An equivariant basis for the cohomology of Springer fibers

Martha Precup and Edward Richmond*

Washington University in St. Louis
Oklahoma State University*

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Fix $n \in \mathbb{Z}_+$ and define the **flag variety**

$$\mathcal{B} := \{V_\bullet = (V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n) \mid \dim(V_i) = i\}.$$

For any nilpotent matrix $X \in M_{n\times n}$, we define the **Springer fiber**:

$$\mathcal{B}_X := \{V_\bullet \in \mathcal{B} \mid XV_i \subseteq V_i \ \forall i\}.$$

Theorem: Springer (1976)

There is an action of the permutation group $W := S_n$ on the cohomology ring $H^*(\mathcal{B}_X)$. Moreover,

- $H^{\text{top}}(\mathcal{B}_X)$ is an irreducible representation of W.
- Every irreducible representation of appears as $H^{\text{top}}(\mathcal{B}_X)$ for some X.

Springer correspondence:

$$\{\text{Nilpotent orbits in } M_{n\times n}\} \Leftrightarrow \{\text{Irreducible } W\text{-representations}\}.$$

Question: Is there a nice combinatorial model for $H^*(\mathcal{B}_X)$?

Example:
If $X = [0]$, then $\mathcal{B}_X = \mathcal{B}$ (full flag variety) and we have Borel's presentation:

$$H^*(\mathcal{B}) \simeq \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle.$$

given by $c_1(V_i/V_{i-1}) \mapsto -x_i$.

Remarks:

- Monomial basis: $H^*(\mathcal{B}) \simeq \bigoplus_{w \in W} \mathbb{C} \cdot \mathbf{x}^{\mathrm{inv}(w)}$, where $\mathbf{x}^{\mathrm{inv}(w)} := \prod_{(i,j) \in \mathrm{inv}(w)} x_i$.
- The Springer action of W on $H^*(\mathcal{B})$ is permuting variables.

What about $H^*(\mathcal{B}_X)$ for other nilpotent elements?

Theorem: Spaltenstein (1976), Hotta-Springer (1977)

Let $i : \mathcal{B}_X \hookrightarrow \mathcal{B}$ denote inclusion. Then $i^* : H^*(\mathcal{B}) \to H^*(\mathcal{B}_X)$ is surjective.

Therefore

$$H^*(\mathcal{B}_X) \simeq \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n, \ker i^* \rangle.$$

Remarks:

- The ideal, $\langle \ker i^* \rangle$, is generated by elementary symmetric functions determined by the Jordan type of X (De Concini-Procesi (1981), Tanisaki (1982)).
- The Springer action of W on $H^*(\mathcal{B}_X)$ is permuting variables.
- Monomial basis??

Row-strict tableaux:

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ be a partition of n and let $\mathtt{RST}(\lambda)$ denote the set of row-strict tableaux of shape λ (decreasing across columns).

Example: $\lambda = (3, 2, 2)$ and

$$T = \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 4 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 7 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

We say (i, j) is a **Springer inversion** of $T \in \mathtt{RST}(\lambda)$ if there exists j' in row j such that $i < j'$ and either

1. j' appears above i and in the same column, or
2. j' appears in a column strictly to the right of the column containing i.

Let $\mathrm{inv}(T)$ denote the set of Springer inversions of T.

Example (con't):

$$\mathrm{inv}(T) = \{(2, 1), (4, 1), (4, 3), (5, 3)\}.$$

Theorem: De Concini-Procesi (1981), Garsia-Procesi (1991)

Let X be of Jordan type λ. Then the cohomology ring

$$H^*(\mathcal{B}_X) \simeq \bigoplus_{T \in \mathtt{RST}(\lambda)} \mathbb{C} \cdot \mathbf{x}^T,$$

where $\mathbf{x}^T := \prod_{(i,j) \in \mathrm{inv}(T)} x_i.$

Example: $\lambda = (2,2)$

T	$\begin{array}{\|c\|c\|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 1 \\ \hline 3 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3 & 2 \\ \hline 4 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array}$
$\mathbf{x}^T$	1	x_3	x_2	x_1	x_1x_3	x_1x_2

Problem: Given $F \in \mathbb{C}[x_1, \ldots, x_n]$, compute the coefficients given by

$$i^*(F) = \sum_{T \in \mathtt{RST}(\lambda)} c_T \, \mathbf{x}^T.$$

Solution: Use equivariant cohomology.

- Let T denote the standard torus acting on the flag variety $\mathcal{B}$.
- Let $S := Z(L)_0 \subseteq T$ (X is regular in the Levi L).
- The subtorus S acts on $\mathcal{B}_X$ and we have a surjective map

$$H_T^*(\mathcal{B}) \twoheadrightarrow H_S^*(\mathcal{B}_X).$$

Presenting S-equivariant cohomology: Let $\mathfrak{s} \times \mathfrak{t}$ denote the Lie algebra of $S \times T$. Consider the reduced closed subvariety of $\mathfrak{s} \times \mathfrak{t}$,

$$Z_\lambda := \{(h, wh) \mid h \in \mathfrak{s}, w \in W\}.$$

Theorem: Kumar-Procesi (2012), Abe-Horguchi (2016)

Let X be of Jordan type λ. Then the cohomology ring

$$H_S^*(\mathcal{B}_X) \simeq \mathbb{C}[Z_\lambda] = \mathbb{C}[y_1, \ldots, y_k; x_1, \ldots, x_n]/I(Z_\lambda).$$

