Pattern avoidance and fiber bundle structures on Schubert varieties

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Let $r < n \in \mathbb{Z}_+$ and $\mathbb{C}^n = \mathrm{Span}_{\mathbb{C}}\{e_1, \ldots, e_n\}$.

Complete flag variety:

$$\mathrm{Fl}(n) := \{V_\bullet = (V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n) \mid \dim(V_i) = i\}$$

Grassmannian:

$$\mathrm{Gr}(r,n) := \{V \subset \mathbb{C}^n) \mid \dim(V) = r\}$$

Consider the projection $\pi_r : \mathrm{Fl}(n) \twoheadrightarrow \mathrm{Gr}(r,n)$ given by

$$\pi_r(V_\bullet) = V_r.$$

The map π_r is a fiber bundle map on $\mathrm{Fl}(n)$ with fibers

$$\begin{aligned}\pi_r^{-1}(V) &\simeq (V_1 \subset \cdots \subset V_{r-1} \subset V) \times (V_{r+1}/V \subset \cdots \subset V_{n-1}/V \subset \mathbb{C}^n/V) \\ &\simeq \mathrm{Fl}(r) \times \mathrm{Fl}(n-r).\end{aligned}$$

Question: When is π_r restricted to a Schubert variety of $\mathrm{Fl}(n)$ a fiber bundle?

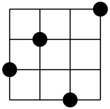
For any $n \times n$ permutation matrix w, define the **Schubert variety:**

$$X(w) := \{V_\bullet \mid \dim(E_i \cap V_j) \geq \mathrm{rk}(w[i,j])\}$$

where $E_i := \mathrm{Span}\{e_1, \ldots, e_i\}$ and $w[i,j]$ is the $(i \times j)$ NW-submatrix of w.

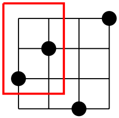
Conventions:

- The matrix entries of w mark the points $(w(i), i)$.
- $(1,1)$ represents the NW corner of the matrix.

Example: Let $n = 4$ and $w = 3241 =$ 

Example: Consider the Schubert variety

$$X(3241) = \{V_\bullet \mid \dim(E_3 \cap V_2) \geq 2\} = \{V_\bullet \mid V_2 \subset E_3\}$$



and the projection $\pi_3 : (V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset \cancel{V_2} \subset V_3 \subset \mathbb{C}^4)$.

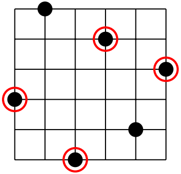
Restricting to $X(3241)$, we get $\pi_3 : X(3241) \twoheadrightarrow \mathrm{Gr}(3,4)$.

The fiber over V is

$$\begin{aligned}\pi_3^{-1}(V) &= \{(V_1 \subset V_2 \subset V \subset \mathbb{C}^4) \mid V_1 \subset V_2 \subseteq E_3 \cap V\} \\ &\cong \begin{cases} \mathrm{Fl}(2) & \text{if } \dim(E_3 \cap V) = 2 \\ \mathrm{Fl}(3) & \text{if } E_3 = V. \end{cases}\end{aligned}$$

So π_3 is not a fiber bundle on $X(3241)$. (But π_1 and π_2 are fiber bundles!)

Pattern avoidance: Let $m \leq n$. We say a permutation $w = w(1) \cdots w(n)$ *contains* the pattern $u = u(1) \cdots u(m)$ if there is a subsequence of w with the same relative order as u. Otherwise, w *avoids* the pattern u.

Example: $w = 416253 =$ 

contains the pattern 3412, but avoids the pattern 1234.

Remark: Pattern avoidance has been a useful tool to describe many geometric properties of Schubert varieties.

History of Pattern avoidance and Schubert varieties:

- $X(w)$ is smooth iff w avoids 3412 and 4231 (Lakshmibai-Sandhya 1990).
- $X(w)$ is defined by inclusions iff w avoids 4231, 35142, 42513, 351624 (Gasharov-Reiner 2002).
- The B-S resolution of $X(w)$ is small iff w avoids 321, 46718235, 46781235, 56718234, 56781234 (Deodhar 1990, Billey-Warrington 2003).
- $X(w)$ is factorial iff w avoids $3\underline{41}2$ and 4231 (Bousquet-Mélou-Butler 2007).
- The B-S resolution of $X(w)$ is isomorphic to $X(w)$ iff w avoids 321 and 3412 (Tenner 2007).

