Coxeter groups and palindromic Poincaré polynomials

Edward Richmond
(joint work with W. Slofstra)

University of British Columbia

January 10, 2013

Let W be a Coxeter group with finite simple reflection set S.

By definition, W is the group generated by S where for any $s, t \in S$,

$$s^2 = e \qquad \text{and} \qquad (st)^{m_{st}} = e \text{ for some } m_{st} \in \{2, 3, \ldots, \infty\}.$$

Examples:

- The symmetric group $W = S_n$, with $S = \{s_1, \ldots, s_{n-1}\}$ and

$$(s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ where } |i - j| > 1.$$

- The crystallographic Coxeter groups where W is the Weyl group of a Lie group or Kac Moody group G. Here we have

$$m_{st} \in \{2, 3, 4, 6, \infty\}.$$

Length:

For any $w \in W$, the *length* $\ell(w)$ is the smallest number of simple reflections needed to express w.

Any expression

$$w = s_{i_1} \cdots s_{i_{\ell(w)}}$$

is called a *reduced word* of w.

Bruhat partial order:

For any $w, u \in W$, we say that $u \leq w$ if there exist reduced words

$$u = s_{j_1} \cdots s_{j_{\ell(u)}} \quad \text{and} \quad w = s_{i_1} \cdots s_{i_{\ell(w)}}$$

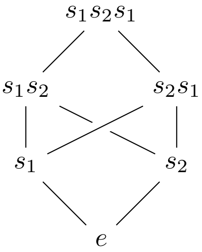
where $j_1, \ldots, j_{\ell(u)}$ is a subsequence of $i_1, \ldots, i_{\ell(w)}$.

The Poincaré polynomial:

For any $w \in W$, define

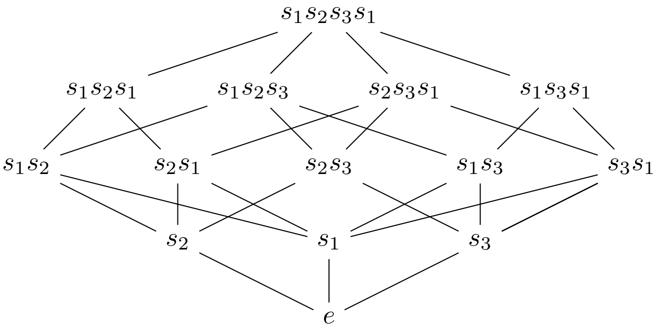
$$P_w(q) := \sum_{x \leq w} q^{\ell(x)}.$$

Example: $W = S_3$ and $w = s_1 s_2 s_1$.



$$P_w(q) = 1 + 2q + 2q^2 + q^3$$

Example: The group $W = \langle s_1, s_2, s_3 \mid s_i^2 = e \rangle$ and $w = s_1 s_2 s_3 s_1$.



$$P_w(q) = 1 + 3q + 5q^2 + 4q^3 + q^4$$

Question: When is $P_w(q)$ a palindromic polynomial?

Definition: A polynomial $\sum_{i=0}^{\ell} a_i q^i$ is *palindromic* if $a_i = a_{\ell-i} \quad \forall\, i$.

Example: $W = S_3$ and $w = s_1 s_2 s_1$.

$$P_w(q) = 1 + 2q + 2q^2 + q^3$$

In this case, $P_w(q)$ is palindromic!

Example: $W = \langle s_1, s_2, s_3 \mid s_i^2 = e \rangle$ and $w = s_1 s_2 s_3 s_1$.

$$P_w(q) = 1 + 3q + 5q^2 + 4q^3 + q^4$$

In this case, $P_w(q)$ is not palindromic!

Motivation from Algebraic Geometry:

When W is the Weyl group of some Kac-Moody group G (i.e cystallographic), then each $w \in W$ corresponds to a Schubert variety X_w in the flag variety G/B.

It is well known that

$$P_w(q^2) = \sum_{i \geq 0} \dim H^i(X_w, \mathbb{C})\; q^i.$$

Theorem: Carrell-Peterson '94

Let W be the Weyl group of some Kac-Moody group G.

Then X_w is rationally smooth if and only if $P_w(q)$ is palindromic.

Suppose G is simply laced of finite type. Then

X_w is smooth if and only if X_w is rationally smooth.

