Littlewood Richardson coefficients for reflection groups

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Preliminaries: Schubert Calculus of G/B

- Let G be a Kac-Moody group over $\mathbb{C}$ (or a simple Lie group).

  Fix $T \subseteq B \subseteq G$ a maximal torus and Borel subgroup of G.

  Let $W := N(T)/T$ denote the Weyl group G.

- Let G/B be the flag variety (projective ind-variety).

- For any $w \in W$, we have the Schubert variety $X_w = \overline{BwB}/B \subseteq G/B$.

  Denote the cohomology class of X_w by

$$\sigma_w \in H^{2\ell(w)}(G/B).$$

Additively, we have that $H^*(G/B) \simeq \bigoplus_{w\in W} \mathbb{Z}\,\sigma_w$.

Goal (Schubert Calculus)

Compute the structure (Littlewood-Richardson) coefficients $c_{u,v}^w$ with respect to the Schubert basis defined by the product

$$\sigma_u \cdot \sigma_v = \sum_{w\in W} c_{u,v}^w \sigma_w.$$

- Note that if $\ell(w) \neq \ell(u) + \ell(v)$, then $c_{u,v}^w = 0$.
- For any $w, u, v \in W$, we have that $c_{u,v}^w \geq 0$. (proofs are geometric)

For example, if G is a finite Lie group, then the cardinality

$$|g_1 X_u \cap g_2 X_v \cap g_3 X_{w_0 w}| = c_{u,v}^w$$

for a generic choice of $g_1, g_2, g_3 \in G$.

Algebraic approach to Schubert calculus

Let $A = A(G)$ denote the Cartan matrix of G.

Alternatively, fix a finite index set I and let $A = \{a_{ij}\}$ be an $I \times I$ matrix such that

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$.

The matrix A defines an action of a Coxeter group W generated by reflections $\{s_i\}_{i \in I}$ on the vector space $V := \mathrm{Span}_{\mathbb{C}}\{\alpha_i\}_{i \in I}$ given by

$$s_i(v) := v - \langle v, \alpha_i^\vee \rangle \alpha_i$$

where $\langle \alpha_i, \alpha_j^\vee \rangle := a_{ij}$.

In particular, Coxeter groups of this type are crystallographic.

Algebraic approach to Schubert calculus

If we abandon the group G, we can consider a matrix A as follows:

Fix a finite index set I and let $A = \{a_{ij}\}$ be an $I \times I$ matrix such that

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{R}_{\leq 0}$ if $i \neq j$
- $a_{ij} = 0 \;\Leftrightarrow\; a_{ji} = 0.$

The matrix A defines an action of a Coxeter group W generated by reflections $\{s_i\}_{i \in I}$ on the vector space $V := \mathrm{Span}_{\mathbb{C}}\{\alpha_i\}_{i \in I}$ given by

$$s_i(v) := v - \langle v, \alpha_i^\vee \rangle \alpha_i$$

where $\langle \alpha_i, \alpha_j^\vee \rangle := a_{ij}$.

Every Coxeter group can be represented as above.

Some examples

- $G = SL(4)$ (type A_3)

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } W = S_4 \text{ (symmetric group)}$$

- $G = Sp(4)$ (type C_2)

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \text{ and } W = I_2(4) \text{ (dihedral group of 8 elements)}$$

- $G = \widehat{SL(2)}$ (affine type A)

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \text{ and } W = I_2(\infty) \text{ (free dihedral group)}$$

- Let $\rho = 2\cos(\pi/5)$

$$A = \begin{bmatrix} 2 & -\rho \\ -\rho & 2 \end{bmatrix} \text{ and } W = I_2(5) \text{ (dihedral group of 10 elements)}$$

Notation on sequences and subsets

- Any sequence $\mathbf{i} := (i_1, \ldots, i_m) \in I^m$ has a corresponding element

$$s_{i_1} \cdots s_{i_m} \in W.$$

If $s_{i_1} \cdots s_{i_m}$ is a reduced word of some $w \in W$, then we say that $\mathbf{i} \in R(w)$, the collection of reduced words.

- For each $\mathbf{i} \in I^m$ and subset $K = \{k_1 < k_2 < \cdots < k_n\}$ of the interval $[m] := \{1, 2, \ldots, m\}$ let the subsequence

$$\mathbf{i}_K := (i_{k_1}, \ldots, i_{k_n}) \in I^n.$$

We say a sequence $\mathbf{i}$ is *admissible* if $i_j \neq i_{j+1}$ for all $j \in [m-1]$.

