A combinatorial characterization of tight fusion frames using Littlewood-Richardson coefficients

Edward Richmond
(joint work with M. Bownik and K. Luoto)

University of British Columbia

May 19, 2012

1. Tight fusion frames
2. Littlewood-Richardson coefficients
3. Consequences and applications
4. Proof

A question from functional analysis

Definition: A sequence of $N \times N$ orthogonal projection matrices $(P_1, \ldots, P_K)$ is called a *tight fusion frame* if

$$P_1 + \cdots + P_K = \alpha I_N$$

for some positive real number α.

Question: For which integer sequences $\mathbf{L} = (L_1, \ldots, L_K)$ does there exist a tight fusion frame $(P_1, \ldots, P_K)$ such that $\mathrm{rank}(P_i) = L_i$?

A question from functional analysis

Definition: A sequence of $N \times N$ orthogonal projection matrices $(P_1, \ldots, P_K)$ is called a *tight fusion frame* if

$$P_1 + \cdots + P_K = \alpha I_N$$

for some positive real number α.

Question: For which integer sequences $\mathbf{L} = (L_1, \ldots, L_K)$ does there exist a tight fusion frame $(P_1, \ldots, P_K)$ such that $\mathrm{rank}(P_i) = L_i$?

Motivation: Sensor networks

Fusion frames are used to model sensor networks.

Sensors with limited range are placed through an area.

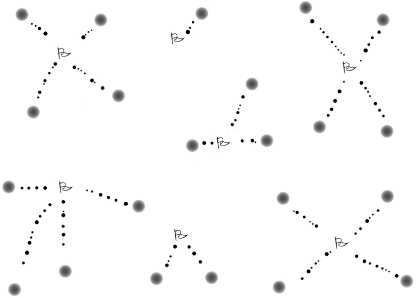
Local receivers are placed to record and package data (P_i).

A main processing center then combines the data. $(\alpha I_n = P_1 + \cdots + P_K)$

The eigenvalue α measures the "redundancy" in the system.

Motivation: Sensor networks

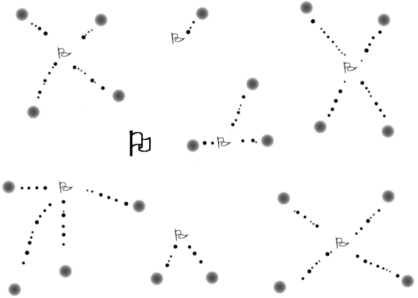
Fusion frames are used to model sensor networks.



Sensors with limited range are placed through an area.
Local receivers are placed to record and package data (P_i).
A main processing center then combines the data. $(\alpha I_n = P_1 + \cdots + P_K)$
The eigenvalue α measures the "redundancy" in the system.

Motivation: Sensor networks

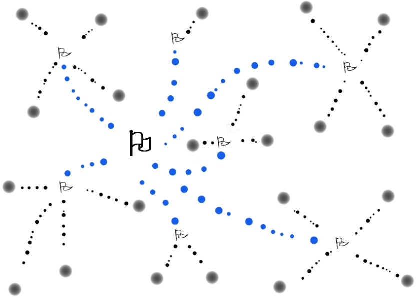
Fusion frames are used to model sensor networks.



Sensors with limited range are placed through an area.
Local receivers are placed to record and package data (P_i).
A main processing center then combines the data. $(\alpha I_n = P_1 + \cdots + P_K)$
The eigenvalue α measures the "redundancy" in the system.

Motivation: Sensor networks

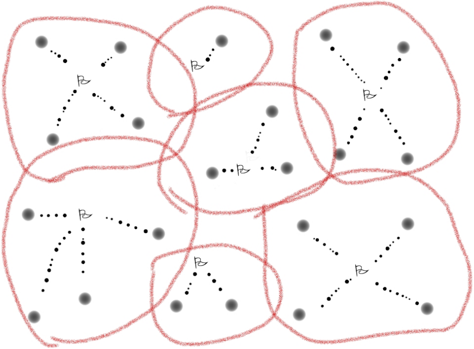
Fusion frames are used to model sensor networks.



Sensors with limited range are placed through an area.
Local receivers are placed to record and package data (P_i).
A main processing center then combines the data. $(\alpha I_n = P_1 + \cdots + P_K)$
The eigenvalue α measures the "redundancy" in the system.

Motivation: Sensor networks

Fusion frames are used to model sensor networks.



Sensors with limited range are placed through an area.
Local receivers are placed to record and package data (P_i).
A main processing center then combines the data. $(\alpha I_n = P_1 + \cdots + P_K)$
The eigenvalue α measures the "redundancy" in the system.

Examples

Ex1. $N = 3$ and $\mathbf{L} = (2, 2, 1, 1)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex2. $N = 3$ and $\mathbf{L} = (1, 1, 1, 1)$

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$+ \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Examples

Ex1. $N = 3$ and $\mathbf{L} = (2,2,1,1)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex2. $N = 3$ and $\mathbf{L} = (1,1,1,1)$

$$\begin{aligned} \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad &+ \quad \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \quad + \quad \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \\ &+ \quad \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad = \quad \frac{4}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Examples

Constructing tight fusion frames for the following sequences is not possible.

- $N = 5,\ \mathbf{L} = (2, 1, 1)$
- $N = 5,\ \mathbf{L} = (5, 2, 1, 1)$
- $N = 5,\ \mathbf{L} = (3, 3, 2, 1)$

Definition: Let $\mathrm{TFF}(N)$ denote the set of integer sequences for which N-dimensional tight fusion frames exist.

By the examples, we have

$$(2, 2, 1, 1), (1, 1, 1, 1) \in \mathrm{TFF}(3) \quad \text{and} \quad (5, 2, 1, 1) \notin \mathrm{TFF}(5).$$

Examples

Constructing tight fusion frames for the following sequences is not possible.