History of characterizing palindromic Poincaré polynomials:

- For W of a classical type (ABCD), $P_w(q)$ is palindromic if and only if w avoids a certain list of patterns (Lakshmibai-Sandhya '90, Billey '98).

- For W of finite Lie type, $P_w(q)$ is palindromic if and only if the inversion set of w avoids a certain list of root system patterns (Billey-Postnikov '05).

Weaker notion of palindromic:

Definition: A polynomial $\sum_{i=0}^{\ell} a_i q^i$ is *k-palindromic* if $a_i = a_{\ell-i} \quad \forall\, i < k$.

Example: $1 + q + 2q^2 + 3q^3 + q^4 + q^5$ is 2-palindromic, but not 3-palindromic.

Observation: Billey-Postnikov '05

For $W = S_n$,

$P_w(q)$ is $(n-1)$-palindromic if and only if $P_w(q)$ is palindromic.

Question: Is this a good criterion for detecting palindromic Poincaré polynomials for general Coxeter groups?

Theorem 1: Slofstra-R. '12

Let W be a Coxeter group with generating set S.

- Suppose that $m_{st} \neq 2 \quad \forall\, s, t \in S$. Then

$$P_w(q) \text{ is 4-palindromic if and only if } P_w(q) \text{ is palindromic.}$$

- Suppose that $m_{st} \neq 2, 3 \quad \forall\, s, t \in S$. Then

$$P_w(q) \text{ is 2-palindromic if and only if } P_w(q) \text{ is palindromic.}$$

Example: Let $W = \langle s_1, s_2, s_3, s_4 \mid s_i^2 = e \rangle$ and $w = s_1 s_2 s_1 s_3 s_1 s_3 s_4 s_3$.

We observe that $P_w(q) = \sum_{i=0}^{8} a_i q^i$ is palindromic since

$$a_0 = a_8 = 1 \text{ and } a_1 = a_7 = 4.$$

In particular, $P_w(q) = 1 + 4q + 9q^2 + 14q^3 + 16q^4 + 14q^5 + 9q^6 + 4q^7 + q^8$.

The theorem follows from a much stronger result where we can explicitly factor Poincaré polynomials given that they are 2-palindromic.

Enumeration results:

We can explicitly enumerate the number of palindromic elements in uniform Coxeter groups. For $m, n \in \mathbb{Z}_+$, let $W(m,n)$ denote the Coxeter group with $|S| = n$ and $m_{s,t} = m \quad \forall\, s, t \in S$.

Define the generating series

$$\Phi_m(q,t) := \sum_{n,k \geq 0} P_{n,k} \, \frac{q^k t^n}{n!}$$

where $P_{n,k}$ denotes the number of $w \in W(m,n)$ of length k with a palindromic Poincaré polynomial.

Corollary: Slofstra-R. '12

The generating series for the number of palindromic elements is

$$\Phi_m(q,t) = \frac{\exp(t)}{1-\phi_m(q,t)}$$

where

$$\phi_m(q,t) = \begin{cases} \dfrac{(2q-2q^3)t-(3q^3+q^5)t^2}{2-2q^2-4q^2t} & \text{for } m=3 \\[2ex] \dfrac{2qt-3q^m t^2 - q^{m+2}[m-3]_q t^3}{2-2q^2t([m-2]_q+q^{m-3})} & \text{for } 4 \le m < \infty \\[2ex] \dfrac{qt-q^2t}{1-q-q^2t} & \text{for } m=\infty. \end{cases}$$

Example:

$$\Phi_4(q,t) = 1 + (1+q)\,t + (1+2q+2q^2+2q^3+q^4)\,\frac{t^2}{2} + (1+3q+6q^2+12q^3+15q^4+12q^5+12q^6+6q^7)\,\frac{t^3}{6} + (1+4q+12q^2+36q^3+78q^4+120q^5 + 156q^6+168q^7+150q^8+120q^9+48q^{10})\,\frac{t^4}{24} + O(t^5).$$

Evaluating $\Phi_m(1,t)$ gives the following table on the total number of palindromic elements in $W(m,n)$.

$m \diagdown n$	1	2	3	4	5	6	7
4	2	8	67	893	15596	330082	8165963
5	2	10	115	2057	47356	1314292	42584795
6	2	12	175	3893	110436	3768982	150113447
7	2	14	247	6545	219956	8884312	418725119
8	2	16	331	10157	393916	18351562	997538291