Littlewood Richardson coefficients for reflection groups

Arkady Berenstein and Edward Richmond*

University of British Columbia

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Let $G$ be a Kac-Moody group over $\mathbb{C}$ (or a simple Lie group).

Fix $T \subseteq B \subseteq G$ a maximal torus and Borel subgroup of $G$.

Let $W := N(T)/T$ denote the Weyl group $G$.

Let $G/B$ be the flag variety (projective ind-variety).

For any $w \in W$, we have the Schubert variety $X_w = \overline{BwB}/B \subseteq G/B$.

Denote the cohomology class of $X_w$ by

$$\sigma_w \in H^{2\ell(w)}(G/B).$$
Additively, we have that $H^*(G/B) \cong \bigoplus_{w \in W} \mathbb{Z} \sigma_w$.

Goal (Schubert Calculus)

Compute the structure (Littlewood-Richardson) coefficients $c_{u,v}^w$ with respect to the Schubert basis defined by the product

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^w \sigma_w.$$  

- Note that if $\ell(w) \neq \ell(u) + \ell(v)$, then $c_{u,v}^w = 0$.
- For any $w, u, v \in W$, we have that $c_{u,v}^w \geq 0$. (proofs are geometric)

For example, if $G$ is a finite Lie group, then the cardinality

$$|g_1X_u \cap g_2X_v \cap g_3X_{w_0w}| = c_{u,v}^w$$

for a generic choice of $g_1, g_2, g_3 \in G$. 
Let $A = A(G)$ denote the Cartan matrix of $G$.

Alternatively, fix a finite index set $I$ and let $A = \{a_{ij}\}$ be an $I \times I$ matrix such that

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$.

The matrix $A$ defines an action of a Coxeter group $W$ generated by reflections $\{s_i\}_{i \in I}$ on the vector space $V := \text{Span}_\mathbb{C}\{\alpha_i\}_{i \in I}$ given by

$$s_i(v) := v - \langle v, \alpha_i^\vee \rangle \alpha_i$$

where $\langle \alpha_i, \alpha_j^\vee \rangle := a_{ij}$.

In particular, Coxeter groups of this type are crystallographic.
If we abandon the group $G$, we can consider a matrix $A$ as follows:

Fix a finite index set $I$ and let $A = \{a_{ij}\}$ be an $I \times I$ matrix such that

- $a_{ii} = 2$
- $a_{ij} \in \mathbb{R}_{\leq 0}$ if $i \neq j$
- $a_{ij} = 0 \iff a_{ji} = 0$.

The matrix $A$ defines an action of a Coxeter group $W$ generated by reflections $\{s_i\}_{i \in I}$ on the vector space $V := \text{Span}_\mathbb{C}\{\alpha_i\}_{i \in I}$ given by

$$s_i(v) := v - \langle v, \alpha_i^\vee \rangle \alpha_i$$

where $\langle \alpha_i, \alpha_j^\vee \rangle := a_{ij}$.

Every Coxeter group can be represented as above.
Some examples

- \( G = SL(4) \) (type \( A_3 \))
  \[
  A = \begin{bmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 2 \\
  \end{bmatrix}
  \quad \text{and} \quad W = S_4 \) (symmetric group)

- \( G = Sp(4) \) (type \( C_2 \))
  \[
  A = \begin{bmatrix}
  2 & -2 \\
  -1 & 2 \\
  \end{bmatrix}
  \quad \text{and} \quad W = I_2(4) \) (dihedral group of 8 elements)

- \( G = \hat{SL}(2) \) (affine type \( A \))
  \[
  A = \begin{bmatrix}
  2 & -2 \\
  -2 & 2 \\
  \end{bmatrix}
  \quad \text{and} \quad W = I_2(\infty) \) (free dihedral group)

- Let \( \rho = 2 \cos(\pi/5) \)
  \[
  A = \begin{bmatrix}
  2 & -\rho \\
  -\rho & 2 \\
  \end{bmatrix}
  \quad \text{and} \quad W = I_2(5) \) (dihedral group of 10 elements)
Notation on sequences and subsets

- Any sequence $\mathbf{i} := (i_1, \ldots, i_m) \in I^m$ has a corresponding element
  \[ s_{i_1} \cdots s_{i_m} \in W. \]

  If $s_{i_1} \cdots s_{i_m}$ is a reduced word of some $w \in W$, then we say that $\mathbf{i} \in R(w)$, the collection of reduced words.

