Staircase diagrams and the enumeration of smooth Schubert varieties

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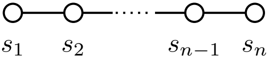
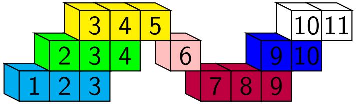
Let Γ be a Dynkin diagram of finite type with vertex set $S = \{s_1, \ldots, s_n\}$. Let $\mathcal{D}$ be a collection of subsets of S.

Type A Dynkin diagram:



Type A example:

$$\mathcal{D} = \{[s_1, s_3], [s_2, s_4], [s_3, s_5], [s_6], [s_7, s_9], [s_9, s_{10}], [s_{10}, s_{11}]\}$$



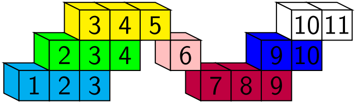
For any $s \in S$, define $\mathcal{D}_s := \{B \in \mathcal{D} \mid s \in B\}$.

$$\mathcal{D}_{s_3} = \{[s_1, s_3], [s_2, s_4], [s_3, s_5]\}$$

Definition: We say a partially ordered set $(\mathcal{D}, \prec)$ is a *staircase diagram* over Γ if

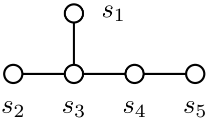
- Each $B \in \mathcal{D}$ is connected.
- If B covers B', then $B \cup B'$ is connected.
- For each $s \in S$, $\mathcal{D}_s$ is a saturated chain.
- If $s \,\mathrm{adj}\, t$, then $\mathcal{D}_s \cup \mathcal{D}_t$ is a chain.
- Each is $B \in \mathcal{D}$ is maximal (resp. minimal) in $\mathcal{D}_s$ for some $s \in S$.

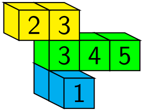
Type A example:



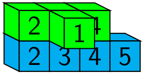
$$\{[s_1, s_3] \prec [s_2, s_4] \prec [s_3, s_5] \succ [s_6] \succ [s_7, s_9] \prec [s_9, s_{10}] \prec [s_{10}, s_{11}]\}$$

Type D examples:



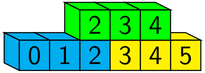


$\{[s_1, s_3] \prec [s_3, s_5] \prec [s_2, s_3]\}$



$\{[s_2, s_5] \prec ([s_2, s_4] \cup \{s_1\})\}$

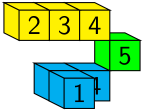
Non-example 1:



Violates:

- If $s \operatorname{adj} t$, then $\mathcal{D}_s \cup \mathcal{D}_t$ is a chain.
- Each is $B \in \mathcal{D}$ is maximal (resp. minimal) in $\mathcal{D}_s$ for some $s \in S$.

Non-example 2:



Violates:

- For each $s \in S$, $\mathcal{D}_s$ is a saturated chain.

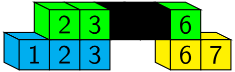
Let G be a finite-type Lie group with Weyl group (W, S) and Dynkin diagram Γ.

For any $J \subseteq S$, let u_J denote the maximal element in W_J.

Let $\mathcal{D}$ be a staircase diagram on Γ. For any $B \in \mathcal{D}$, define

$$J(B) := \{s \in B \mid B \neq \min \mathcal{D}_s\}$$

Example:



$$J([s_2, s_6]) = \{s_2, s_3, s_6\}$$

For each $B \in \mathcal{D}$, define

$$\lambda(B) := u_B u_{J(B)} \in W.$$

Remark: $\lambda(B)$ is the maximal element of $W_B \cap W^{J(B)}$

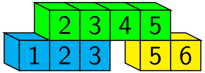
Remark: The map $\lambda : \mathcal{D} \to W$ is called the *maximal labelling* of $\mathcal{D}$.

Definition: Let $(B_1 < B_2 < \cdots < B_n)$ be a linear extension of $\mathcal{D}$. Define

$$\Lambda(\mathcal{D}) := \lambda(B_n) \cdot \lambda(B_{n-1}) \cdots \lambda(B_1).$$

If B, B' are incomparable, then they are disjoint and non-adjacent. Thus $\lambda(B), \lambda(B')$ commute and hence $\Lambda(\mathcal{D})$ is well defined.

Example: Let $\mathcal{D} = \{[s_1, s_3], [s_5, s_6], [s_2, s_5]\}$.



