)}$ with their associated projections $P^{(n)}$, indexed by $\mathbb{N}$ with the usual partial order $\le$, satisfy the conditions of Theorem 7, and thus we have orthogonal projections $T^{(n)}$ and an orthogonal decomposition

$$\mathcal{F} = \bigoplus_{n=1}^{\infty} T^{(n)}(\mathcal{F}). \tag{2.13}$$

The operators $T^{(n)}$ have a particularly simple form in terms of the $P^{(n)}$ projections. The Möbius function for $\mathbb{N}$ under the partial order $\leq$ is well-known and is merely

$$\mu_{\mathbb{N}}(m,n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n-1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $T^{(1)} = P^{(1)} = P_{\mathbb{Q}}$ and

$$T^{(n)} = P^{(n)} - P^{(n-1)} \quad \text{for all} \quad n > 1.$$

We call $T^{(n)}f$ the *degree $n$ component of $f$*. The following proposition is now obvious from the above constructions:

**Proposition 2.26.** *Each $f \in \mathcal{F}$ has a unique finite expansion into its degree $n$ components, $f^{(n)} = T^{(n)}f \in \mathcal{F}$*

$$f = \sum_{n\in\mathbb{N}} f^{(n)}.$$

*Each $f^{(n)}$ term can be written as a finite sum $f^{(n)} = \sum_i f_i^{(n)}$ where $f_i^{(n)} \in \mathcal{F}$ and $\delta(f_i^{(n)}) = n$ for each $i$, and $f^{(n)}$ cannot be expressed as a finite sum $\sum_j f_j^{(n)}$ with $\delta(f_j^{(n)}) \leq n$ for each $j$ and $\delta(f_j^{(n)}) < n$ for some $j$.*

This completes the proof of Theorem 2. It remains to prove Theorem 3.

*Proof of Theorem 3.* From Proposition 2.23, we see that the operators $T_K$ and $P^{(n)}$ commute for $K \in \mathcal{K}^G$ and $n \in \mathbb{N}$. But $T^{(n)} = P^{(n)} - P^{(n-1)}$ for $n > 1$ and $T^{(1)} = P^{(1)}$, so by the commutativity of $T_K$ with $P^{(n)}$ we have the desired result. In particular, the map $T_K^{(n)} = T^{(n)}T_K : \mathcal{F} \to \mathcal{F}$ is also a projection, and thus we can combine equations (2.6) and (2.13) to obtain the orthogonal decomposition

$$\mathcal{F} = \bigoplus_{n=1}^{\infty} \bigoplus_{K\in\mathcal{K}^G} T_K^{(n)}(\mathcal{F}). \tag{2.14}$$ □

## 3. Reducing the Lehmer problem

3.1. **Representability.** The concept of representability was introduced in [6, §2.2]. In this section we develop the idea in greater depth. Let us recall that we defined in Section 2.6 the function $\delta : \mathcal{F} \to \mathbb{N}$ by

$$\delta(f) = \#\{L_\sigma f : \sigma \in G\} = [G : \mathrm{Stab}_G(f)] = [K_f : \mathbb{Q}].$$

Observe that since nonzero scaling of $f$ does not affect its $\mathbb{Q}$-vector space span or the minimal field $K_f$ that the function $\delta$ is invariant under nonzero scaling in $\mathcal{F}$, that is,

$$\delta(rf) = \delta(f) \quad \text{for all} \quad f \in \mathcal{F} \text{ and } 0 \neq r \in \mathbb{Q}.$$

In order to better understand the relationship between our functions in $\mathcal{F}$ and the algebraic numbers from which they arise, we need to understand when a function $f_\alpha \in V_K$ has a representative $\alpha \in K^\times$ or is merely an $n$th root of an element of $K^\times$ for some $n > 1$. Naturally, the choice of coset representative modulo torsion affects this question, and we would like to avoid such considerations. Therefore we define the function $d : \mathcal{F} \to \mathbb{N}$ by

$$d(f_\alpha) = \min\{\deg(\zeta\alpha) : \zeta \in \mathrm{Tor}(\overline{\mathbb{Q}}^\times)\}. \tag{3.1}$$

Notice that the minimum is invariant under the choice of coset representative $\alpha \in \overline{\mathbb{Q}}^\times$ for $f_\alpha \in \mathcal{F}$.

Notice that a function $f \in \mathcal{F}$ can then be written as $f = f_\alpha$ with $\alpha \in K_f^\times$ if and only if $d(f) = \delta(f)$. We therefore make the following definition:

**Definition 3.1.** We define the set of *representable* elements of $\mathcal{F}$ to be the set

$$\mathcal{R} = \{f \in \mathcal{F} : \delta(f) = d(f)\}. \tag{3.2}$$

The set $\mathcal{R}$ consists precisely of the functions $f$ such that $f = f_\alpha$ for some $\alpha$ of degree equal to the degree of the minimal field of definition $K_f$ of $f$.

We recall the terminology from [4] that a number $\alpha \in \overline{\mathbb{Q}}^\times$ is *torsion-free* if $\alpha/\sigma\alpha \notin \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$ for all distinct Galois conjugates $\sigma\alpha$. As we observed above in the proof of Proposition 2.8, torsion-free numbers give rise to distinct functions $f_{\sigma\alpha} = L_\sigma f_\alpha$ for each distinct Galois conjugate $\sigma\alpha$ of $\alpha$.

The goal of this subsection is to prove the following result relating $\delta$ and $d$:

**Proposition 3.2.** *Let $0 \neq f \in \mathcal{F}$ and $r, s \in \mathbb{Z}$ with $(r, s) = 1$. Then the set $R(f) = \{q \in \mathbb{Q} : qf \in \mathcal{R}\}$ satisfies*

$$R(f) = \frac{\ell}{n}\mathbb{Z}$$

*where $\ell, n \in \mathbb{N}$, $(\ell, n) = 1$, and*

$$d((r/s)f) = \frac{\ell s}{(\ell, r)(n, s)}\delta(f). \tag{3.3}$$

*In particular, $d(f) = \ell \cdot \delta(f)$.*

The proof of Proposition 3.2 consists of showing that $R(f)$ is a fractional ideal of $\mathbb{Q}$ which scales according to $R(qf) = (1/q)R(f)$, and that when $f$ is scaled so that $R(f) = \mathbb{Z}$ we have $d((r/s)f) = s\delta(f)$. We establish these results in a series of lemmas below. We begin by demonstrating the most basic results concerning representability:

**Lemma 3.3.** *We have the following results:*

1. *For each $f \in \mathcal{F}$, there is a unique minimal exponent $\ell = \ell(f) \in \mathbb{N}$ such that $\ell f \in \mathcal{R}$.*
2. *For any $\alpha \in \overline{\mathbb{Q}}^\times$, we have $\delta(f_\alpha) | \deg \alpha$.*
3. *$f \in \mathcal{R}$ if and only if it has a representative in $\overline{\mathbb{Q}}^\times$ which is torsion-free.*
4. *Every torsion-free representative of $f \in \mathcal{R}$ lies in the same field $K_f$, the minimal field of $f$.*

*Proof.* Choose a representative $\alpha \in \overline{\mathbb{Q}}^\times$ such that $f = f_\alpha$ and let

$$\ell = \mathrm{lcm}\{\mathrm{ord}(\alpha/\sigma\alpha) : \sigma \in G \text{ and } \alpha/\sigma\alpha \in \mathrm{Tor}(\overline{\mathbb{Q}}^\times)\}$$

where $\mathrm{ord}(\zeta)$ denotes the order of an element $\zeta \in \mathrm{Tor}(\overline{\mathbb{Q}}^\times)$. Then observe that $\alpha^\ell$ is torsion-free. Clearly, $\mathbb{Q}(\alpha^\ell) \subset \mathbb{Q}(\alpha)$ so $[\mathbb{Q}(\alpha^\ell) : \mathbb{Q}] | [\mathbb{Q}(\alpha) : \mathbb{Q}]$. Now if a number