- For each $\mathbf{i} \in I^m$ and subset $K = \{k_1 < k_2 < \cdots < k_n\}$ of the interval $[m] := \{1, 2, \ldots, m\}$ let the subsequence
  \[ \mathbf{i}_K := (i_{k_1}, \ldots, i_{k_n}) \in I^n. \]

  We say a sequence $\mathbf{i}$ is admissible if $i_j \neq i_{j+1}$ for all $j \in [m - 1]$.

- Observe that any reduced sequence is admissible. (In general, the converse is false.)
**Definition**

Let \( m > 0 \) and let \( K, L \) be subsets of \([m] := \{1, 2, \ldots, m\}\) such that \(|K| + |L| = m\). We say that a bijection

\[
\phi : K \rightarrow [m] \setminus L
\]

is *bounded* if \( \phi(k) < k \) for each \( k \in K \).

**Definition**

Given a reduced sequence \( i = (i_1, \ldots, i_m) \in I^m \), we say that a bounded bijection

\[
\phi : K \rightarrow [m] \setminus L
\]

is *i-admissible* if the sequence \( i_L \) and the sequences \( i_{L(k)} \) are admissible for all \( k \in K \) where

\[
L(k) := L \cup \phi(K_{\leq k}).
\]
Recall that the Littlewood-Richardson coefficients $c^w_{u,v}$ are defined by the product

\[ \sigma_u \cdot \sigma_v = \sum_{w \in W} c^w_{u,v} \sigma_w. \]

**Theorem: Berenstein-R, 2010**

Let $u, v, w \in W$ such that $\ell(w) = \ell(u) + \ell(v)$ and let $i = (i_1, \ldots, i_m) \in R(w)$. Then

\[ c^w_{u,v} = \sum p_\phi \]

where the summation is over all triples $(\hat{u}, \hat{v}, \phi)$, where

- $\hat{u}, \hat{v} \subset [m]$ such that $i_{\hat{u}} \in R(u), i_{\hat{v}} \in R(v)$.
- $\phi : \hat{u} \cap \hat{v} \to [m] \setminus (\hat{u} \cup \hat{v})$ is an $i$-admissible bounded bijection.

- The Theorem is still true without $i$-admissible.
- The Theorem generalizes to structure coefficients of $T$-equivariant cohomology $H^*_T(G/B)$.
**Definition**

For any $k \in [m]$ and $i = (i_1, \ldots, i_m)$, denote $\alpha_k := \alpha_{i_k}$ and $s_k := s_{i_k}$. For any bounded bijection

$$\phi : K \to [m] \setminus L$$

we define the monomial $p_\phi \in \mathbb{Z}$ by the formula

$$p_\phi := (-1)^{|K|} \prod_{k \in K} \langle w_k(\alpha_k), \alpha_\phi^\vee(k) \rangle \quad \text{where} \quad w_k := \prod_{r \in L(k) \atop \phi(k) < r < k} s_r$$

where the product $\prod$ is taken in the natural order induced by the sequence $[m]$ and if the product is empty, we set $w_k = 1$. Also, if $K = \emptyset$, then $p_\phi = 1$.

**Theorem: Berenstein-R, 2010**

If the Cartan matrix $A = (a_{ij})$ satisfies

$$a_{ij} \cdot a_{ji} \geq 4 \quad \forall \quad i \neq j,$$

then $p_\phi \geq 0$ when $\phi$ is an i-admissible bounded bijection.
Examples where $G = \hat{SL}(2)$, $i = (1, 2, 1, 2)$

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

Compute $c^w_{u,v}$ where $w = s_1 s_2 s_1 s_2$, $u = v = s_1 s_2$.