Then

$$\lambda([s_1, s_3]) = s_1 s_2 s_3 s_1 s_2 s_1, \quad \lambda([s_5, s_6]) = s_5 s_6 s_5,$$

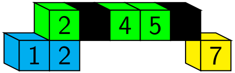
$$\lambda([s_2, s_5]) = (s_3 s_2 s_4 s_3 s_5 s_4 s_5 s_2 s_3 s_2)(s_2 s_3 s_2 s_5) = s_3 s_2 s_4 s_3 s_5 s_4$$

and

$$\Lambda(\mathcal{D}) = (s_3 s_2 s_4 s_3 s_5 s_4)(s_5 s_6 s_5)(s_1 s_2 s_3 s_1 s_2 s_1).$$

Define $D_R(\mathcal{D}) := \{s \in S \mid s \text{ is not a "lower inner corner" of } \mathcal{D}\}$.

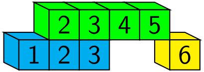
Ex:

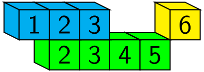


$$D_R(\mathcal{D}) = \{s_1, s_2, s_4, s_5, s_7\}.$$

Let $\text{flip}(\mathcal{D})$ denote the staircase diagram $\mathcal{D}$ with the reserve partial order.

Ex:





Coxeter group properties of $\Lambda(\mathcal{D})$: R-Slofstra (arXiv15)

- $\ell(\Lambda(\mathcal{D})) = \ell(\lambda(B_1)) + \cdots + \ell(\lambda(B_n))$
- $D_R(\mathcal{D})$ is the right-decent set of $\Lambda(\mathcal{D})$
- $\Lambda(\mathcal{D})^{-1} = \Lambda(\text{flip}(\mathcal{D}))$

Connection with geometry: Let G be a finite group with Weyl group W and let $X(w) \subseteq G/B$ denote the **Schubert variety** indexed by $w \in W$. Let Γ denote the Dynkin diagram of G.

Theorem: R-Slofstra (arXiv15)

If G is a simply-laced, then the map $\mathcal{D} \mapsto X(\Lambda(\mathcal{D}))$ defines a bijection:

$$\left\{ \text{ staircase diagrams over } \Gamma \right\} \Rightarrow \left\{ \text{ smooth Schubert varieties in } G/B \right\}$$

If $\lambda : \mathcal{D} \to W$ is a (rationally) smooth labelling, then define $\Lambda(\mathcal{D}, \lambda) \in W$ accordingly.

Theorem: R-Slofstra (arXiv15)

If G is of finite type, then the map $\mathcal{D} \mapsto X(\Lambda(\mathcal{D}, \lambda))$ defines a bijection:

$$\left\{ \begin{array}{c} \text{staircase diagrams over } \Gamma \\ \text{with (rationally) smooth labellings} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{(rationally) smooth Schubert} \\ \text{varieties in } G/B \end{array} \right\}$$

Theorem: Ryan (87), Wolper (89), R-Slofstra (arVix14)

- (Rationally) smooth Schubert varieties are iterated fiber bundles of (rationally) smooth "Grassmannian Schubert varieties".
- (Rationally) smooth Grassmannian Schubert varieties are classified.

Let $P \subseteq G$ and consider the fibration

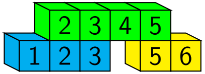
$$P/B \hookrightarrow G/B \twoheadrightarrow G/P.$$

The labelling map $\Lambda(\mathcal{D}) = \lambda(B_n) \cdot \Lambda(\mathcal{D} \setminus \{B_n\})$ corresponds to a fibration of Schubert varieties

$$X(\Lambda(\mathcal{D} \setminus \{B_n\})) \hookrightarrow X(\Lambda(\mathcal{D})) \twoheadrightarrow X^P(\lambda(B_n))$$

Where the parabolic P is defined by the support of $\mathcal{D} \setminus \{B_n\}$ in S.

Example:



The support of $\mathcal{D} \setminus \{B_3\}$ is $\{s_1, s_2, s_3\} \sqcup \{s_5, s_6\}$.

Application to enumeration: Define generating series

$$A(t) := \sum_{n=0}^{\infty} a_n\, t^n, \qquad B(t) := \sum_{n=0}^{\infty} b_n\, t^n, \qquad C(t) := \sum_{n=0}^{\infty} c_n\, t^n,$$

$$D(t) := \sum_{n=3}^{\infty} d_n\, t^n, \qquad BC(t) := \sum_{n=0}^{\infty} bc_n\, t^n,$$

where the coefficients a_n, b_n, c_n, d_n denote the number of smooth Schubert varieties of types A_n, B_n, C_n, D_n respectively, and bc_n denotes the number of rationally smooth Schubert varieties of type B_n or C_n.

Theorem: Haiman (90s), Bona (98), R-Slofstra (arXiv15)

Let $W(t) := \sum w_n\, t^n$ where $W = A$, B, C, D, or BC. Then

$$W(t) = \frac{P_W(t) + Q_W(t)\sqrt{1-4t}}{(1-t)^2(1-6t+8t^2-4t^3)}$$

for some polynomials $P_W(t)$ and $Q_W(t)$.