- $N = 5$, $\mathbf{L} = (2, 1, 1)$
- $N = 5$, $\mathbf{L} = (5, 2, 1, 1)$
- $N = 5$, $\mathbf{L} = (3, 3, 2, 1)$

Definition: Let $\mathrm{TFF}(N)$ denote the set of integer sequences for which N-dimensional tight fusion frames exist.

By the examples, we have

$$(2, 2, 1, 1), (1, 1, 1, 1) \in \mathrm{TFF}(3) \quad \text{and} \quad (5, 2, 1, 1) \notin \mathrm{TFF}(5).$$

A characterization in the uniform rank case

Suppose that $L_1 = \cdots = L_K$. We denote these sequences by (L^K).

Theorem: Casazza, Fickus, Mixon, Wang, Zhou (2010)

Suppose L divides N. Then

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad KL \geq N.$$

Otherwise, suppose $2L < N$. Then the following are true:

- If $(L^K) \in \mathrm{TFF}(N)$, then $K \geq \lceil \frac{N}{L} \rceil + 1$.
- If $K \geq \lceil \frac{N}{L} \rceil + 2$, then $(L^K) \in \mathrm{TFF}(N)$.

Moreover (Naimark and spatial duality),

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad (L^K) \in \mathrm{TFF}(LK - N)$$

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad ((N-L)^K) \in \mathrm{TFF}(N).$$

A characterization in the uniform rank case

Suppose that $L_1 = \cdots = L_K$. We denote these sequences by (L^K).

Theorem: Casazza, Fickus, Mixon, Wang, Zhou (2010)

Suppose L divides N. Then

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad KL \geq N.$$

Otherwise, suppose $2L < N$. Then the following are true:

- If $(L^K) \in \mathrm{TFF}(N)$, then $K \geq \lceil \frac{N}{L} \rceil + 1$.
- If $K \geq \lceil \frac{N}{L} \rceil + 2$, then $(L^K) \in \mathrm{TFF}(N)$.

Moreover (Naimark and spatial duality),

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad (L^K) \in \mathrm{TFF}(LK - N)$$

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad ((N-L)^K) \in \mathrm{TFF}(N).$$

A characterization in the uniform rank case

Suppose that $L_1 = \cdots = L_K$. We denote these sequences by (L^K).

Theorem: Casazza, Fickus, Mixon, Wang, Zhou (2010)

Suppose L divides N. Then

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad KL \geq N.$$

Otherwise, suppose $2L < N$. Then the following are true:

- If $(L^K) \in \mathrm{TFF}(N)$, then $K \geq \lceil \frac{N}{L} \rceil + 1$.
- If $K \geq \lceil \frac{N}{L} \rceil + 2$, then $(L^K) \in \mathrm{TFF}(N)$.

Moreover (Naimark and spatial duality),

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad (L^K) \in \mathrm{TFF}(LK - N)$$

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad ((N-L)^K) \in \mathrm{TFF}(N).$$

A characterization in the uniform rank case

Suppose that $L_1 = \cdots = L_K$. We denote these sequences by (L^K).

Theorem: Casazza, Fickus, Mixon, Wang, Zhou (2010)

Suppose L divides N. Then

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad KL \geq N.$$

Otherwise, suppose $2L < N$. Then the following are true:

- If $(L^K) \in \mathrm{TFF}(N)$, then $K \geq \lceil \frac{N}{L} \rceil + 1$.
- If $K \geq \lceil \frac{N}{L} \rceil + 2$, then $(L^K) \in \mathrm{TFF}(N)$.

Moreover (Naimark and spatial duality),

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad (L^K) \in \mathrm{TFF}(LK - N)$$

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad ((N-L)^K) \in \mathrm{TFF}(N).$$

A characterization in the uniform rank case

Suppose that $L_1 = \cdots = L_K$. We denote these sequences by (L^K).

Theorem: Casazza, Fickus, Mixon, Wang, Zhou (2010)

Suppose L divides N. Then

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad KL \geq N.$$

Otherwise, suppose $2L < N$. Then the following are true:

- If $(L^K) \in \mathrm{TFF}(N)$, then $K \geq \lceil \frac{N}{L} \rceil + 1$.
- If $K \geq \lceil \frac{N}{L} \rceil + 2$, then $(L^K) \in \mathrm{TFF}(N)$.

Moreover (Naimark and spatial duality),

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad (L^K) \in \mathrm{TFF}(LK - N)$$

$$(L^K) \in \mathrm{TFF}(N) \quad \text{if and only if} \quad ((N-L)^K) \in \mathrm{TFF}(N).$$

Representations of $GL_N(\mathbb{C})$

Question: Can we characterize TFF sequences in general?

Let $G = GL_N(\mathbb{C})$ and define the *Littlewood-Richardson coefficients* $c^{\mu}_{\lambda,\nu}$ as the tensor product multiplicities

$$V_\lambda \otimes V_\nu = \bigoplus_{\mu} c^{\mu}_{\lambda,\nu}\, V_\mu$$

where V_λ denotes the fd. irr. representation of G with highest weight λ.

In general, for any collection of weights $\lambda^1, \cdots, \lambda^K, \mu$, define the coefficients $c^{\mu}_{\lambda^1,\cdots,\lambda^K}$ by the tensor product

$$V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^K} = \bigoplus_{\mu} c^{\mu}_{\lambda^1,\cdots,\lambda^K}\, V_\mu.$$

Representations of $GL_N(\mathbb{C})$

Question: Can we characterize TFF sequences in general?

Let $G = GL_N(\mathbb{C})$ and define the *Littlewood-Richardson coefficients* $c^{\mu}_{\lambda,\nu}$ as the tensor product multiplicities

$$V_\lambda \otimes V_\nu = \bigoplus_{\mu} c^{\mu}_{\lambda,\nu} \; V_\mu$$

where V_λ denotes the fd. irr. representation of G with highest weight λ.