- Observe that any reduced sequence is admissible. (In general, the converse is false.)

Definition

Let $m > 0$ and let K, L be subsets of $[m] := \{1, 2, \ldots, m\}$ such that $|K| + |L| = m$. We say that a bijection

$$\phi : K \to [m] \setminus L$$

is *bounded* if $\phi(k) < k$ for each $k \in K$.

Definition

Given a reduced sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, we say that a bounded bijection

$$\phi : K \to [m] \setminus L$$

is $\mathbf{i}$-*admissible* if the sequence $\mathbf{i}_L$ and the sequences $\mathbf{i}_{L(k)}$ are admissible for all $k \in K$ where

$$L(k) := L \cup \phi(K_{\leq k}).$$

Recall that the Littlewood-Richardson coefficients $c_{u,v}^w$ are defined by the product

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^w \ \sigma_w.$$

Theorem: Berenstein-R, 2010

Let $u, v, w \in W$ such that $\ell(w) = \ell(u) + \ell(v)$ and let $\mathbf{i} = (i_1, \ldots, i_m) \in R(w)$. Then

$$c_{u,v}^w = \sum p_\phi$$

where the summation is over all triples $(\hat{u}, \hat{v}, \phi)$, where

- $\hat{u}, \hat{v} \subset [m]$ such that $\mathbf{i}_{\hat{u}} \in R(u)$, $\mathbf{i}_{\hat{v}} \in R(v)$.
- $\phi : \hat{u} \cap \hat{v} \ \rightarrow \ [m] \setminus (\hat{u} \cup \hat{v})$ is an $\mathbf{i}$-admissible bounded bijection.

- The Theorem is still true without $\mathbf{i}$-admissible.
- The Theorem generalizes to structure coefficients of T-equivariant cohomology $H_T^*(G/B)$.

Definition

For any $k \in [m]$ and $\mathbf{i} = (i_1, \ldots, i_m)$, denote $\alpha_k := \alpha_{i_k}$ and $s_k := s_{i_k}$. For any bounded bijection

$$\phi : K \ \to [m] \setminus L$$

we define the monomial $p_\phi \in \mathbb{Z}$ by the formula

$$p_\phi := (-1)^{|K|} \prod_{k \in K} \langle w_k(\alpha_k), \alpha^\vee_{\phi(k)} \rangle \quad \text{where} \quad w_k := \overrightarrow{\prod_{\substack{r \in L(k) \\ \phi(k) < r < k}}} s_r$$

where the product $\overrightarrow{\prod}$ is taken in the natural order induced by the sequence $[m]$ and if the product is empty, we set $w_k = 1$. Also, if $K = \emptyset$, then $p_\phi = 1$.

Theorem: Berenstein-R, 2010

If the Cartan matrix $A = (a_{ij})$ satisfies

$$a_{ij} \cdot a_{ji} \geq 4 \qquad \forall \quad i \neq j,$$

then $p_\phi \geq 0$ when ϕ is an $\mathbf{i}$-admissible bounded bijection.

Examples where $G = \widehat{SL}(2)$, $\mathbf{i} = (1,2,1,2)$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Compute $c^w_{u,v}$ where $w = s_1 s_2 s_1 s_2, \quad u = v = s_1 s_2.$

- Find $\hat{u}, \hat{v} \subseteq [4] = \{1,2,3,4\}$

$$\color{blue}{1\ 2\ \underline{\overline{3}}\ \underline{\overline{4}}} \quad \underline{1}\ 2\ \overline{3}\ \underline{\overline{4}} \quad \underline{1}\ \underline{2}\ \overline{3}\ \overline{4} \quad \overline{1}\ 2\ \underline{3}\ \underline{\overline{4}} \quad \underline{\overline{1}}\ 2\ 3\ \underline{\overline{4}}$$

$$\underline{\overline{1}}\ \underline{2}\ 3\ \overline{4} \quad \overline{1}\ \overline{2}\ \underline{3}\ \underline{4} \quad \underline{\overline{1}}\ \overline{2}\ 3\ \underline{4} \quad \underline{\overline{1}}\ \underline{\overline{2}}\ 3\ 4$$

- For $\color{blue}{1\ 2\ \underline{\overline{3}}\ \underline{\overline{4}}}$, we have

$$\hat{u} \cap \hat{v} = \{3,4\} \quad \text{and} \quad [4] \setminus (\hat{u} \cup \hat{v}) = \{1,2\}$$

with bounded bijections

$$\color{red}{\phi_1 : (3,4) \to (1,2)} \qquad \phi_2 : (3,4) \to (2,1).$$

NOTE: ϕ_1 is not $\mathbf{i}$-admissible.