$\beta \in \overline{\mathbb{Q}}^\times$ is torsion-free, then since each distinct conjugate $\sigma\beta$ gives rise to a distinct function in $\mathcal{F}$, we have

$$\deg \beta = [G : \mathrm{Stab}_G(f_\beta)] = [K_{f_\beta} : \mathbb{Q}] = \delta(f_\beta).$$

Thus $\deg \alpha^\ell = \delta(f_\alpha)$ and we have proven existence in the first claim. The existence of a minimum value follows since $\mathbb{N}$ is discrete. To prove the second claim it now suffices to observe that since $\delta$ is invariant under scaling, with the choice of $\ell$ as above, we have $\delta(f_\alpha) = \delta(f_\alpha^\ell) | \deg \alpha$ for all $\alpha \in \overline{\mathbb{Q}}^\times$. The third claim now follows immediately. Lastly, since any representative of $f$ differs by a root of unity, each representative has some power which lies in (and generates) the minimal field, and thus each torsion-free representative generates the minimal field. $\square$

We note the following easy corollary for its independent interest:

**Corollary 3.4.** *Let $\alpha \in \overline{\mathbb{Q}}^\times$ have minimal polynomial $F(x) \in \mathbb{Z}[x]$. Let $G(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of smallest degree in $\mathbb{Z}[x]$ such that there exists some $k \in \mathbb{N}$ with $F(x)|G(x^k)$. Then $\delta(f_\alpha) = \deg G$.*

(We observe in passing that $\delta(f) = 1$ if and only if $f \in V_{\mathbb{Q}}$, in which case, $f = f_\alpha$ where $\alpha^n \in \mathbb{Q}^\times$ and so $f$ represents a *surd*, that is, a root of a rational number.)

**Lemma 3.5.** *If $0 \neq f \in \mathcal{F}$, then $R(f) = \{r \in \mathbb{Q} : rf \in \mathcal{R}\}$ is a fractional ideal of $\mathbb{Q}$, that is, $R(f) = r\mathbb{Z}$ for some $r \in \mathbb{Q}$.*

*Proof.* We can assume $\delta(f) > 1$, otherwise $f$ arises from a surd and the proof is trivial. First we show that $R(f)$ is a $\mathbb{Z}$-module. It is trivial that if $r \in R(f)$ then $-r \in R(f)$ as inversion does not affect degree. Suppose now that we have $r, s \in R(f)$ and choose torsion-free representatives $\beta \in \overline{\mathbb{Q}}^\times$ of $rf$ and $\gamma \in \overline{\mathbb{Q}}^\times$ of $sf$. If $r + s = 0$ the result is trivial, so suppose not. By Lemma 3.3 (4), we have $\beta, \gamma \in K_f$. But then $\beta\gamma \in K_f$ as well, and hence is a representative of $f_{\beta\gamma} = f_\beta + f_\gamma = rf + sf = (r + s)f$ of degree $[K_f : \mathbb{Q}] = \delta(f)$, and thus we have $r + s \in R(f)$ as well.

If we can now show that $R(f)$ is finitely generated the proof will be complete, as it is easy to check that any finitely generated $\mathbb{Z}$-submodule of $\mathbb{Q}$ is indeed a fractional ideal. But were it to require an infinite number of generators, we would have to have elements of arbitrarily large denominator. Further, we could fix an $N$ sufficiently large so that for a sequence of $n_i \to \infty$, we would have some $r_i/n_i \in R(f)$ and $|r_i/n_i| \leq N$. (For example, given $r_1/n_1$, we can take $N = r_1/n_1$ by appropriately subtracting off multiples of $r_1/n_1$ from any other $r_i/n_i$.) But then we would have torsion-free representatives $\alpha^{r_i/n_i}$ satisfying $h(\alpha^{r_i/n_i}) \leq N\, h(\alpha)$, and as representable representatives, each representative has the same degree $\delta(f)$, and thus we have an infinite number of algebraic numbers with bounded height and degree, contradicting Northcott's theorem. $\square$

**Lemma 3.6.** *Let $0 \neq q \in \mathbb{Q}$. Then $R(qf) = \frac{1}{q}R(f)$.*

*Proof.* This is clear from the definition. $\square$

**Lemma 3.7.** *Let $f \in \mathcal{R}$ with $R(f) = \mathbb{Z}$ and let $m, n \in \mathbb{Z}$ where $(m, n) = 1$ and $n > 0$. Let $\alpha$ be a torsion-free representative of $f$ and denote by $\alpha^{m/n}$ any* *representative of the class of* $f_{\alpha^{m/n}} = (m/n)f$ *modulo torsion of minimal degree. Then* $\deg \alpha^{m/n} = n \deg \alpha$. *In particular, we have*

$$d((m/n)f) = n\, d(f) = n\, \delta(f) \quad \textit{if} \quad R(f) = \mathbb{Z}. \tag{3.4}$$

*Proof.* Since $R(f) = \mathbb{Z}$, our choice of torsion-free representative $\beta$ in $\overline{\mathbb{Q}}^{\times}$ has degree $\delta(f)$. Clearly, we can say that $d((m/n)f) \leq n \deg \alpha = n\, \delta(f)$ because any root of $x^n - \alpha^m$ over $\mathbb{Q}(\alpha)$ will be a representative of the class of $(m/n)f$. Observe that the minimal field $K_f = \mathbb{Q}(\alpha)$ is, as we observed above, unique, and thus the choice of $\alpha$ differs at most by some torsion element of $\mathbb{Q}(\alpha)^{\times}$. Further, any choice of representative $\beta \in \overline{\mathbb{Q}}^{\times}$ of $(m/n)f$ will satisfy $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\beta)$ since some power of $\beta$ will make it torsion-free and therefore it will be a power of $\alpha$.

Let us show that the degree of $\beta$ cannot satisfy $\deg \beta < n \deg \alpha$ if $R(f) = \mathbb{Z}$. Suppose it did, so that $k = [\mathbb{Q}(\beta) : \mathbb{Q}(\alpha)] < n$. Then observe that by taking the algebraic norm down to $\mathbb{Q}(\alpha)$, we have

$$\mathrm{Norm}^{\mathbb{Q}(\beta)}_{\mathbb{Q}(\alpha)}(\beta) = \zeta \alpha^{km/n} \in \mathbb{Q}(\alpha)$$

where $\zeta$ is a root of unity. As $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \delta(f)$ the existence of the representative $\zeta\alpha^{km/n}$ would imply that $km/n \in R(f)$, but since $(m, n) = 1$ and $k < n$, we have $km/n \notin \mathbb{Z}$. This contradicts our assumption that $R(f) = \mathbb{Z}$. □

Combining the above lemmas, we now see that we have proven Proposition 3.2.

3.2. **Reduction to representable numbers.** We will now show that we can reduce questions related to lower bounds for the $L^p$ Mahler measure to the set of representable elements. We begin with two lemmas regarding the relationship between the projection operators $P_K$ and the degree functions $d$ and $\delta$ which will be used below:

**Lemma 3.8.** *If* $f \in \mathcal{F}$ *and* $K \subset K_f$, *then* $d(P_K f) \leq d(f)$.

*Proof.* Let $f = f_\alpha$ and let $\alpha \in \overline{\mathbb{Q}}^{\times}$ be a minimal degree representative of $f$, and choose $\ell \in \mathbb{N}$ such that $\alpha^\ell$ is torsion-free. Then $\mathbb{Q}(\alpha^\ell) = K_f$, so in particular, we see that

$$K \subseteq K_f \subseteq \mathbb{Q}(\alpha).$$

Observe that the norm $N_K^{K(\alpha)}$ from $K(\alpha)$ to $K$ is well-defined on the class $f_\alpha \in \mathcal{F}$. Since for some choice of root $(N_K^{K(\alpha)}\alpha)^{1/[K(\alpha):K]}$ is a representative of $P_K f$ modulo torsion, it follows from the fact that $N_K^{K(\alpha)}\alpha \in K$ that

$$\begin{aligned} d(P_K f) \leq \deg(N_K^{K(\alpha)}\alpha)^{1/[K(\alpha):K]} &\leq [K(\alpha) : K] \cdot [K : \mathbb{Q}] \\ &= [\mathbb{Q}(\alpha) : \mathbb{Q}] = d(f). \quad \square \end{aligned}$$

**Lemma 3.9.** *If* $K \in \mathcal{K}$ *and* $K \subset K_f$ *for* $f \in \mathcal{F}$, *we have* $\delta(P_K f) \leq \delta(f)$.