In general, for any collection of weights $\lambda^1, \cdots, \lambda^K, \mu$, define the coefficients $c^{\mu}_{\lambda^1,\cdots,\lambda^K}$ by the tensor product

$$V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^K} = \bigoplus_{\mu} c^{\mu}_{\lambda^1,\cdots,\lambda^K} \; V_\mu.$$

Representations of $GL_N(\mathbb{C})$

Question: Can we characterize TFF sequences in general?

Let $G = GL_N(\mathbb{C})$ and define the *Littlewood-Richardson coefficients* $c^{\mu}_{\lambda,\nu}$ as the tensor product multiplicities

$$V_\lambda \otimes V_\nu = \bigoplus_{\mu} c^{\mu}_{\lambda,\nu}\ V_\mu$$

where V_λ denotes the fd. irr. representation of G with highest weight λ.

In general, for any collection of weights $\lambda^1, \cdots, \lambda^K, \mu$, define the coefficients $c^{\mu}_{\lambda^1,\cdots,\lambda^K}$ by the tensor product

$$V_{\lambda^1} \otimes \cdots \otimes V_{\lambda^K} = \bigoplus_{\mu} c^{\mu}_{\lambda^1,\cdots,\lambda^K}\ V_\mu.$$

Statement of results for general sequences

Theorem: Bownik-Luoto-R

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let

$$M := \sum_{i=1}^{K} L_i.$$

The following are equivalent:

1. The sequence $\mathbf{L} \in \mathrm{TFF}(N)$.
2. The Littlewood-Richardson coefficient

$$c^{(M^N)}_{(N^{L_1}),\cdots,(N^{L_K})} \neq 0.$$

where (a^b) denotes the rectangular partition $(\underbrace{a,\ldots,a}_{b})$.

Statement of results for general sequences

Theorem: Bownik-Luoto-R

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let

$$M := \sum_{i=1}^{K} L_i.$$

The following are equivalent:

1. The sequence $\mathbf{L} \in \mathrm{TFF}(N)$.
2. The Littlewood-Richardson coefficient

$$c^{(M^N)}_{(N^{L_1}),\cdots,(N^{L_K})} \neq 0.$$

where (a^b) denotes the rectangular partition $(\underbrace{a, \ldots, a}_{b})$.

Statement of results for general sequences

Theorem: Bownik-Luoto-R

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let

$$M := \sum_{i=1}^{K} L_i.$$

The following are equivalent:

1. The sequence $\mathbf{L} \in \mathrm{TFF}(N)$.
2. The Littlewood-Richardson coefficient

$$c^{(M^N)}_{(N^{L_1}),\cdots,(N^{L_K})} \neq 0.$$

where (a^b) denotes the rectangular partition $(\underbrace{a, \ldots, a}_{b})$.

Applying the LR-rule for skew tableau

Consider $N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$.

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

Goal: Fill the rectangle below with skew diagrams according to the "rules".

Rules: Each rectangle gives a skew diagram λ/μ where μ is the partition consisting of the union of previous rectangles.

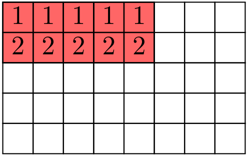
(LR-skew tableau) In each skew diagram we have:

- Content across rows is weakly increasing.
- Content down columns is strictly increasing.
- The content is a Yamanouchi word.

Applying the LR-rule for skew tableau

Consider $N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$.

Goal: Fill the rectangle below with skew diagrams according to the "rules".



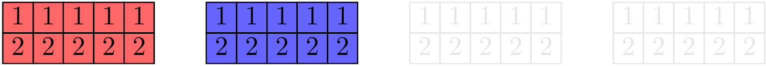
Rules: Each rectangle gives a skew diagram λ/μ where μ is the partition consisting of the union of previous rectangles.

(LR-skew tableau) In each skew diagram we have:

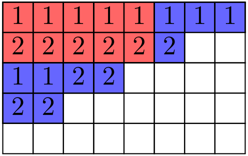
- Content across rows is weakly increasing.
- Content down columns is strictly increasing.
- The content is a Yamanouchi word.

Applying the LR-rule for skew tableau

Consider $N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$.



Goal: Fill the rectangle below with skew diagrams according to the “rules”.



Rules: Each rectangle gives a skew diagram λ/μ where μ is the partition consisting of the union of previous rectangles.

(LR-skew tableau) In each skew diagram we have:

- Content across rows is weakly increasing.
- Content down columns is strictly increasing.
- The content is a Yamanouchi word.

Applying the LR-rule for skew tableau

Consider $N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$.

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

Goal: Fill the rectangle below with skew diagrams according to the "rules".

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1					
2	2	2					

Rules: Each rectangle gives a skew diagram λ/μ where μ is the partition consisting of the union of previous rectangles.

(LR-skew tableau) In each skew diagram we have:

- Content across rows is weakly increasing.
- Content down columns is strictly increasing.
- The content is a Yamanouchi word.

Applying the LR-rule for skew tableau

Consider $N = 5$ and $\mathbf{L} = (2, 2, 2, 2)$.

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

1	1	1	1	1
2	2	2	2	2

Goal: Fill the rectangle below with skew diagrams according to the "rules".

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

Rules: Each rectangle gives a skew diagram λ/μ where μ is the partition consisting of the union of previous rectangles.

(LR-skew tableau) In each skew diagram we have:

- Content across rows is weakly increasing.
- Content down columns is strictly increasing.
- The content is a Yamanouchi word.

Applying the LR-rule for skew tableau

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

The existence of such tableaux implies that the Littlewood Richardson coefficient

$$c^{(8^5)}_{(5^2),(5^2),(5^2),(5^2)} \neq 0$$

and hence $(2,2,2,2) \in \mathrm{TFF}(5)$.

In fact, the number of such tableaux is equal to the corresponding LR coefficient.

Applying the LR-rule for skew tableau

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

The existence of such tableaux implies that the Littlewood Richardson coefficient

$$c^{(8^5)}_{(5^2),(5^2),(5^2),(5^2)} \neq 0$$

and hence $(2,2,2,2) \in \mathrm{TFF}(5)$.

In fact, the number of such tableaux is equal to the corresponding LR coefficient.