Examples where $G = \widehat{SL}(2)$, $\mathbf{i} = (1,2,1,2)$

- For $1\ 2\ \overline{\underline{3}}\ \overline{\underline{4}}$, we have bounded bijections

$$\phi_1 : (3,4) \to (1,2) \qquad \phi_2 : (3,4) \to (2,1).$$

$$p_{\phi_1} = \langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle \qquad p_{\phi_2} = \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle$$

- Totaling over all bounded bijections, we have

$$\begin{aligned} c_{u,v}^w &= \xcancel{\langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle} + \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle \\ &\quad - \xcancel{\langle s_3(\alpha_4), \alpha_2^\vee \rangle} - \xcancel{\langle s_3(\alpha_4), \alpha_2^\vee \rangle} + 1 + 1 \\ &= -\cancel{4} + 4 + \cancel{2} + \cancel{2} + 1 + 1 = 6 \end{aligned}$$

With only $\mathbf{i}$-admissible terms.

Examples where $G = \widehat{SL}(2)$, $\mathbf{i} = (1,2,1,2,1)$

Compute $c_{u,v}^w$ where $w = s_1 s_2 s_1 s_2 s_1, \quad u = s_1 s_2 s_1, \quad v = s_2 s_1.$

- Find $\hat{u}, \hat{v} \subseteq [5] = \{1,2,3,4,5\}$

$$1\ 2\ \overline{3}\ \overline{\underline{4}}\ \overline{\underline{5}} \quad \overline{1}\ 2\ 3\ \overline{\underline{4}}\ \overline{\underline{5}} \quad \overline{1}\ \overline{2}\ 3\ \underline{4}\ \overline{\underline{5}} \quad \overline{1}\ \overline{2}\ \overline{3}\ \underline{4}\ \underline{5} \quad 1\ \underline{2}\ \overline{3}\ \overline{4}\ \overline{\underline{5}} \quad \overline{1}\ \underline{2}\ 3\ \overline{4}\ \overline{\underline{5}}$$

$$\overline{1}\ \overline{\underline{2}}\ 3\ 4\ \overline{\underline{5}} \quad \overline{1}\ \overline{\underline{2}}\ \overline{3}\ 4\ \underline{5} \quad 1\ \underline{2}\ \overline{\underline{3}}\ \overline{4}\ \overline{5} \quad \overline{1}\ \underline{2}\ \underline{3}\ \overline{4}\ \overline{5} \quad \overline{1}\ \overline{\underline{2}}\ \underline{3}\ 4\ \overline{5} \quad \overline{1}\ \overline{\underline{2}}\ \overline{\underline{3}}\ 4\ 5$$

- Totaling over all bounded bijections, we have

$$\begin{aligned}
c_{u,v}^w &= \xcancel{\langle \alpha_4, \alpha_2^\vee \rangle \cdot \langle s_4(\alpha_5), \alpha_3^\vee \rangle} + \xcancel{\langle \alpha_4, \alpha_3^\vee \rangle \cdot \langle s_3 s_4(\alpha_5), \alpha_2^\vee \rangle} \\
&+ \xcancel{\langle s_3(\alpha_4), \alpha_1^\vee \rangle \cdot \langle s_3 s_4(\alpha_5), \alpha_2^\vee \rangle} + {\color{blue}\langle s_3(\alpha_4), \alpha_2^\vee \rangle \cdot \langle s_2 s_3 s_4(\alpha_5), \alpha_1^\vee \rangle} \\
&- {\color{blue}\langle s_2(\alpha_3), \alpha_1^\vee \rangle} - \xcancel{\langle s_4(\alpha_5), \alpha_3^\vee \rangle} - \xcancel{\langle s_4(\alpha_5), \alpha_3^\vee \rangle} - {\color{blue}\langle s_2 s_3 s_4(\alpha_5), \alpha_1^\vee \rangle} \\
&+ {\color{blue}1 + 1} \\
&= -\xcancel{4} + \xcancel{4} - \xcancel{4} {\color{blue}+ 4 + 2} + \xcancel{2} + \xcancel{2} {\color{blue}+ 2 + 1 + 1 = 10}
\end{aligned}$$

With only $\mathbf{i}$-admissible terms.