*Proof.* Since we can rescale $f$ without affecting either $\delta$ value, we can assume $f \in \mathcal{R}$ so $d(f) = \delta(f)$. Let $F = K_f$. Then by Lemma 3.8 above, we have

$$\delta(P_K f) \leq d(P_K f) \leq d(f) = \delta(f). \quad \square$$

From the construction of $d$ above, it is easy to see that:

**Proposition 3.10.** *Let $m_p : \mathcal{F} \to [0,\infty)$ be given by $m_p(f) = d(f) \cdot \|f\|_p$. Fix $0 \neq f \in \mathcal{F}$. Then*

$$m_p(f) = \min\{(\deg \alpha) \cdot h_p(\alpha) : \alpha \in \overline{\mathbb{Q}}^\times, f_\alpha = f\}.$$

*The right hand side of this equation is the minimum of the $L^p$ analogue of the usual logarithmic Mahler measure on $\overline{\mathbb{Q}}^\times$ taken over all representatives of $f$ modulo torsion.*

We now prove the reduction to $\mathcal{R} \subset \mathcal{F}$:

**Proposition 3.11.** *Let $m_p(f) = d(f) \cdot \|f\|_p$. Then $m_p(\mathcal{F}) = m_p(\mathcal{R})$, so in particular, $\inf m_p(\mathcal{F} \setminus \{0\}) > 0$ if and only if $\inf m_p(\mathcal{R} \setminus \{0\}) > 0$.*

*Proof.* Let $f \in \mathcal{F}$ and $\ell = \ell(f)$. Then by Proposition 3.2 we have $\delta(f) = d(\ell f)$ and $\ell\,\delta(f) = d(f)$, and thus

$$m_p(\ell f) = \delta(f) \cdot \|\ell f\|_p = \ell\,\delta(f)\|f\|_p = d(f) \cdot \|f\|_p = m_p(f).$$ □

*Remark* 3.12. Proposition 3.11, which will be used below in the proof of Theorem 4, is a key step in constructing equivalent statements of Lehmer's conjecture for heights which scale, such as $\delta\, h_p$ and particularly for the norms we will construct. Consider for example that if $\alpha = 2^{1/n}$ then $\delta(f_\alpha) = 1$ for all $n \in \mathbb{N}$ and $h_1(2^{1/n}) = (2\log 2)/n \to 0$.

3.3. **Projection irreducibility.** In this section we introduce the last criterion which we will require to reduce the Lehmer conjectures to a small enough set of algebraic numbers to prove our main results.

**Definition 3.13.** We say $f \in \mathcal{F}$ is *projection irreducible* if $P_K(f) = 0$ for all proper subfields $K$ of the minimal field $K_f$. We denote the collection of projection irreducible elements by $\mathcal{P} \subset \mathcal{F}$.

*Remark* 3.14. Notice that we cannot in general require that $P_K(f) = 0$ for all $K \neq K_f$, as an element with a minimal field which is not Galois will typically have nontrivial projections to the conjugates of its minimal field. See Remark 2.21 above for more details.

We now prove that we can reduce questions about lower bounds on the Mahler measure $m_p$ to elements of $\mathcal{P}$:

**Proposition 3.15.** *We have*

$$\inf_{f \in \mathcal{F}\setminus\{0\}} m_p(f) > 0 \quad\iff\quad \inf_{f \in \mathcal{P}\setminus\{0\}} m_p(f) > 0.$$

*Proof.* Let $f \in \mathcal{F}$. Notice that for any $K \in \mathcal{K}$ that by Lemma 3.8 we have $d(P_K f) \leq d(f)$ and by Lemma 2.11 we have $h_p(P_K f) \leq h_p(f)$, so $m_p(P_K \alpha) \leq m_p(f)$. Let $\operatorname{supp}_{\mathcal{K}}(f) = \{K \in \mathcal{K} : P_K f \neq 0\}$. Notice that if $K \subset L$ and $K \in \operatorname{supp}_{\mathcal{K}}(f)$, then $L \in \operatorname{supp}_{\mathcal{K}}(f)$. Let $E$ denote the Galois closure of $K_f$, and observe that $P_K f = P_K(P_E f) = P_{K\cap E} f$ by Lemma 2.16, so since we have only a finite number of subfields of $E$, we can write $\operatorname{supp}_{\mathcal{K}}(f) = \bigcup_{i=1}^n [K_i,\ )$ where $[K_i,\ ) = \{L \in \mathcal{K} : K_i \subseteq L\}$, and each $K_i \subseteq E$ is minimal in the sense that $[K_i,\ ) \not\subseteq [K_j,\ )$ for all $i \neq j$. Thus, for each $i$, $P_F f = 0$ for all $F \subsetneq K_i$, and so $P_{K_i} f \in \mathcal{P} \setminus \{0\}$. Then $0 < m_p(P_{K_i} f) \leq m_p(f)$, and so we have shown $\inf_{f\in\mathcal{P}\setminus\{0\}} m_p(f) \leq \inf_{f\in\mathcal{F}\setminus\{0\}} m_p(f)$. The reverse inequality is trivial. □

## 4. The Mahler $p$-norm

4.1. **An $L^p$ analogue of Northcott's theorem.** We begin by proving an analogue of Northcott's theorem for the $L^p$ Weil heights which we will make use of in this chapter. We begin with some easy lemmas which relate the $L^p$ height to the $L^1$ height.

**Lemma 4.1.** *Let $f \in \mathcal{F}$ and suppose $\operatorname{supp}(f) \subseteq Y(\mathbb{Q}, \pi)$ for some rational prime $\pi$ (possibly infinity). Then for $1 < p \leq \infty$, we have*

$$\|f\|_1 \leq \|f\|_p \leq \delta(f)^{1-1/p} \|f\|_1.$$

(We follow the usual convention for exponents and let $1/p = 0$ when $p = \infty$ for convenience.)

*Proof.* The first inequality in fact is a well-known fact of $L^p$ norms on measure one spaces, however, we will give another proof in this case as it is useful to do so. Let $K = K_f$ be the minimal field, so in particular, $[K : \mathbb{Q}] = \delta(f)$. Let $n = \delta(f)$ denote this common value. Then $Y(\mathbb{Q}, \pi)$ can be partitioned into a disjoint union of the sets $Y(K, v)$ for $v|\pi$. Notice that $\lambda(Y(K, v)) = d_v/n$ for each $v$, where $d_v = [K_v : \mathbb{Q}_v]$ is the local degree. Enumerate the set of $v$ lying over $\pi$ as $v_1, \ldots, v_n$, counting each place $d_v$ times, so that if, for example, $d_v = 3$, then there will be three places $v_k, v_{k+1}, v_{k+2}$ corresponding to $v$ (for some number $k$). Let $c_i$ denote the value of $f(y)$ on $Y(K, v_i)$. Let $q$ be the usual conjugate exponent determined by $1/p + 1/q = 1$. Then observe that:

$$\|f\|_1 = \frac{1}{n} \sum_{i=1}^{n} |c_i| \leq \frac{1}{n} \cdot n^{1/q} \left( \sum_{i=1}^{n} |c_i|^p \right)^{1/p} = \frac{1}{n^{1/p}} \left( \sum_{i=1}^{n} |c_i|^p \right)^{1/p} = \|f\|_p.$$

where we have applied Hölder's inequality. For the upper bound, we compute

$$\|f\|_p = \frac{1}{n^{1/p}} \left( \sum_{i=1}^{n} |c_i|^p \right)^{1/p} \leq \frac{1}{n^{1/p}} \sum_{i=1}^{n} |c_i| = n^{1-1/p} \cdot \frac{1}{n} \sum_{i=1}^{n} |c_i| = n^{1-1/p} \|f\|_1$$

from which the result now follows. □

We now bound our heights without assuming that $f$ is supported on a single prime:

**Proposition 4.2.** *Let $f \in \mathcal{F}$ and $1 < p \leq \infty$. Then we have the following inequalities:*

$$\|f\|_1 \leq \lambda(\operatorname{supp} f)^{1-1/p} \|f\|_p \quad \textit{and} \quad \|f\|_p \leq \delta(f)^{1-1/p} \|f\|_1. \tag{4.1}$$

*Proof.* Let $q$ be given by $1/p + 1/q = 1$ as usual. Then the first inequality is just the usual application of Hölder's inequality:

$$\|f\|_1 = \int_{\operatorname{supp} f} |f(y)|\, d\lambda(y) \leq \left( \int_{\operatorname{supp} f} 1^q\, d\lambda(y) \right)^{1/q} \left( \int_{\operatorname{supp} f} |f(y)|^p\, d\lambda(y) \right)^{1/p}$$
$$= \lambda(\operatorname{supp} f)^{1/q} \|f\|_p.$$

For the second inequality, let us write $f|_\pi$ for the restriction of $f$ to the set $Y(\mathbb{Q}, \pi)$. Then $f|_\pi$ is a function on a measure one space, so locally we can make use of the above lemma at each place $\pi$:

$$\begin{aligned}\|f\|_p &= \Bigg(\sum_{\pi\in M_{\mathbb{Q}}} \|f|_\pi\|_p^p\Bigg)^{1/p} \le \Bigg(\sum_{\pi\in M_{\mathbb{Q}}} \delta(f)^{p/q}\|f|_\pi\|_1^p\Bigg)^{1/p}\\ &= \delta(f)^{1/q}\Bigg(\sum_{\pi\in M_{\mathbb{Q}}} \|f|_\pi\|_1^p\Bigg)^{1/p} \le \delta(f)^{1/q}\sum_{\pi\in M_{\mathbb{Q}}}\|f|_\pi\|_1 = \delta(f)^{1-1/p}\|f\|_1. \quad \square\end{aligned}$$

where we make use of the general fact that for any sequence $x \in \ell^p(\mathbb{N})$, we have $\|x\|_{\ell^p} \le \|x\|_{\ell^1}$. (In fact, each $f \in \mathcal{F}$ is supported on a finite number of rational primes, so there is no issue of convergence here.)

The classical Northcott theorem tells us that any set of algebraic numbers of bounded height and degree is finite. As $2h(\alpha) = \|f_\alpha\|_1$, this translates to a bound on the $L^1$ height. Naturally, as we are working in $\mathcal{F}$, we count modulo torsion, but even so we must be careful about the choice of our notion of degree (indeed, it is easy to see that the number of elements of $\mathcal{F}$ with bounded $\delta$ and $L^p$ norm is not finite).

**Theorem 8** ($L^p$ Northcott)**.** *For any $C, D > 0$, we have*

$$\#\{f \in \mathcal{R} : \|f\|_p \le C \text{ and } \delta(f) \le D\} < \infty, \tag{4.2}$$

*and*

$$\#\{f \in \mathcal{F} : \|f\|_p \le C \text{ and } d(f) \le D\} < \infty. \tag{4.3}$$

*Proof.* Notice that $f \in \mathcal{R}$ implies that $d(f) = \delta(f)$ by definition, so that the first set is a subset of the second. Thus, it suffices to show that the second set is finite. Each element $f_\alpha$ of the second set gives rise to a representative $\alpha \in \overline{\mathbb{Q}}^\times$ with degree $d(f_\alpha)$, so if we can show that $h(\alpha)$ is bounded, then Northcott's theorem will give us the desired result. Notice that if $f \in \mathcal{F}$ has nontrivial support at a rational prime $\pi$, then

$$\|f\|_p \ge \|f|_{Y(\mathbb{Q},\pi)}\|_p \ge \|f|_{Y(\mathbb{Q},\pi)}\|_1 \ge \frac{\log \pi}{d(f)} \ge \frac{\log \pi}{D}.$$

As we assume that $\|f\|_p \le C$, this tells us that $\log \pi \le CD$. Since the measure $\lambda$ assigns measure 1 to any rational prime, we see that

$$\lambda(\operatorname{supp}(f)) \le 1 + \pi(\exp(CD))$$

where $\pi(x)$ is the usual prime counting function. Thus, by Proposition 4.2, we see that

$$\|f\|_1 \le \lambda(\operatorname{supp}(f))^{1-1/p}\,\|f\|_p \le (1 + \pi(\exp(CD)))^{1-1/p}\, C. \tag{4.4}$$

As $2h(\alpha) = \|f_\alpha\|_1$, this gives a bound on the classical Weil height for any representative of an element of our set. Northcott's theorem then applies and gives us the desired result, as we find we have a finite number of possible coset representatives and therefore a finite number of elements of $\mathcal{F}$. $\square$

4.2. **The Mahler $p$-norms and proof of Theorem 4.** We will now make use of our orthogonal decomposition (2.13) to define one of the main operators of our study. Let

$$\begin{aligned} M : \mathcal{F} &\to \mathcal{F} \\ f &\mapsto \sum_{n=1}^{\infty} n\, T^{(n)} f. \end{aligned} \tag{4.5}$$

The $M$ operator serves the purpose of allowing us to scale a function in $\mathcal{F}$ by its appropriate degree while still being linear. As each element of $\mathcal{F}$ has a finite expansion in terms of $T^{(n)}$ components, the above map is well-defined. Further, it is easily seen to be linear by the linearity of the $T^{(n)}$, and it is also a bijection. However, it is not a bounded operator (and thus, in particular, $M$ is not well-defined on the space $L^p(Y)$):

**Proposition 4.3.** *The linear operator $M : \mathcal{F} \to \mathcal{F}$ is unbounded in any $L^p$ norm.*

*Proof.* Below in Propositions 4.7, 4.8, and 4.5 we will prove that every Salem number $\tau > 1$ is representable, projection irreducible, and therefore, an eigenvector of the $M$ operator of eigenvalue $\delta(f_\tau)$, that is, $f_\tau \in \mathcal{R} \cap \mathcal{P}$, $K_\tau = \mathbb{Q}(\tau)$, and $Mf_\tau = \delta(f_\tau) \cdot f_\tau = d(f_\tau) \cdot f_\tau$. As there exist Salem numbers of arbitrarily large degree, $M$ has eigenvectors of arbitrarily large eigenvalue and we obtain the desired result. □

We recall that *the Mahler $p$-norm* on $\mathcal{F}$ is defined to be

$$\|f\|_{m,p} = \|Mf\|_p \tag{4.6}$$

where $\|\cdot\|_p$ denotes the usual $L^p$ norm as defined above.