More examples

$N = 3$

$(2,2,1,1)$

1	1	1	1	1	1
2	2	2	1	1	1
2	2	2	1	1	1

$(2,1,1,1)$

1	1	1	1	1
2	2	2	1	1
1	1	1	1	1

$(1,1,1,1)$

1	1	1	1
1	1	1	1
1	1	1	1

$N = 5$

$(2,2,2,2)$

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

$(3,3,3,3)$

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	1	2	2	2
3	3	3	3	3	3	1	1	1	1	1	1
2	2	3	3	1	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3

More examples

$N = 3$

$(2, 2, 1, 1)$

1	1	1	1	1	1
2	2	2	1	1	1
2	2	2	1	1	1

$(2, 1, 1, 1)$

1	1	1	1	1
2	2	2	1	1
1	1	1	1	1

$(1, 1, 1, 1)$

1	1	1	1
1	1	1	1
1	1	1	1

$N = 5$

$(2, 2, 2, 2)$

1	1	1	1	1	1	1	1
2	2	2	2	2	2	1	1
1	1	2	2	1	1	2	2
2	2	1	1	1	1	1	1
2	2	2	2	2	2	2	2

$(3, 3, 3, 3)$

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	1	2	2	2
3	3	3	3	3	3	1	1	1	1	1	1
2	2	3	3	1	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3

More examples

$N = 7$

$(4, 3, 3, 1, 1)$

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	1	1	1
4	4	4	4	4	4	4	1	1	2	2	2
1	1	3	3	3	1	1	2	2	3	3	3
2	2	2	2	3	3	1	1	1	1	1	1
3	3	3	3	1	1	1	1	1	1	1	1

$(3, 2, 2, 2, 1)$

1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	1	1	1
1	1	1	1	1	1	1	2	2	2
2	2	2	2	2	1	1	1	1	1
1	2	2	1	1	2	2	2	2	2
2	2	2	1	1	1	1	1	1	1

Restrictions on tight fusion frames

Young tableaux can be used to prove non-existence of tight fusion frames as well.

Consider $N = 4$ and $(3, 1, 1, 1)$.

1	1	1	1	1	1
2	2	2	2	1	1
3	3	3	3		
1	1	1	1		

Last column cannot be completed with remaining partitions.

Consider $N = 5$ and $(3, 3, 2, 1)$.

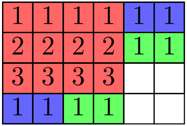
1	1	1	1	1				
2	2	2	2	2				
3	3	3	3	3	1	1	1	1
				2	2	2	2	2
				3	3	3	3	3

First two partitions are too large.

Restrictions on tight fusion frames

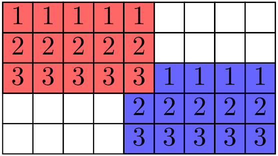
Young tableaux can be used to prove non-existence of tight fusion frames as well.

Consider $N = 4$ and $(3, 1, 1, 1)$.



Last column cannot be completed with remaining partitions.

Consider $N = 5$ and $(3, 3, 2, 1)$.



First two partitions are too large.

Restrictions on tight fusion frames

Theorem: Bownik-Luoto-R

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let $M := \sum_{i=1}^{K} L_i$.

If $\mathbf{L} \in \mathrm{TFF}(N)$ and $M < 2N$, then

1. $L_1 \leq M - N$
2. $L_1 + L_2 \leq N$
3. If $2M > 3N$, then $L_1 + L_2 + L_3 \leq 2(M - N)$
4. If $2M = 3N$, then $L_1 + L_2 + L_3 \leq 3N/2$
5. If $2M < 3N$, then $L_1 + L_2 + L_3 \leq N$

Conversely, if $L_4 = \cdots = L_K = 1$ and L_1, L_2, L_3 satisfy the conditions above, then $\mathbf{L} \in \mathrm{TFF}(N)$.

Restrictions on tight fusion frames

Theorem: Bownik-Luoto-R

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let $M := \sum_{i=1}^{K} L_i$.

If $\mathbf{L} \in \mathrm{TFF}(N)$ and $M < 2N$, then

1. $L_1 \leq M - N$
2. $L_1 + L_2 \leq N$
3. If $2M > 3N$, then $L_1 + L_2 + L_3 \leq 2(M - N)$
4. If $2M = 3N$, then $L_1 + L_2 + L_3 \leq 3N/2$
5. If $2M < 3N$, then $L_1 + L_2 + L_3 \leq N$

Conversely, if $L_4 = \cdots = L_K = 1$ and L_1, L_2, L_3 satisfy the conditions above, then $\mathbf{L} \in \mathrm{TFF}(N)$.

Dominance partial order on TFFs

Using Okada's theorem (1998) on multiplying rectangular Schur functions, one can show that any integers $a > b$, we have the following Schur positive inequality:

$$c^{\lambda}_{(N^a),\,(N^b)} \leq c^{\lambda}_{(N^{a-1}),\,(N^{b+1})}.$$

Suppose that $\mathbf{L} = (L_1 \geq L_2 \geq \cdots \geq L_K)$ and $\mathbf{L}' = (L'_1 \geq L'_2 \geq \cdots \geq L'_K)$. We say that $\mathbf{L}' \succeq \mathbf{L}$ if

$$\sum_{i=1}^{K} L_i = \sum_{i=1}^{K} L'_i \qquad \text{and} \qquad \sum_{i=1}^{k} L_i \leq \sum_{i=1}^{k} L'_i,$$

for all $k \leq K$.

Corollary: Bownik-Luoto-R

If $\mathbf{L}' \succeq \mathbf{L}$, then $\mathbf{L}' \in \mathrm{TFF}(N) \Rightarrow \mathbf{L} \in \mathrm{TFF}(N)$.