Examples where $G = \widehat{SL}(2)$, $\mathbf{i} = \underbrace{(2,1,\dots,2,1)}_{2n}$

Question: Bounded bijections vs. $\mathbf{i}$-admissible bounded bijections?

$$c(n) := c^{w}_{u,v}, \qquad w = \underbrace{s_2 s_1 \cdots s_2 s_1}_{2n}, \quad u = v = \underbrace{\cdots s_2 s_1}_{n}$$

	#{Bounded bijections}	#{$\mathbf{i}$-admissible bounded bijections}
$c(2)$	6	3
$c(3)$	20	7
$c(4)$	190	19
$c(5)$	1110	51
$c(6)$	14348	141
$c(n)$	??	largest coeff of $(1+x+x^2)^n$??

Remark: $c(n) = \binom{2n}{n}$

General rank 2 group G.

Let $a, b \neq 0$,

$$A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}.$$

Define "Chebyshev" sequences

$$A_k := aB_{k-1} - A_{k-2} \quad \text{and} \quad B_k := bA_{k-1} - B_{k-2}$$

where $A_0 = B_0 = 0$ and $A_1 = B_1 = 1$. Let

$$u_k = \underbrace{\cdots s_2 s_1}_{k} \quad \text{and} \quad v_k = \underbrace{\cdots s_1 s_2}_{k}.$$

Corollary: Binomial formula (Kitchloo, 2008)

The rank 2 Littlewood-Richardson coefficients

$$c_{u_k, u_{n-k}}^{u_n} = c_{v_{k+1}, u_{n-k}}^{v_{n+1}} = \frac{A_n \cdots A_2 A_1}{(A_k \cdots A_2 A_1)(A_{n-k} \cdots A_2 A_1)}$$

$$c_{v_k, v_{n-k}}^{v_n} = c_{u_{k+1}, v_{n-k}}^{u_{n+1}} = \frac{B_n \cdots B_2 B_1}{(B_k \cdots B_2 B_1)(B_{n-k} \cdots B_2 B_1)}$$

Examples where $G = SL(4)$, $\mathbf{i} = (3, 2, 1, 3, 2)$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Compute $c_{u,v}^w$ where

$$w = s_3 s_2 s_1 s_3 s_2, \quad u = s_3 s_1 s_2 = s_1 s_3 s_2, \quad v = s_3 s_1 = s_1 s_3.$$

- Totaling over all bounded bijections, we have

$$\begin{aligned} c_{u,v}^w &= \langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle + \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle \\ &\quad -\langle \alpha_3, \alpha_2^\vee \rangle - \langle \alpha_3, \alpha_2^\vee \rangle \\ &= 0 - 1 + 1 + 1 = 1 \end{aligned}$$

- In this case: {Bounded bijections}={$\mathbf{i}$-admissible bounded bijections}
- In general, the $\mathbf{i}$-admissible formula is not nonnegative.

Construction of Kostant and Kumar

Recall the the matrix A gives an action of the Coxeter group W on V and thus W acts on the algebras $S := S(V)$ and $Q := Q(V)$ (polynomials and rational functions).

Define

$$Q_W := Q \rtimes \mathbb{C}[W]$$

with product structure

$$(q_1 w_1)(q_2 w_2) := q_1 w_1(q_1) w_1 w_2$$

and a Q-linear coproduct

$$\Delta : Q_W \to Q_W \otimes_Q Q_W$$

by

$$\Delta(qw) := qw \otimes w = w \otimes qw.$$

For any $i \in I$, define

$$x_i := \frac{1}{\alpha_i}(s_i - 1).$$

If $\mathbf{i} = (i_1, \ldots, i_m) \in R(w)$, then define $x_w := x_{i_1} \cdots x_{i_m}$.

If A is a Cartan matrix of some Kac-Moody group G, then (by Kostant-Kumar 1986)

- x_w is independent of $\mathbf{i} \in R(w)$
- $x_i^2 = 0 \quad \forall\, i \in I$.

Define the Nil-Hecke ring $H_W := \bigoplus_{w \in W} S\, x_w \subseteq Q_W$.

We have that $\Delta(H_W) \subseteq H_W \otimes_S H_W$ (Kostant-Kumar, 1986).

Define the coproduct structure constants $p^w_{u,v} \in S$ by

$$\Delta(x_w) = \sum_{u,v \in W} p^w_{u,v}\, x_u \otimes x_v.$$

Consider the T-equivariant cohomology ring $H^*_T(G/B) \simeq \bigoplus_{w \in W} S\,\sigma_w$ and L-R coefficients $c^w_{u,v} \in S$ defined by the cup product

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c^w_{u,v} \sigma_w.$$

Theorem: Kostant-Kumar 1986

The coefficients $c^w_{u,v} = p^w_{u,v}$.