- Find $\hat{u}, \hat{v} \subseteq [4] = \{1, 2, 3, 4\}$

  \[
  \begin{align*}
  &1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 \\
  &1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 \\
  &1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 \\
  &1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 & 1 \ 2 \ 3 \ 4 \\
  \end{align*}
  \]

- For $1 \ 2 \ 3 \ 4$, we have

  $\hat{u} \cap \hat{v} = \{3, 4\}$ and $[4] \setminus (\hat{u} \cup \hat{v}) = \{1, 2\}$

  with bounded bijections

  \[
  \phi_1 : (3, 4) \to (1, 2) \quad \phi_2 : (3, 4) \to (2, 1).
  \]

NOTE: $\phi_1$ is not $i$-admissible.
Examples where $G = \hat{SL}(2), \ i = (1, 2, 1, 2)$

- For $1 \ 2 \ 3 \ 4$, we have bounded bijections

$$\phi_1 : (3, 4) \to (1, 2) \quad \phi_2 : (3, 4) \to (2, 1).$$

$$p_{\phi_1} = \langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle \quad p_{\phi_2} = \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle$$

- Totaling over all bounded bijections, we have

$$c^w_{u,v} = \frac{\langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle + \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle}{s_3(\alpha_4), \alpha_2^\vee} - s_3(\alpha_4), \alpha_2^\vee - s_3(\alpha_4), \alpha_2^\vee + 1 + 1$$

$$= -4 + 4 + 2 + 2 + 1 + 1 = 6$$

With only $i$-admissible terms.
Examples where $G = \hat{S}L(2), \ i = (1, 2, 1, 2, 1)$

Compute $c_{u,v}^w$ where $w = s_1s_2s_1s_2s_1, \ u = s_1s_2s_1, \ v = s_2s_1$.

- Find $\hat{u}, \hat{v} \subseteq [5] = \{1, 2, 3, 4, 5\}$

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
$$

- Totaling over all bounded bijections, we have

$$
c_{u,v}^w = \langle \alpha_4, \alpha_2^\vee \rangle \cdot \langle s_4(\alpha_5), \alpha_3^\vee \rangle + \langle \alpha_4, \alpha_3^\vee \rangle \cdot \langle s_3s_4(\alpha_5), \alpha_2^\vee \rangle + \langle s_3(\alpha_4), \alpha_2^\vee \rangle \cdot \langle s_3s_4(\alpha_5), \alpha_1^\vee \rangle + \langle s_3(\alpha_4), \alpha_1^\vee \rangle \cdot \langle s_2s_3s_4(\alpha_5), \alpha_2^\vee \rangle - \langle s_2(\alpha_3), \alpha_1^\vee \rangle - \langle s_4(\alpha_5), \alpha_3^\vee \rangle - \langle s_4(\alpha_5), \alpha_3^\vee \rangle - \langle s_2s_3s_4(\alpha_5), \alpha_1^\vee \rangle + 1 + 1
$$

$$
= -3 + 3 - 3 + 4 + 2 + 3 + 2 + 1 + 1 = 10
$$

With only $i$-admissible terms.
Examples where $G = \hat{SL}(2)$, $i = (2, 1, \ldots, 2, 1)$

Question: Bounded bijections vs. $i$-admissible bounded bijections?

$$c(n) := c_{w,v}, \quad w = s_2 s_1 \cdots s_2 s_1, \quad u = v = \underbrace{s_2 s_1}_{2n}, \quad \underbrace{u = v = \cdots s_2 s_1}_{n}$$

<table>
<thead>
<tr>
<th>$c(n)$</th>
<th>$#{\text{Bounded bijections}}$</th>
<th>$#{\text{i-admissible bounded bijections}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c(2)$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$c(3)$</td>
<td>20</td>
<td>7</td>
</tr>
<tr>
<td>$c(4)$</td>
<td>190</td>
<td>19</td>
</tr>
<tr>
<td>$c(5)$</td>
<td>1110</td>
<td>51</td>
</tr>
<tr>
<td>$c(6)$</td>
<td>14348</td>
<td>141</td>
</tr>
<tr>
<td>$c(n)$</td>
<td>??</td>
<td>largest coeff of $(1 + x + x^2)^n$ ??</td>
</tr>
</tbody>
</table>

Remark: $c(n) = \binom{2n}{n}$
Examples

Example: \( G \) is rank 2.

General rank 2 group \( G \).