Dominance partial order on TFFs

Using Okada's theorem (1998) on multiplying rectangular Schur functions, one can show that any integers $a > b$, we have the following Schur positive inequality:

$$c^{\lambda}_{(N^a),\,(N^b)} \leq c^{\lambda}_{(N^{a-1}),\,(N^{b+1})}.$$

Suppose that $\mathbf{L} = (L_1 \geq L_2 \geq \cdots \geq L_K)$ and $\mathbf{L}' = (L'_1 \geq L'_2 \geq \cdots \geq L'_K)$.
We say that $\mathbf{L}' \succeq \mathbf{L}$ if

$$\sum_{i=1}^{K} L_i = \sum_{i=1}^{K} L'_i \qquad \text{and} \qquad \sum_{i=1}^{k} L_i \leq \sum_{i=1}^{k} L'_i,$$

for all $k \leq K$.

Corollary: Bownik-Luoto-R

If $\mathbf{L}' \succeq \mathbf{L}$, then $\mathbf{L}' \in \mathrm{TFF}(N) \Rightarrow \mathbf{L} \in \mathrm{TFF}(N)$.

Dominance partial order on TFFs

Using Okada's theorem (1998) on multiplying rectangular Schur functions, one can show that any integers $a > b$, we have the following Schur positive inequality:

$$c^{\lambda}_{(N^a),\,(N^b)} \leq c^{\lambda}_{(N^{a-1}),\,(N^{b+1})}.$$

Suppose that $\mathbf{L} = (L_1 \geq L_2 \geq \cdots \geq L_K)$ and $\mathbf{L}' = (L'_1 \geq L'_2 \geq \cdots \geq L'_K)$. We say that $\mathbf{L}' \succeq \mathbf{L}$ if

$$\sum_{i=1}^{K} L_i = \sum_{i=1}^{K} L'_i \qquad \text{and} \qquad \sum_{i=1}^{k} L_i \leq \sum_{i=1}^{k} L'_i,$$

for all $k \leq K$.

Corollary: Bownik-Luoto-R

If $\mathbf{L}' \succeq \mathbf{L}$, then $\mathbf{L}' \in \mathrm{TFF}(N) \Rightarrow \mathbf{L} \in \mathrm{TFF}(N)$.

Dominance partial order on TFFs

Consider $(4,3,3,1,1) \in \mathrm{TFF}(7)$.

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	1	1	1
4	4	4	4	4	4	4	1	1	2	2	2
1	1	3	3	3	1	1	2	2	3	3	3
2	2	2	2	3	3	1	1	1	1	1	1
3	3	3	3	1	1	1	1	1	1	1	1

By the dominance order we have that

(4,3,2,2,1), (4,3,2,1,1,1), (4,3,1,1,1,1,1), (4,2,2,2,2), (4,2,2,2,1,1), (4,2,2,1,1,1,1), (4,2,1,1,1,1,1,1), (4,1,1,1,1,1,1,1,1), (3,3,3,2,1), (3,3,3,1,1,1), (3,3,2,2,2), (3,3,2,2,1,1), (3,3,2,1,1,1,1), (3,3,1,1,1,1,1,1), (3,2,2,2,2,1), (3,2,2,2,1,1,1), (3,2,2,1,1,1,1,1), (3,2,1,1,1,1,1,1,1), (3,1,1,1,1,1,1,1,1,1), (2,2,2,2,2,1), (2,2,2,2,1,1,1), (2,2,2,1,1,1,1,1), (2,2,1,1,1,1,1,1,1), (2,1,1,1,1,1,1,1,1,1), (1,1,1,1,1,1,1,1,1,1,1)

are also elements of $\mathrm{TFF}(7)$.

Dominance partial order on TFFs

Consider $(4,3,3,1,1) \in \mathrm{TFF}(7)$.

1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	1	1	1
4	4	4	4	4	4	4	1	1	2	2	2
1	1	3	3	3	1	1	2	2	3	3	3
2	2	2	2	3	3	1	1	1	1	1	1
3	3	3	3	1	1	1	1	1	1	1	1

By the dominance order we have that

(4,3,2,2,1), (4,3,2,1,1,1), (4,3,1,1,1,1,1), (4,2,2,2,2), (4,2,2,2,1,1),
(4,2,2,1,1,1,1), (4,2,1,1,1,1,1,1), (4,1,1,1,1,1,1,1,1), (3,3,3,2,1), (3,3,3,1,1,1),
(3,3,2,2,2), (3,3,2,2,1,1), (3,3,2,1,1,1,1), (3,3,1,1,1,1,1,1), (3,2,2,2,2,1),
(3,2,2,2,1,1,1), (3,2,2,1,1,1,1,1), (3,2,1,1,1,1,1,1,1), (3,1,1,1,1,1,1,1,1,1),
(2,2,2,2,2,1), (2,2,2,2,1,1,1), (2,2,2,1,1,1,1,1), (2,2,1,1,1,1,1,1,1),
(2,1,1,1,1,1,1,1,1,1), (1,1,1,1,1,1,1,1,1,1,1)

are also elements of $\mathrm{TFF}(7)$.

Dominance partial order on TFFs

For classification, it suffices to find only the maximal elements under $\succeq$.

$N = 7$	
α	maximal elements
1	(7)
$8/7$	$(1, 1, 1, 1, 1, 1, 1, 1)$
$9/7$	$(2, 2, 2, 1, 1, 1)$
$10/7$	$(3, 3, 1, 1, 1, 1), (3, 2, 2, 2, 1)$
$11/7$	$(4, 3, 1, 1, 1, 1), (4, 2, 2, 2, 1)$
$12/7$	$(5, 2, 2, 1, 1, 1), (4, 3, 3, 1, 1), (3, 3, 3, 3)$
$13/7$	$(6, 1, 1, 1, 1, 1, 1, 1), (5, 2, 2, 2, 2), (4, 3, 3, 3)$
2	$(7, 7)$
$15/7$	$(7, 1, 1, 1, 1, 1, 1, 1), (6, 2, 2, 2, 2, 1), (5, 3, 3, 2, 2), (4, 4, 4, 3)$
$16/7$	$(7, 2, 2, 2, 1, 1, 1), (6, 3, 3, 3, 1), (5, 4, 4, 2, 1), (4, 4, 4, 4)$

Dominance partial order on TFFs

For classification, it suffices to find only the maximal elements under $\succeq$.