- (Evaluation) $H^*(\mathcal{B}_X) \simeq \mathbb{C}[Z_\lambda]_{\mathbf{y}\equiv 0}$.
- (Localization) Any $F(\mathbf{y}, \mathbf{x}) \in \mathbb{C}[Z_\lambda]$ is uniquely determined by the values

$$\{F(\mathbf{y}, w\mathbf{y}) \mid w \in W\}.$$

Localization: Let $\lambda = (2,2)$

$$\mathfrak{s} = \mathrm{diag}(y_1, y_1, y_2, y_2) \subseteq \mathrm{diag}(x_1, x_2, x_3, x_4) = \mathfrak{t}.$$

The action of W on $\mathbf{y} = (y_1, y_1, y_2, y_2)$ gives

(y_1, y_2, y_1, y_2)	(y_1, y_2, y_2, y_1)	(y_1, y_1, y_2, y_2)
$\begin{array}{\|c\|c\|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 1 \\ \hline 3 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}$

(y_2, y_1, y_1, y_2)	(y_2, y_1, y_2, y_1)	(y_2, y_2, y_1, y_1)
$\begin{array}{\|c\|c\|}\hline 3 & 2 \\ \hline 4 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array}$

Identify

$$\{w\mathbf{y} \mid w \in W\} \Leftrightarrow \{\mathbf{y}_T \mid T \in \mathtt{RST}(\lambda)\}.$$

Equivariant Springer monomials: For any $T \in \mathtt{RST}(\lambda)$ define

$$P_T(\mathbf{y},\mathbf{x}) := \prod_{(i,j)\in \mathrm{inv}(T)} (x_i - y_j) \qquad \text{and} \qquad p_{T,T'}(\mathbf{y}) := P_T(\mathbf{y}, \mathbf{y}_{T'}).$$

Theorem: Precup-R. (2021)

The following are true:

- $A := [p_{T,T'}(\mathbf{y})]_{(T,T')\in \mathtt{RST}(\lambda)^2}$ is invertible (as a matrix over $Q(\mathbf{y})$).
- Let $F \in \mathbb{C}[x_1, \ldots, x_{n-1}]$ and write

$$i^*(F) = \sum_{T\in \mathtt{RST}(\lambda)} c_T \, \mathbf{x}^T.$$

Define vectors $\mathbf{f} := [F(\mathbf{y}_T)]_{T\in\mathtt{RST}(\lambda)}$ and $\mathbf{c} := [c_T]_{T\in\mathtt{RST}(\lambda)}$. Then

$$\mathbf{c} = (A^{-1}\cdot \mathbf{f})|_{\mathbf{y}\equiv 0}.$$

Monomials:

Let

$$\mathbf{x}^{\delta} = x_1^{\delta_1} \cdots x_{n-1}^{\delta_{n-1}} \qquad \text{and} \qquad \mathbf{x}^{\gamma} = x_1^{\gamma_1} \cdots x_{n-1}^{\gamma_{n-1}}.$$

We say $\mathbf{x}^{\delta} < \mathbf{x}^{\gamma}$ iff $\delta_i < \gamma_i$ where i denotes the smallest index where the compositions δ and γ differ.

Example: $x_1^2 x_2^3 x_3^2 < x_1^2 x_2^4$ since $\delta = (2,3,2)$ and $\gamma = (2,4,0)$.

Theorem: Precup-R. (2021)

Let $\mathbf{x}^{\delta} \in \mathbb{C}[x_1, \ldots, x_{n-1}]$. Then

$$i^*(\mathbf{x}^{\delta}) = \sum_{\substack{T \in \mathtt{RST}(\lambda), \\ \mathbf{x}^{\delta} \leq \mathbf{x}^{T}}} c_T \; \mathbf{x}^{T}$$

for some coefficients $c_T \in \mathbb{Z}$. In other words, if $\mathbf{x}^{T} < \mathbf{x}^{\delta}$, then $c_T = 0$.

Schubert polynomials: Let $\mathfrak{S}_w(\mathbf{x})$ denote the Schubert polynomial corresponding to the permutation $w \in W$.

Problem: Find a subset $W(\lambda) \subseteq W$ such that

$$\{i^*(\mathfrak{S}_w(\mathbf{x})) \mid w \in W(\lambda)\}$$

is a basis of $H^*(\mathcal{B}_\lambda)$.

Previously answered cases:

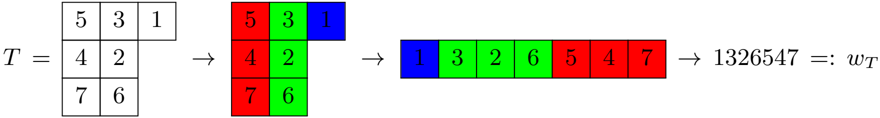
- $\lambda = (1, \dots, 1)$, then $W(\lambda) = W$.
- $\lambda = (n)$, then $W(\lambda) = \{e\}$.
- $\lambda = (n-1, 1)$ case due to Harada-Tymoczko (2017).
- $\lambda = (n-2, 2)$ case due to Dewitt-Harada (2012).

Methods use GKM theory and poset pinball.

Remark: $|W(\lambda)| = |\mathtt{RST}(\lambda)|$.

Permutation associated to $T \in \mathtt{RST}(\lambda)$:

Example: Let $\lambda = (3,2,2)$.



Define

$$W(\lambda) := \{w_T \mid T \in \mathtt{RST}(\lambda)\}.$$

Theorem: Precup-R. (2021)

The set

$$\{i^*(\mathfrak{S}_w(\mathbf{x})) \mid w \in W(\lambda)\}$$

is a basis of $H^*(\mathcal{B}_\lambda)$.