Let \( a, b \neq 0 \),

\[
A = \begin{bmatrix}
2 & -a \\
-b & 2
\end{bmatrix}.
\]

Define “Chebyshev” sequences

\[
A_k := aB_{k-1} - A_{k-2} \quad \text{and} \quad B_k := bA_{k-1} - B_{k-2}
\]

where \( A_0 = B_0 = 0 \) and \( A_1 = B_1 = 1 \). Let

\[
u_k = s_2 s_1 \quad \text{and} \quad v_k = s_1 s_2.
\]

Corollary: Binomial formula (Kitchloo, 2008)

The rank 2 Littlewood-Richardson coefficients

\[
\begin{align*}
C_{u_k, u_{n-k}}^{u_n} &= C_{v_{k+1}, u_{n-k}}^{v_{n+1}} = \frac{A_n \cdots A_2 A_1}{(A_k \cdots A_2 A_1)(A_{n-k} \cdots A_2 A_1)} \\
C_{v_k, v_{n-k}}^{v_n} &= C_{u_{k+1}, v_{n-k}}^{u_{n+1}} = \frac{B_n \cdots B_2 B_1}{(B_k \cdots B_2 B_1)(B_{n-k} \cdots B_2 B_1)}
\end{align*}
\]
Examples where $G = SL(4)$, $i = (3,2,1,3,2)$

Compute $c_{u,v}^w$ where

$$w = s_3 s_2 s_1 s_3 s_2, \quad u = s_3 s_1 s_2 = s_1 s_3 s_2, \quad v = s_3 s_1 = s_1 s_3.$$

- Totaling over all bounded bijections, we have

$$c_{u,v}^w = \langle \alpha_3, \alpha_1^\vee \rangle \cdot \langle s_3(\alpha_4), \alpha_2^\vee \rangle + \langle \alpha_3, \alpha_2^\vee \rangle \cdot \langle s_2 s_3(\alpha_4), \alpha_1^\vee \rangle - \langle \alpha_3, \alpha_2^\vee \rangle - \langle \alpha_3, \alpha_1^\vee \rangle = 0 - 1 + 1 + 1 = 1$$

- In this case: \{Bounded bijections\} = \{i-admissible bounded bijections\}
- In general, the i-admissible formula is not nonnegative.
Recall the matrix $A$ gives an action of the Coxeter group $W$ on $V$ and thus $W$ acts on the algebras $S := S(V)$ and $Q := Q(V)$ (polynomials and rational functions).

Define

$$Q_W := Q \rtimes \mathbb{C}[W]$$

with product structure

$$(q_1 w_1)(q_2 w_2) := q_1 w_1 (q_1) w_1 w_2$$

and a $Q$-linear coproduct

$$\Delta : Q_W \to Q_W \otimes_Q Q_W$$

by

$$\Delta(qw) := qw \otimes w = w \otimes qw.$$
For any $i \in I$, define

$$x_i := \frac{1}{\alpha_i}(s_i - 1).$$

If $i = (i_1, \ldots, i_m) \in R(w)$, then define $x_w := x_{i_1} \cdots x_{i_m}$.

If $A$ is a Cartan matrix of some Kac-Moody group $G$, then (by Kostant-Kumar 1986)

- $x_w$ is independent of $i \in R(w)$
- $x_i^2 = 0 \quad \forall \ i \in I$.

Define the Nil-Hecke ring $H_W := \bigoplus_{w \in W} S x_w \subseteq Q_W$.

We have that $\Delta(H_W) \subseteq H_W \otimes_S H_W$ (Kostant-Kumar, 1986).
Define the coproduct structure constants \( p_{u,v} \in S \) by

\[
\Delta(x_w) = \sum_{u,v \in W} p_{u,v}^w x_u \otimes x_v.
\]

Consider the \( T \)-equivariant cohomology ring \( H^*_T(G/B) \simeq \bigoplus_{w \in W} S \sigma_w \) and L-R coefficients \( c_{u,v}^w \in S \) defined by the cup product

\[
\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^w \sigma_w.
\]

**Theorem: Kostant-Kumar 1986**

The coefficients \( c_{u,v}^w = p_{u,v}^w \).