$N = 7$	
α	maximal elements
1	(7)
$8/7$	$(1,1,1,1,1,1,1,1)$
$9/7$	$(2,2,2,1,1,1)$
$10/7$	$(3,3,1,1,1,1),(3,2,2,2,1)$
$11/7$	$(4,3,1,1,1,1),(4,2,2,2,1)$
$12/7$	$(5,2,2,1,1,1),(4,3,3,1,1),(3,3,3,3)$
$13/7$	$(6,1,1,1,1,1,1,1),(5,2,2,2,2),(4,3,3,3)$
2	$(7,7)$
$15/7$	$(7,1,1,1,1,1,1,1),(6,2,2,2,2,1),(5,3,3,2,2),(4,4,4,3)$
$16/7$	$(7,2,2,2,1,1,1),(6,3,3,3,1),(5,4,4,2,1),(4,4,4,4)$

Identities between LR coefficients

In analysis, there are natural dualities between fusion frames. Let $\mathbf{L} = (L_1, \cdots, L_K)$ and let $M := \sum_{i=1}^{K} L_i$.

Spatial duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow (N - L_1, \cdots, N - L_K) \in \mathrm{TFF}(N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (5, 5, 3, 3, 2) \in \mathrm{TFF}(6).$$

Naimark duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow \mathbf{L} \in \mathrm{TFF}(M - N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (4, 2, 2, 1, 1) \in \mathrm{TFF}(4).$$

Identities between LR coefficients

In analysis, there are natural dualities between fusion frames. Let $\mathbf{L} = (L_1, \cdots, L_K)$ and let $M := \sum_{i=1}^{K} L_i$.

Spatial duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow (N - L_1, \cdots, N - L_K) \in \mathrm{TFF}(N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (5, 5, 3, 3, 2) \in \mathrm{TFF}(6).$$

Naimark duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow \mathbf{L} \in \mathrm{TFF}(M - N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (4, 2, 2, 1, 1) \in \mathrm{TFF}(4).$$

Identities between LR coefficients

In analysis, there are natural dualities between fusion frames. Let $\mathbf{L} = (L_1, \cdots, L_K)$ and let $M := \sum_{i=1}^{K} L_i$.

Spatial duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow (N - L_1, \cdots, N - L_K) \in \mathrm{TFF}(N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (5, 5, 3, 3, 2) \in \mathrm{TFF}(6).$$

Naimark duality

$$\mathbf{L} \in \mathrm{TFF}(N) \Leftrightarrow \mathbf{L} \in \mathrm{TFF}(M - N)$$

Ex.

$$(4, 2, 2, 1, 1) \in \mathrm{TFF}(6) \Leftrightarrow (4, 2, 2, 1, 1) \in \mathrm{TFF}(4).$$

Identities between LR coefficients

Corollary: Combinatorial spatial and Naimark dualities

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let $M := \sum_{i=1}^{K} L_i$.

Then the LR coefficients satisfy:

$$c^{(M^N)}_{(N^{L_1}),\dots,(N^{L_K})} \neq 0 \Leftrightarrow c^{((KN-M)^N)}_{(N^{N-L_1}),\dots,(N^{N-L_K})} \neq 0$$

and

$$c^{(M^N)}_{(N^{L_1}),\dots,(N^{L_K})} \neq 0 \Leftrightarrow c^{(M^{(M-N)})}_{((M-N)^{L_1}),\dots,((M-N)^{L_K})} \neq 0.$$

These numbers are equal.

Proof: Construct a bijection between “unions” of LR-tableaux.

Identities between LR coefficients

Theorem (BLR): Combinatorial spatial and Naimark dualities

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let $M := \sum_{i=1}^{K} L_i$.

Then the LR coefficients satisfy:

$$c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} = c^{((KN-M)^N)}_{(N^{N-L_1}),\ldots,(N^{N-L_K})}$$

and

$$c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} = c^{(M^{(M-N)})}_{((M-N)^{L_1}),\ldots,((M-N)^{L_K})}.$$

These numbers are equal.

Proof: Construct a bijection between “unions” of LR-tableaux.

Identities between LR coefficients

Theorem (BLR): Combinatorial spatial and Naimark dualities

Let $\mathbf{L} = (L_1 \geq \cdots \geq L_K)$ where $L_1 \leq N$ and let $M := \sum_{i=1}^{K} L_i$.

Then the LR coefficients satisfy:

$$c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} = c^{((KN-M)^N)}_{(N^{N-L_1}),\ldots,(N^{N-L_K})}$$

and

$$c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} = c^{(M^{(M-N)})}_{((M-N)^{L_1}),\ldots,((M-N)^{L_K})}.$$

These numbers are equal.

Proof: Construct a bijection between “unions” of LR-tableaux.

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.

1	1	1	1
2	2	2	2

1	1	1	1
2	2	2	2

1	1	1	1
2	2	2	2

1	1	1	1

1	1	1	1	1	1	1
2	2	2	2	2	1	1
1	2	2	1	1	2	2
2	2	2	1	1	1	1

$\longrightarrow$

1	1	1	1			
2	2	2	2			
1				1	1	1
2	2	2		2		
			1	1	1	1
	2	2			2	2
			1	1	1	1

$\rightsquigarrow$

				1	1	1
				2	2	2
	1	1	1			
			2		2	2
1	1	1				
2			2	2		
1	1	1				

$\longrightarrow$

1	1	1	1	1	1	1
2	1	1	2	2	2	2
1	1	1	2	2	2	2

Flipping the tableaux gives

1	1	1	1	1	1	1
2	2	2	2	1	1	2
2	2	2	2	1	1	1

This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

1	1	1
2	2	2

1	1	1
2	2	2

1	1	1
2	2	2

1	1	1

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.