**Proposition 4.4.** *The map $\|\cdot\|_{m,p} : \mathcal{F} \to [0, \infty)$ is a vector space norm on $\mathcal{F}$.*

*Proof.* This follows easily from the fact that $M$ is an invertible linear operator on $\mathcal{F}$. Specifically, we have for all $f, g \in \mathcal{F}$ and $r \in \mathbb{Q}$,

$$\|f\|_{m,p} = \|Mf\|_p = 0 \iff Mf = 0 \iff f = 0$$

because $M$ is invertible and $\|\cdot\|_p$ is a norm on $\mathcal{F}$, and

$$\|f+g\|_{m,p} = \|M(f+g)\|_p = \|Mf + Mg\|_p \le \|Mf\|_p + \|Mg\|_p = \|f\|_{m,p} + \|g\|_{m,p},$$

and

$$\|rf\|_{m,p} = \|M(rf)\|_p = \|rMf\|_p = |r| \cdot \|Mf\|_p = |r| \cdot \|f\|_{m,p}$$

by the linearity of $M$. □

The following proposition, interesting in its own right, will be useful to us below:

**Proposition 4.5.** *If $f \in \mathcal{P}$, then $T^{(\delta(f))} f = f$, and in particular $f$ is an eigenvector of the $M$ operator with eigenvalue $\delta(f)$.*

*Proof.* Let $n = \delta(f)$ and $K = K_f$ be the minimal field of $f$. Obviously, as $f \in V_K$ and $[K_f : \mathbb{Q}] = \delta(f) = n$, we have $P^{(n)} f = f$. Since

$$P^{(n)} = \sum_{k=1}^{n} T^{(k)},$$

we can find a minimal value $1 \le m \le n$ such that $T^{(m)} f \ne 0$. Then $T^{(m)} f = P^{(m)} f$ for this value. We claim that if $m < n$, then $f$ is not projection irreducible.

To see this, observe that from the proof of Proposition 2.25, we found equation (2.12), which, together with the commutativity of $P^{(m)}$ and the $T_K$ operators and expanding the set of primes $S$ appropriately (every element of $V_K$ is an $S$-unit for a large enough set of primes $S$ of $K$) tells us that in fact, the $P^{(m)}$ projection corresponds to the $\mathbb{Q}$-vector space direct sum decomposition:

$$V_K = P^{(m)}(V_K) \oplus Q^{(m)}(V_K) = \Bigg( \sum_{\substack{F \subseteq K \\ [F:\mathbb{Q}] \leq m}} P_F(V_K) \Bigg) \oplus \Bigg( \bigcap_{\substack{F \subseteq K \\ [F:\mathbb{Q}] \leq m}} Q_F(V_K) \Bigg), \tag{4.7}$$

where $Q^{(m)} = I - P^{(m)}$ and $Q_F = I - P_F$ are the complementary projections. (Technically, we should replace $K$ with its Galois closure to match the construction in the proof of Proposition 2.25, but observe that we can repeat the construction starting with $V_{K,S}$ for $K$ any number field instead of using $T_K(V_{K,S})$ for $K$ Galois; the results are the same, as it is only the finite dimensionality of the $S$-unit space $V_{K,S}$ that is essential to the construction). If $P_F(f) = 0$ then $Q_F(f) = f$, so if $f$ had no nontrivial projections to any proper subfields of $f$, it would also have decomposition $f = 0 \oplus f$ and thus $P^{(m)} f = 0$. Thus if $P^{(m)} f \neq 0$ then $P_F(f) \neq 0$ for some $F \subsetneq K$, but this is a contradiction to the projection irreducibility of $f$. Hence we must have had $T^{(n)} f = f$. □

We can complete $\mathcal{F}$ with respect to $\|\cdot\|_{m,p}$ to obtain a real Banach space which we denote $\mathcal{F}_{m,p}$. We are now ready to prove Theorem 4, which we restate for the reader's convenience. First, we recall the $L^p$ analogue of the Lehmer conjecture (Conjecture 1) from above:

$$(*_p) \qquad m_p(\alpha) = (\deg \alpha) \cdot h_p(\alpha) \geq c_p > 0 \quad \text{for all} \quad \alpha \in \overline{\mathbb{Q}}^\times \setminus \operatorname{Tor}(\overline{\mathbb{Q}}^\times).$$

**Theorem 4.** *For each $1 \leq p \leq \infty$, equation $(*_p)$ holds if and only if*

$$(**_p) \qquad \|f\|_{m,p} \geq c_p > 0 \quad \textit{for all} \quad 0 \neq f \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$$

*where $\mathcal{R}$ denotes the set of representable elements, $\mathcal{P}$ the set of projection irreducible elements, and $\mathcal{U}$ the subspace of algebraic units. Further, for $1 \leq p \leq q \leq \infty$, if equation $(*_p)$ holds for $p$ then equation $(*_p)$ holds for $q$ as well.*

*Proof of Theorem 4.* First let us show that it suffices to bound $m_p(f)$ away from zero for $f \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$. Let $f \in \mathcal{F}$. We begin by reducing to the vector space $\mathcal{U} = \{f \in \mathcal{F} : \operatorname{supp}_Y(f) \subseteq Y(\mathbb{Q}, \infty)\}$. If $1 \leq p < \infty$, observe that

$$h_p(f) = \|f\|_p = \Bigg( \sum_{\pi \in M_\mathbb{Q}} \|f|_{Y(\mathbb{Q},\pi)}\|_p^p \Bigg)^{1/p} \geq \|f|_{Y(\mathbb{Q},\pi)}\|_p \geq \|f|_{Y(\mathbb{Q},\pi)}\|_1,$$

since $Y(\mathbb{Q}, \pi)$ is a space of measure 1. Likewise, it is easy to see that

$$h_\infty(f) = \max_{\pi \in M_\mathbb{Q}} \|f|_{Y(\mathbb{Q},\pi)}\|_\infty \geq \|f|_{Y(\mathbb{Q},\pi)}\|_\infty \geq \|f|_{Y(\mathbb{Q},\pi)}\|_1$$

for a specific rational prime $\pi$, so we can let $p = \infty$ as well. Let the rational prime $\pi$ be chosen above so that the norm of the restriction to $Y(\mathbb{Q}, \pi)$ is nonzero, which we can do if $f \notin \mathcal{U}$. Let $\alpha \in \overline{\mathbb{Q}}^\times$ be a representative of minimal degree $d(f)$ for $f$. Then $\alpha$ has a nontrivial valuation over $\pi$, and since the product of $\alpha$ over all of its conjugates must be in $\mathbb{Q}$, we know that we must have $\|f|_{Y(\mathbb{Q},p)}\|_1 \geq (\log \pi)/d(f)$. Thus $h_p(f) \geq (\log 2)/d(f)$, so $m_p(f) \geq \log 2$ for $1 \leq p \leq \infty$ if $f \notin \mathcal{U}$. Now it remains to show that we can reduce to the consideration of $\mathcal{P}$ as well, but this now follows immediately from the technique of the proof in Proposition 3.15, p. 23 above, specifically, by projecting to a minimal field $F$ in the $\mathcal{K}$-support of $f$ to ensure projection irreducibility, and observing that by Lemma 2.10, p. 10, $P_F(f) \in \mathcal{U}$ if $f \in \mathcal{U}$. Now observe that if $f \in \mathcal{U} \cap \mathcal{P}$, then upon scaling $f$ it remains in $\mathcal{U} \cap \mathcal{P}$, so we are free to replace $f$ by $\ell(f)f$ as in the proof of Proposition 3.11, p. 23 without changing the value of $m_p(f)$, and thus we can assume $f \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$, as claimed.

Now let $f \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$, and we will show that $m_p(f) = \|f\|_{m,p}$, completing the proof of the equivalence. Observe that for such an element, by Proposition 4.5 projection irreducibility, we must have $T^{(n)} f = f$ where $n = \delta(f) = [K_f : \mathbb{Q}]$ for $K_f$ the minimal field of $f$, and in particular, $Mf = nf$. Thus

$$\|f\|_{m,p} = \|Mf\|_p = [K_f : \mathbb{Q}] \cdot \|f\|_p = \delta(f) h_p(f) = d(f) h_p(f) = m_p(f)$$

where the second equality follows from the fact that $f \in \mathcal{P}$ and the fourth from the fact that $f \in \mathcal{R}$. This completes the equivalence of the bounds.