Theorem: Haiman (90s), Bona (98), R-Slofstra (arXiv15)

Type	$P_W(t)$	$Q_W(t)$
A	$(1-4t)(1-t)^3$	$t(1-t)^2$
B	$(1-5t+5t^2)(1-t)^3$	$(2t-t^2)(1-t)^3$
C	$1-7t+15t^2-11t^3-2t^4+5t^5$	$t-t^2-t^3+3t^4-t^5$
D	$(-4t+19t^2+8t^3-30t^4+16t^5)(1-t)^2$	$(4t-15t^2+11t^3-2t^5)(1-t)$
BC	$1-8t+23t^2-29t^3+14t^4$	$2t-6t^2+7t^3-2t^4$

	a_n	b_n	c_n	d_n	bc_n
$n=1$	2	2	2		2
$n=2$	6	7	7		8
$n=3$	22	28	28	22	34
$n=4$	88	116	114	108	142
$n=5$	366	490	472	490	596
$n=6$	1552	2094	1988	2164	2530
$n=7$	6652	9014	8480	9474	10842
$n=8$	28696	38988	36474	41374	46766

Asymptotics: The smallest singularity of $W(t)$ is the root

$$\alpha := \frac{1}{6}\left(4 - \sqrt[3]{17 + 3\sqrt{33}} + \sqrt[3]{-17 + 3\sqrt{33}}\right) \approx 0.228155$$

of the polynomial $1 - 6t + 8t^2 - 4t^3$ appearing in the denominator.

Corollary: R-Slofstra (arXiv15)

Let $W(t) = \sum w_n t^n$, where $W = A$, B, C, D, or BC. Then

$$w_n \sim \frac{W_\alpha}{\alpha^{n+1}},$$

where $W_\alpha := \lim_{t\to\alpha} (\alpha - t) W(t)$. In particular,

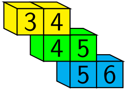
$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = \alpha^{-1} \approx 4.382985$$

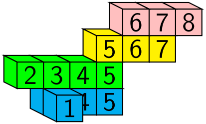
	A	B	C	D	BC
$W_\alpha \approx$	0.045352	0.062022	0.057301	0.067269	0.073972

Proof of enumeration: We say a staircase diagram $\mathcal{D}$ is elementary if:

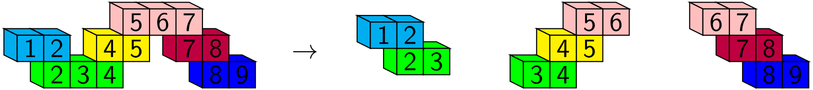
- The support of $\mathcal{D}$ is connected.
- If $|\mathcal{D}_s| = 1$, then s is a leaf of the support.

Examples:



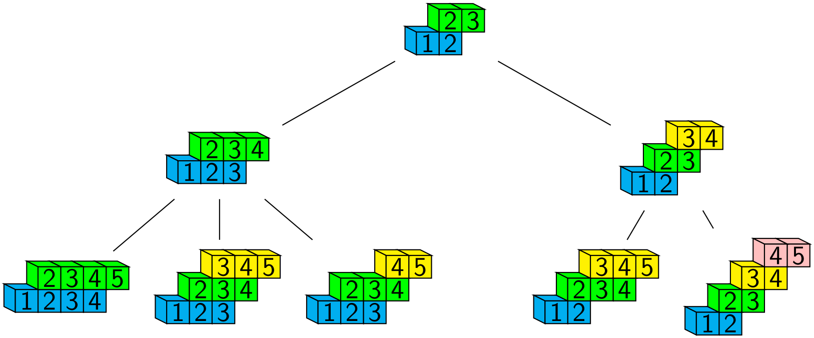


Step 1: Decompose a staircase diagram into elementary diagrams.



Step 2: Count elementary diagrams.

Key observation: Elementary diagrams "grow" recursively at the rate of Catalan numbers!



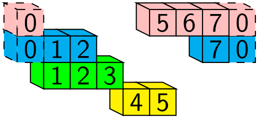
Step 3: Use the generating series for Catalan numbers. Let $\mathsf{c}_n := \frac{1}{n+1}\binom{2n}{n}$ and

$$\mathsf{Cat}(t) := \sum_{n=0}^{\infty} \mathsf{c}_n\, t^n = \frac{1-\sqrt{1-4t}}{2t}.$$

Further directions:

- Analogous enumerative results hold for affine type A (R-Slofstra, in progress).

  Example:



- What about other affine classical Lie types? Kac-Moody types?
- Find a generating series for the number of staircase diagrams over the Dynkin diagrams of E_n.

Thanks!