1	1	1	1
2	2	2	2

1	1	1	1
2	2	2	2

1	1	1	1
2	2	2	2

1	1	1	1

1	1	1	1	1	1	1
2	2	2	2	2	1	1
1	2	2	1	1	2	2
2	2	2	1	1	1	1

$\longrightarrow$

1	1	1	1			
2	2	2	2			
1				1	1	1
2	2	2		2		
			1	1	1	1
	2	2			2	2
			1	1	1	1

$\rightsquigarrow$

				1	1	1
				2	2	2
	1	1	1			
			2		2	2
1	1	1				
2			2	2		
1	1	1				

$\longrightarrow$

1	1	1	1	1	1	1
2	1	1	2	2	2	2
1	1	1	2	2	2	2

Flipping the tableaux gives

1	1	1	1	1	1	1
2	2	2	2	1	1	2
2	2	2	2	1	1	1

This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

1	1	1
2	2	2

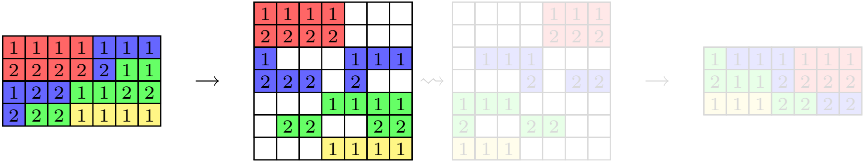
1	1	1
2	2	2

1	1	1
2	2	2

1	1	1

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.

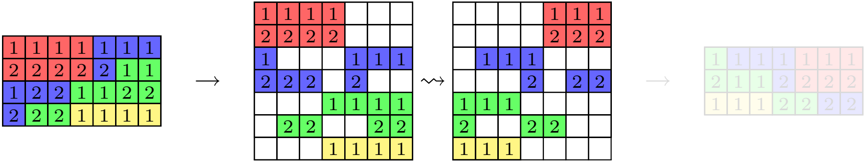


Flipping the tableaux gives

This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.

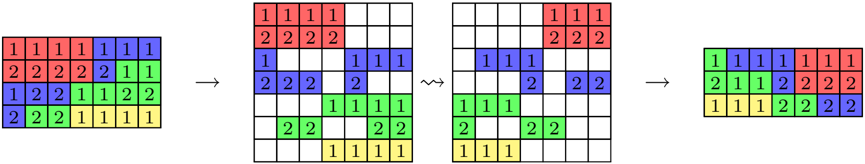


Flipping the tableaux gives

This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.

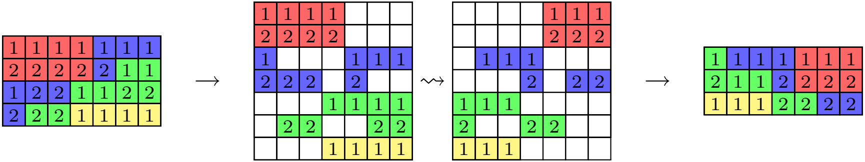


Flipping the tableaux gives

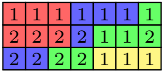
This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.



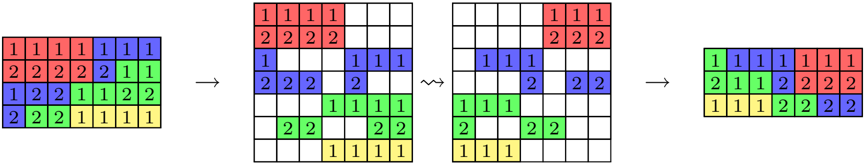
Flipping the tableaux gives



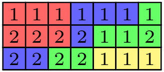
This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

Bijection for Naimark duality

Consider $N = 4, M = 7$ and $\mathbf{L} = (2, 2, 2, 1)$.



Flipping the tableaux gives



This is a tableaux for $N = 3$ and $\mathbf{L} = (2, 2, 2, 1)$.

Proof

Step1: Find a connection with representation theory.

Theorem (Mumford-Fogarty-Kirwan '94, Knutson '99, Kylachko '98):

Let $\lambda^1, \dots, \lambda^K$ be weakly decreasing sequences of integers. Then the following are equivalent:

1. There exist $N \times N$ hermitian matrices $B_1, \dots, B_K$ with spectra $\lambda^1, \dots, \lambda^K$ such that
$$\sum_{i=1}^{K} B_i = 0.$$
2. There exists an integer $p > 0$ such that the G-invariant subspace
$$(V(p\lambda^1) \otimes \cdots \otimes V(p\lambda^K))^G \neq 0.$$

The proof of this theorem requires techniques in symplectic geometry and geometric invariant theory.

Proof

Step1: Find a connection with representation theory.

Theorem (Mumford-Fogarty-Kirwan '94, Knutson '99, Kylachko '98):

Let $\lambda^1, \ldots, \lambda^K$ be weakly decreasing sequences of integers. Then the following are equivalent:

1. There exist $N \times N$ hermitian matrices $B_1, \ldots, B_K$ with spectra $\lambda^1, \ldots, \lambda^K$ such that
$$\sum_{i=1}^{K} B_i = 0.$$
2. There exists an integer $p > 0$ such that the G-invariant subspace
$$(V(p\lambda^1) \otimes \cdots \otimes V(p\lambda^K))^G \neq 0.$$

The proof of this theorem requires techniques in symplectic geometry and geometric invariant theory.

Proof

Step1: Find a connection with representation theory.

Theorem (Mumford-Fogarty-Kirwan '94, Knutson '99, Kylachko '98):

Let $\lambda^1, \ldots, \lambda^K$ be weakly decreasing sequences of integers. Then the following are equivalent:

1. There exist $N \times N$ hermitian matrices $B_1, \ldots, B_K$ with spectra $\lambda^1, \ldots, \lambda^K$ such that
$$\sum_{i=1}^{K} B_i = 0.$$
2. There exists an integer $p > 0$ such that the G-invariant subspace
$$(V(p\lambda^1) \otimes \cdots \otimes V(p\lambda^K))^G \neq 0.$$

The proof of this theorem requires techniques in symplectic geometry and geometric invariant theory.