To show that for $1 \leq p \leq q \leq \infty$ the result for $p$ implies the result for $q$, we observe that having reduced the problem to the study of algebraic units $\mathcal{U}$, that these numbers are of the form

$$\mathcal{U} = \{f \in \mathcal{F} : \operatorname{supp}_Y(f) \subseteq Y(\mathbb{Q}, \infty)\}$$

and since $\lambda(Y(\mathbb{Q}, \infty)) = 1$, we are reduced to the consideration of measurable functions on the probability space $(Y(\mathbb{Q}, \infty), \lambda)$. But on such a space one has the usual inequality $\|f\|_p \leq \|f\|_q$ and thus $\|f\|_{m,p} = \|Mf\|_p \leq \|Mf\|_q = \|f\|_{m,q}$. □

Lastly, we note for its own interest:

**Proposition 4.6.** *Equation $(*_p)$ for $p = 1$ is equivalent to the Lehmer conjecture, and for $p = \infty$, $(*_p)$ is equivalent to the Schinzel-Zassenhaus conjecture.*

*Proof.* Since $h = 2h_1$ it is obvious that $m_1 = 2m$ so we exactly have the statement of the Lehmer conjecture when $p = 1$. Let us now show that when $p = \infty$, equation $(*_p)$ is equivalent to the Schinzel-Zassenhaus conjecture. Recall that the house $\overline{|\alpha|} = \max\{|\sigma\alpha| : \sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}\}$ where $|\cdot|$ denotes the usual Euclidean absolute on $\mathbb{C}$. The Schinzel-Zassenhaus conjecture [9] states that for an algebraic integer $\alpha$, $(\deg \alpha) \cdot \log \overline{|\alpha|}$ is bounded away from zero by an absolute constant. Observe that by Smyth's well-known theorem [11], we have $m_1(\alpha) \geq c > 0$ for an absolute constant $c$ if $\alpha$ is not reciprocal. Since $\|f\|_{m,\infty} \geq \|f\|_{m,1} = m_1(f)$ for the numbers under consideration, we see that if $\alpha$ is not reciprocal, then there is nothing more to show by the previous theorem. If $\alpha$ is reciprocal, then observe that $\alpha$ and $\alpha^{-1}$ are conjugate, and so $\overline{|\alpha|} = \max\{\overline{|\alpha|}, \overline{|\alpha^{-1}|}\}$, where $\max\{\overline{|\alpha|}, \overline{|\alpha^{-1}|}\}$ is called the *symmetric house*. Now, it is easy to see that $h_\infty(\alpha) = \log \max\{\overline{|\alpha|}, \overline{|\alpha^{-1}|}\}$ is the logarithmic symmetric house of $\alpha$ for $f_\alpha \in \mathcal{U}$, so we do indeed recover the Schinzel-Zassenhaus conjecture when $p = \infty$.[1] □

4.3. **Explicit values.** We now evaluate the Mahler $p$-norms for two classes of algebraic numbers, surds and Salem numbers. Salem numbers are conjectured to be of minimal Mahler measure for the classical Lehmer conjecture. This is in part due to the fact that the minimal value for the Mahler measure known, dating back to Lehmer's original 1933 paper [8], is that of the Salem number called Lehmer's

[1] We remark in passing that while $h_\infty$ agrees with the logarithmic symmetric house on $\mathcal{U}$, $h_\infty$ seems to be a better choice for non-integers as well, as, for example, $h_\infty(3/2) = \log 3$ while the logarithmic symmetric house of $3/2$ is $\log(3/2)$.

$\tau > 1$, the larger positive real root of the irreducible polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. Here we show that, in fact, Salem numbers belong to the set $\mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$.

4.3.1. *Surds.* Recall that a *surd* is an algebraic number which is a root of a rational number. In particular, if $\alpha \in \overline{\mathbb{Q}}^\times$ is a surd, then $\alpha^n \in \mathbb{Q}^\times$ for some $n$. Therefore, an element $f \in \mathcal{F}$ is represented by a surd if and only if $f \in V_\mathbb{Q}$, or equivalently $\delta(f) = 1$. As $\mathbb{Q}$ has no proper subfields, all surds are trivially projection irreducible. Thus, for a surd $f$,

$$\|f\|_{m,p} = \delta(f)\|f\|_p = \|f\|_p = h_p(f).$$

4.3.2. *Pisot and Salem numbers.* We say that $f_\tau \in \mathcal{F}$ is Pisot or Salem number if it has a representative $\tau \in \overline{\mathbb{Q}}^\times$ which is a Pisot or Salem number, respectively. Recall that $\tau > 1$ is said to be a Pisot number if $\tau$ is an algebraic integer whose conjugates in the complex plane all lie strictly within the unit circle, and that $\tau > 1$ is a Salem number if $\tau$ is algebraic unit which is reciprocal and has all conjugates except $\tau$ and $\tau^{-1}$ on the unit circle in the complex plane (with at least one pair of conjugates on the circle).

**Proposition 4.7.** *Every Pisot or Salem number $f_\tau$ is representable, that is, $f_\tau \in \mathcal{R}$.*

*Proof.* This follows from [6, Prop. 3.10]. □

**Proposition 4.8.** *Every Salem number $\tau$ is projection irreducible and an algebraic unit, and therefore $f_\tau \in \mathcal{R} \cap \mathcal{U} \cap \mathcal{P}$.*

*Proof.* That $\tau$ is a unit is well-known and follows immediately from being a reciprocal algebraic integer. Suppose $f_\tau$ has its distinguished representative $\tau \in K^\times$, where $K = K_f = \mathbb{Q}(\tau)$. Then there are precisely two real places of $K$, call them $v_1, v_2 | \infty$, where $\tau$ has nontrivial valuation, and the remaining archimedean places are complex. By the definition of projection irreducibility, we need to show that $P_F(f_\tau) = 0$ for all $F \subsetneqq K$. Now, since $\lambda(Y(K, v_1)) = \lambda(Y(K, v_2)) = 1/[K : \mathbb{Q}]$, we know that for our subfield $F \subsetneqq K$, either $Y(K, v_1) \cup Y(K, v_2) \subseteq Y(F, w)$ for some place $w$ of $F$, in which case $P_F(f_\tau) = 0$ because the two valuations sum to zero by the product formula, or else $v_1$ and $v_2$ lie over distinct places of $F$, call them $w_1$ and $w_2$. Then the algebraic norm $\beta = \mathrm{N}^K_F\, \tau$ has nontrivial valuations at precisely the two archimedean places $w_1, w_2$. Observe that $w_1, w_2$ must be real, as the completions are $\mathbb{Q}_\infty = \mathbb{R} \subset F_{w_i} \subset K_{v_i} = \mathbb{R}$ for $i = 1, 2$. Thus $\beta$ must be a nontrivial Salem number or a quadratic unit. In either case, if we assume without loss of generality that $\log \|\beta\|_{w_1} > 0$, observe that

$$\beta = \|\beta\|_{w_1}$$

But it is easy to see that

$$\log \|\beta\|_{w_1} = \frac{1}{[K : F]} \log \|\tau\|_{v_1}$$

and thus $\beta^{[K:F]} = \tau$. But this is a contradiction, as then the minimal field of $f_\beta$ must also be $K$, but $\beta \in F \subsetneqq K$. That it is also in $\mathcal{R} \cap \mathcal{U}$ follows from the preceding proposition. □

**Corollary 4.9.** *Every Salem number $\tau > 1$ gives rise to an eigenvector $f_\tau$ of the $M$ operator with eigenvalue $\delta(f_\tau) = [\mathbb{Q}(\tau) : \mathbb{Q}]$.*

*Proof.* This now follows from the above results and Proposition 4.5. $\square$

Thus, if $\tau > 1$ is a Salem number, we have $f_\tau \in \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}$, so we can compute explicitly:

$$\|f_\tau\|_{m,p} = \delta(f_\tau)\|f_\tau\|_p = \delta(f_\tau)^{1-1/p} 2^{1/p} |\log \tau|. \tag{4.8}$$

When $p = 1$ this is, of course, twice the classical logarithmic Mahler measure of $\tau$, and when $p = \infty$, this is precisely the degree times the logarithmic house of $\tau$.

4.4. **The group $\Gamma$ and proof of Theorem 5.** We now construct an additive subgroup $\Gamma \leq \langle \mathcal{R} \cap \mathcal{P} \cap \mathcal{U} \rangle$ which is bounded away from 0 if and only if the $L^p$ Lehmer conjecture is true and thus establish Theorem 5. For $K \in \mathcal{K}^G$, let $W_K = T_K(\mathcal{U}) \cap \mathcal{P} \cap \mathcal{R}$. Notice first $W_K$ is not empty if $T_K(\mathcal{U})$ is not empty, as any element $f \in T_K(\mathcal{U})$ can be projected to a minimal element of its $\mathcal{K}$-support $\{F \in \mathcal{K} : P_F(f) \neq 0\}$, and that by construction such a projected element $P_F(f)$ will be an element of $T_K(\mathcal{U}) \cap \mathcal{P}$, and since $T_K(\mathcal{U})$ and $\mathcal{P}$ are both closed under scaling, we can ensure such an element is representable. (We remark in passing that we may in fact have $T_K(\mathcal{U}) = \{0\}$, for example, when $K = \mathbb{Q}(i)$ where $i^2 = -1$; see Remark 2.9.) By our $L^p$ Northcott analogue Theorem 8, we see that the set $\{f \in W_K : \|f\|_p \leq C\}$ is finite (notice that $f \in W_K \implies \delta(f) = [K_f : \mathbb{Q}] \leq [K : \mathbb{Q}]$) for any $C > 0$. As $W_K \subset \mathcal{P}$ we have $\|f\|_{m,p} = \delta(f) \cdot \|f\|_p$ (see Proposition 4.5 above), so we may choose an element $f_K \in W_K$ of minimal Mahler $p$-norm for each $K \in \mathcal{K}^G$, letting $f_K = 0$ if $T_K(\mathcal{U}) = \{0\}$. Notice that

$$m_p(f_K) = \|f_K\|_{m,p}$$

by construction (this follows from the usual argument following Proposition 4.5 and using $\mathcal{R} = \{d = \delta\}$). We let $\Gamma = \Gamma_p$ be the additive subgroup generated by these elements (notice that our choices may depend on $p$):

$$\Gamma = \langle \{f_K : K \in \mathcal{K}^G\} \rangle \leq \mathcal{R} \cap \mathcal{P} \cap \mathcal{U}. \tag{4.9}$$

Notice that $\Gamma$ is, by construction, clearly a free group, as by Theorem 1, p. 3, we have the direct sum $\Gamma = \bigoplus_{K \in \mathcal{K}^G} \mathbb{Z} \cdot f_K$.

Let $\mathcal{U}_{m,p}$ denote the completion of $\mathcal{U}$ with respect to the Mahler $p$-norm $\|\cdot\|_{m,p}$. Our goal is now to prove Theorem 5, which we recall here:

**Theorem 5.** *Equation $(*_p)$ holds if and only if the additive subgroup $\Gamma \subset \mathcal{U}_{m,p}$ is closed.*

We begin by proving a basic result about additive subgroups of Banach spaces, following the remarks and proofs in [2, Remark 5.6] and [10, Theorem 2 et seq.]. (We only need the second part of this lemma for our theorem, however, we prove both directions for their own interest.)

**Lemma 4.10.** *Let $\Lambda$ be a countable additive subgroup of a Banach space $\mathcal{B}$. If $\Lambda$ is discrete, then it is closed and free abelian. If $\Lambda$ is closed, then it is discrete.*

*Proof.* We restrict our attention to real Banach spaces, as this is the case that interests us, but note that the result continues to be true in the complex setting under suitable assumptions (see the discussion in [10]).

We will first show that if $\Lambda \subset \mathcal{B}$ is countable and discrete then it is also closed and free. That it is closed is trivial, so let us show that it is free by exhibiting a basis as a $\mathbb{Z}$-module. Let $\{v_i\}_{i=1}^{\infty}$ be an enumeration of the non-zero elements of $\Lambda$. Choose $b_1 = tv_1$ where $t > 0$ is the smallest number such that $tv_1 \in \Lambda$; clearly such a choice exists, else $\Lambda$ would not be discrete. Let $B_1 = \{b_1\}$ and let $X_1 = \operatorname{span}_{\mathbb{R}} B_1$. Then $B_1$ is a basis for $\Lambda \cap X_1$. Suppose now we have chosen basis vectors $B_n = \{b_1, \ldots, b_n\}$ such that $B_n$ is a basis for $\Lambda \cap X_n$ where $X_n = \operatorname{span}_{\mathbb{R}} B_n$. If $\Lambda \subset X_n$, then $\Lambda$ has finite rank and we are done, so suppose $\Lambda \not\subset X_n$. Let $v = v_k$ be the first element of the enumeration $\{v_i\}$ which is not in $\Lambda \cap X_n$, so that $v_i \in \Lambda \cap X_n$ for all $i < k$. Let $X_{n+1} = \operatorname{span}_{\mathbb{R}}(B_n \cup \{v\})$. Observe that the set

$$T = \{t \in \mathbb{R} : tv \in X_n + \Lambda\}$$

is an additive subgroup of $\mathbb{R}$, and further, there must exist a minimal element $t_0 > 0$, as otherwise, we could find a sequence $t_n \to 0$ such that $0 < t_n < 1$, $x_n + t_n v \in \Lambda$, and $x_n = \sum_{i=1}^{n} r_i b_i \in X_n$ where $r_i \in [0, 1)$ for each $1 \leq i \leq n$ by adding appropriate elements of $\Lambda \cap X_n$ to $x_n$. But then

$$\|x_n + t_n v\| \leq \max_{r \in [0,1]^n} \left\| \sum_{i=1}^{n} r_i b_i \right\| + \|v\|$$

so the vectors $x_n + t_n v$ give an infinite subset of $\Lambda \cap X_{n+1}$ of bounded norm in the finite dimensional vector space $X_{n+1}$ (with the norm from $\mathcal{B}$) and this contradicts the fact that $\Lambda \cap X_{n+1}$ is discrete (which follows from the fact that $\Lambda$ is discrete). Thus, there must exist a minimal positive element $t_0 \in T$ such that $T = \mathbb{Z} t_0$. Let $b_{n+1} = x_0 + t_0 v$ where $x_0 = \sum_{i=1}^{n} r_i b_i \in X_n$ for some $r_i \in [0, 1)$. We claim that $B_{n+1} = \{b_1, \ldots, b_{n+1}\}$ is a basis for $\Lambda \cap X_{n+1}$ such that $v = v_k \in \Lambda \cap X_{n+1}$. To see this, observe that by our construction of the set $T$, every $\lambda \in \Lambda \cap X_{n+1}$ has the form $\lambda = x + m_{n+1} t_0 v$ for some $m_{n+1} \in \mathbb{Z}$. Then

$$\lambda - m_{n+1} b_{n+1} = x - m_{n+1} x_0 \in \Lambda \cap X_n \implies \lambda - m_{n+1} b_{n+1} = \sum_{i=1}^{n} m_i b_i \quad (m_i \in \mathbb{Z}).$$

But then $\lambda = \sum_{i=1}^{n+1} m_i b_i$ and so $B_{n+1}$ is indeed a basis for $\Lambda \cap X_{n+1}$. Now, either this process continues indefinitely and each nonzero element $v_k$ of $\Lambda$ is contained in some $B_n$, in which case $\bigcup_n B_n$ is a basis for $\Lambda$, or else $\Lambda \subset X_n$ for some $n$, in which case $\Lambda$ has a basis $B_n$. In either case, we have constructed a basis for $\Lambda$ as a $\mathbb{Z}$-module, and thus $\Lambda$ is free.