Proof

Step 2: Apply to TFFs. Suppose $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$.

Then there exists orthogonal projections matrices $P_1, \ldots, P_K$ such that

$$P_1 + \cdots + P_K - \alpha I_N = 0.$$

Note that $\alpha = M/N$ is rational and thus the matrices in the sum

$$NP_1 + \cdots + NP_K - MI_N = 0.$$

have integral eigenvalues.

The previous theorem implies that there exists an integer $p > 0$ such that

$$(V(p(N)^{L_1}) \otimes \cdots \otimes V(p(N)^{L_K}) \otimes V(p(M^N))^*)^G \neq 0.$$

Hence $\quad \mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N) \Leftrightarrow c^{p(M^N)}_{p(N^{L_1}), \ldots, p(N^{L_K})} > 0.$

Proof

Step 2: Apply to TFFs. Suppose $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$.

Then there exists orthogonal projections matrices $P_1, \ldots, P_K$ such that

$$P_1 + \cdots + P_K - \alpha I_N = 0.$$

Note that $\alpha = M/N$ is rational and thus the matrices in the sum

$$NP_1 + \cdots + NP_K - MI_N = 0.$$

have integral eigenvalues.

The previous theorem implies that there exists an integer $p > 0$ such that

$$(V(p(N)^{L_1}) \otimes \cdots \otimes V(p(N)^{L_K}) \otimes V(p(M^N))^*)^G \neq 0.$$

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N) \Leftrightarrow c^{p(M^N)}_{p(N^{L_1}), \ldots, p(N^{L_K})} > 0.$

Proof

Step 2: Apply to TFFs. Suppose $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$.

Then there exists orthogonal projections matrices $P_1, \ldots, P_K$ such that

$$P_1 + \cdots + P_K - \alpha I_N = 0.$$

Note that $\alpha = M/N$ is rational and thus the matrices in the sum

$$NP_1 + \cdots + NP_K - MI_N = 0.$$

have integral eigenvalues.

The previous theorem implies that there exists an integer $p > 0$ such that

$$(V(p(N)^{L_1}) \otimes \cdots \otimes V(p(N)^{L_K}) \otimes V(p(M^N))^*)^G \neq 0.$$

Hence $\quad \mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N) \Leftrightarrow c^{p(M^N)}_{p(N^{L_1}), \ldots, p(N^{L_K})} > 0.$

Proof

Step 2: Apply to TFFs. Suppose $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$.

Then there exists orthogonal projections matrices $P_1, \ldots, P_K$ such that

$$P_1 + \cdots + P_K - \alpha I_N = 0.$$

Note that $\alpha = M/N$ is rational and thus the matrices in the sum

$$NP_1 + \cdots + NP_K - MI_N = 0.$$

have integral eigenvalues.

The previous theorem implies that there exists an integer $p > 0$ such that

$$(V(p(N)^{L_1}) \otimes \cdots \otimes V(p(N)^{L_K}) \otimes V(p(M^N))^*)^G \neq 0.$$

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N) \Leftrightarrow c^{p(M^N)}_{p(N^{L_1}), \ldots, p(N^{L_K})} > 0.$

Proof

Step 2: Apply to TFFs. Suppose $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$.

Then there exists orthogonal projections matrices $P_1, \ldots, P_K$ such that

$$P_1 + \cdots + P_K - \alpha I_N = 0.$$

Note that $\alpha = M/N$ is rational and thus the matrices in the sum

$$NP_1 + \cdots + NP_K - MI_N = 0.$$

have integral eigenvalues.

The previous theorem implies that there exists an integer $p > 0$ such that

$$(V(p(N)^{L_1}) \otimes \cdots \otimes V(p(N)^{L_K}) \otimes V(p(M^N))^*)^G \neq 0.$$

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N) \Leftrightarrow c^{p(M^N)}_{p(N^{L_1}),\ldots,p(N^{L_K})} > 0.$

Proof

Step 3: Apply saturation.

Saturation theorem (Knutson-Tao 1999):

For any integer $p > 0$, the Littlewood-Richardson coefficients satisfy

$$c^{\mu}_{\lambda^1,\ldots,\lambda^K} > 0 \Leftrightarrow c^{p\mu}_{p\lambda^1,\ldots,p\lambda^K} > 0.$$

The proof of saturation uses the honeycomb model.

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$ if and only if $c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} > 0$.

Proof

Step 3: Apply saturation.

Saturation theorem (Knutson-Tao 1999):

For any integer $p > 0$, the Littlewood-Richardson coefficients satisfy

$$c^{\mu}_{\lambda^1,\ldots,\lambda^K} > 0 \Leftrightarrow c^{p\mu}_{p\lambda^1,\ldots,p\lambda^K} > 0.$$

The proof of saturation uses the honeycomb model.

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$ if and only if $c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} > 0$.

Proof

Step 3: Apply saturation.

Saturation theorem (Knutson-Tao 1999):

For any integer $p > 0$, the Littlewood-Richardson coefficients satisfy

$$c^{\mu}_{\lambda^1,\ldots,\lambda^K} > 0 \Leftrightarrow c^{p\mu}_{p\lambda^1,\ldots,p\lambda^K} > 0.$$

The proof of saturation uses the honeycomb model.

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$ if and only if $c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} > 0.$

Proof

Step 3: Apply saturation.

Saturation theorem (Knutson-Tao 1999):

For any integer $p > 0$, the Littlewood-Richardson coefficients satisfy

$$c^{\mu}_{\lambda^1,\ldots,\lambda^K} > 0 \Leftrightarrow c^{p\mu}_{p\lambda^1,\ldots,p\lambda^K} > 0.$$

The proof of saturation uses the honeycomb model.

Hence $\mathbf{L} = (L_1, \cdots, L_K) \in \mathrm{TFF}(N)$ if and only if $c^{(M^N)}_{(N^{L_1}),\ldots,(N^{L_K})} > 0$.