Now, let us show that if $\Lambda$ is countable and closed then it must be discrete. If $\Lambda$ were not discrete, then we could choose a sequence of vectors $v_n \to 0$ such that $\|v_{n+1}\| \leq \frac{1}{3}\|v_n\|$ for all $n \in \mathbb{N}$. To every subset $S \subset \mathbb{N}$ we associate the vector $v_S = \sum_{n \in S} v_n$. Notice that each $v_S$ is an absolutely convergent series, and belongs to $\Lambda$ since $\Lambda$ is closed. We claim that the elements $v_S$ are distinct for distinct subsets of $\mathbb{N}$. To see this, observe that for $S \neq T \subset \mathbb{N}$,

$$v_S - v_T = \sum_{n \in S \setminus T} v_n - \sum_{m \in T \setminus S} v_m = \sum_{n=1}^{\infty} \epsilon_n v_n$$

where $\epsilon_n \in \{-1, 0, +1\}$, and for at least one $n$ we have $\epsilon_n \neq 0$. Let $k$ be the first such number. Then if $v_S - v_T = 0$ we must have $-\epsilon_k v_k = \sum_{n=k+1}^{\infty} \epsilon_n v_n$, but

$$\left\| \sum_{n=k+1}^{\infty} \epsilon_n v_n \right\| \leq \left( \sum_{n=1}^{\infty} \frac{1}{3^n} \right) \|v_k\| = \frac{\|v_k\|}{2} < \|v_k\| = \|\epsilon_k v_k\|,$$

which is impossible. Thus each $v_S$ is uniquely associated to $S$, but this gives an uncountable number of elements of $\Lambda$, a contradiction. $\square$

We remark that countability is essential in the above lemma, as the uncountable subgroup $\{f : [0,1] \to \mathbb{Z} : \|f\|_\infty < \infty\} \subset L^\infty[0,1]$ is discrete and closed but not free.

We are now prepared to prove Theorem 5:

*Proof of Theorem 5.* By Proposition 3.11 and Theorem 4, we know that $(*_p)$ holds if and only if there exists a constant $c_p$ such that $m_p(f) \geq c_p > 0$ for all $f \in \mathcal{R} \cap \mathcal{U} \cap \mathcal{P}$. Given any $f \in \mathcal{R} \cap \mathcal{U} \cap \mathcal{P}$, let $A(f) = \{K \in \mathcal{K}^G : P_K(f) \neq 0\}$. $A(f)$ clearly contains a minimal element $K$ which satisfies $F \subsetneqq K, F \in \mathcal{K}^G \implies P_F(f) = 0$. Let $K$ be any such minimal element. Then observe that $P_K(f) = T_K(f)$, as

$$P_K(f) = \sum_{\substack{F \subseteq K \\ F \in \mathcal{K}^G}} T_F(f),$$

but $P_F(f) = 0 \implies T_F(f) = 0$ for all $F \subsetneqq K, F \in \mathcal{K}^G$. Observe that $m_p(P_K f) \leq m_p(f)$ by Proposition 2.11 and Lemma 3.8. But then, by construction of $\Gamma$, $\|f_K\|_{m,p} = m_p(f_K) \leq m_p(P_K f)$ since $P_K f = T_K f \in T_K(\mathcal{U}) \cap \cap \mathcal{P}$. Thus, if $\Gamma$ is discrete, we gain $(**_p)$ and by Theorem 4 we gain the $L^p$ Lehmer conjecture $(*_p)$.

Likewise, supposing $(**_p)$, we can repeat the same procedure as above given an arbitrary element $f \in \Gamma$ to obtain $P_K(f) = f_K$ for some minimal $K$ in $A(f)$. Now, by the fact that $P_K$ is a norm one projection with respect to the $L^p$ norm (Propositon 2.11), and the fact that it commutes with the $T^{(n)}$ operators (Proposition 2.23), we see that it commutes with the $M$ operator well, and therefore, by the definition of the Mahler norm,

$$\|P_K f\|_{m,p} = \|M P_K f\|_p = \|P_K(Mf)\|_p \leq \|Mf\|_p = \|f\|_{m,p}.$$

Since $f_K \in \mathcal{R} \cap \mathcal{U} \cap \mathcal{P}$, we see that $(**_p)$ implies that $\|P_K f\|_{m,p} \geq c_p > 0$ and thus $\Gamma$ is discrete.

Finally, observe that as a countable (free abelian) additive subgroup of the Banach space $\mathcal{U}_{m,p}$, by Lemma 4.10 above, $\Gamma$ is discrete if and only it is closed. $\square$

4.5. **The Mahler 2-norm and proof of Theorem 6.** Recall that we define the Mahler 2-norm for $f \in \mathcal{F}$ to be:

$$\|f\|_{m,2} = \|Tf\|_2 = \left\| \sum_{n=1}^{\infty} n\, T^{(n)} f \right\|_2.$$

The goal of this section is to prove Theorem 6, which we recall here for the convenience of the reader:

**Theorem 6.** *The Mahler* 2*-norm satisfies*

$$\|f\|_{m,2}^2 = \sum_{n=1}^{\infty} n^2 \, \|T^{(n)}(f)\|_2^2 = \sum_{K\in\mathcal{K}^G} \sum_{n=1}^{\infty} n^2 \, \|T_K^{(n)}(f)\|_2^2.$$

*Further, the Mahler* 2*-norm arises from the inner product*

$$\langle f,g\rangle_m = \langle Mf, Mg\rangle = \sum_{n=1}^{\infty} n^2 \, \langle T^{(n)}f, T^{(n)}g\rangle = \sum_{K\in\mathcal{K}^G} \sum_{n=1}^{\infty} n^2 \, \langle T_K^{(n)}f, T_K^{(n)}g\rangle$$

*where* $\langle f,g\rangle = \int_Y fg\,d\lambda$ *denotes the usual inner product in* $L^2(Y)$*, and therefore the completion* $\mathcal{F}_{m,2}$ *of* $\mathcal{F}$ *with respect to the Mahler* 2*-norm is a Hilbert space.*

*Proof of Theorem 6.* The first part of the theorem follows easily from the fact that the $T_K^{(n)}$ form an orthogonal decomposition of $\mathcal{F}$. Indeed, for $f\in\mathcal{F}$, we have:

$$\|f\|_{m,2}^2 = \left\| \sum_{K\in\mathcal{K}^G} \sum_{n=1}^{\infty} n\, T_K^{(n)}(f) \right\|_2^2 = \sum_{K\in\mathcal{K}^G} \sum_{n=1}^{\infty} n^2 \, \|T_K^{(n)}(f)\|_2^2.$$

The above sums are, of course, finite for each $f\in\mathcal{F}$. That the specified inner product $\langle f,g\rangle_m$ defines this norm is then likewise immediate. Therefore, the completion of $\mathcal{F}$ with respect to the norm $\|\cdot\|_{m,2}$ is a Hilbert space, as claimed. □

Lastly, we note that $\|\cdot\|_{m,2} \le \delta h_2 \le m_2$. The authors suspect that this inequality is not true for general $p\neq 2$, but we know of no examples proving such a result. To see that the desired inequality holds for $p=2$, let us recall (Proposition 2.26) that for a given $f\in\mathcal{F}$, we have an expansion into degree $n$ components given by $f = T^{(1)}f + \cdots + T^{(N)}f$ with $T^{(N)}f \neq 0$. Then observe that $\delta(f)\ge N$, for otherwise, $T^{(N)}f = 0$ since $f$ itself would have $[K_f:\mathbb{Q}] = n < N$ and thus $f\in V^{(n)}$, and so it would have no essential projection to $V^{(N)}$. Thus

$$\begin{aligned}\|f\|_{m,2} = \left(\sum_{n=1}^{N} n^2 \|T^{(n)}f\|_2^2\right)^{1/2} &\le \left(\sum_{n=1}^{N} N^2 \|T^{(n)}f\|_2^2\right)^{1/2} \\ &= N\|f\|_2 \le \delta(f)\|f\|_2 = \delta h_2(f).\end{aligned}$$

That $\delta h_2 \le m_2 = d\,h_2$ follows from the inequality $\delta \le d$.

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