History of Pattern avoidance and Schubert varieties (con't):

- $X(w)$ is Gorenstein iff w interval-avoids a certain list of patterns (Woo-Yong 2008).
- $X(w)$ is LCI iff w avoids 53241, 52341, 52431, 35142, 42513, 426153 (Úlfarsson-Woo 2013).

More remarks:

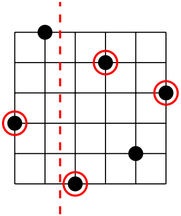
- Notions of pattern avoidance exist for Schubert varieties in other types (Billey 98, Billey-Postnikov 2005).
- Tenner's database for permutation pattern avoidance: `http://math.depaul.edu/bridget/patterns.html`

Split pattern avoidance:

We say a permutation w *contains the split pattern* $u = u_1|u_2$ with respect to position r if

- there is a subsequence of w with the same relative order as u such that
- $w(1)\cdots w(r)$ contains u_1 and $w(r+1)\cdots w(n)$ contains u_2.

Otherwise, w *avoids the split pattern* $u = u_1|u_2$ with respect to position r.

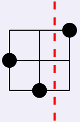
Example: $w = 416253 =$ 

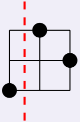
contains the split pattern $3|412$ with respect to positions $r = 1, 2$ but avoids $3|412$ with respect to $r = 3, 4, 5$.

Theorem 1: Alland-R (arXiv 2016)

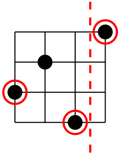
The following are equivalent:

- The projection π_r is a fiber bundle on $X(w)$.
- w avoids the split patterns $23|1$ and $3|12$ with respect to position r.





Example: Let $w = 3241$ and consider $X(w) = \{V_\bullet \mid V_2 \subset E_3\}$.

We have $w =$  containing $23|1$ with respect to position $r = 3$.

Hence π_3 is not a fiber bundle on $X(w)$.

Key result needed in the proof of Theorem 1:

Theorem: R-Slofstra (2016)

Let $\{s_1, \ldots, s_{n-1}\}$ denote the simple transpositions and let $w = vu$ be the parabolic decomposition corresponding to the projection π_r.

The following are equivalent:

- The projection π_r is a fiber bundle on $X(w)$.
- The support of v is contained in $D_L(u) \cup \{s_r\}$.

Remarks:

- Either condition is equivalent to $w = vu$ being a Billey-Postnikov (BP) decomposition (i.e. satisfies a certain factoring condition on Poincaré polynomials of w, v and u).

- This is a Coxeter theoretic condition and is true for Schubert varieties of any finite or Kac-Moody type.

Let $[n-1] := \{1, 2, \cdots, n-1\}$.

Partial flag varieties: For any $\mathbf{r} = \{r_1 < \cdots < r_k\} \subseteq [n-1]$ define

$$\mathrm{Fl}(\mathbf{r}, n) := \{V_1 \subset \cdots \subset V_k \subset \mathbb{C}^n \mid \dim(V_i) = r_i\}.$$

Any sequence of subsets $\mathbf{r}_1 \subset \mathbf{r}_2 \subset \cdots \subset \mathbf{r}_{n-2} \subset [n-1]$ where $|\mathbf{r}_i| = i$ induces an iterated fiber bundle structure

$$\mathrm{Fl}(n) \twoheadrightarrow \mathrm{Fl}(\mathbf{r}_{n-2}, n) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Fl}(\mathbf{r}_1, n).$$

Example: If $n = 4$, then the sequence $\{2\} \subset \{2, 3\} \subset \{1, 2, 3\}$ gives

$$(V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset V_2 \subset V_3 \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset V_2 \subset \cancel{V_3} \subset \mathbb{C}^4).$$

Question: When does such a sequence induce an iterated fiber bundle structure on a Schubert variety $X(w)$?

Definition: If such a sequence exists, then we say $X(w)$ has a *complete parabolic bundle structure*.

Theorem: Ryan (1987), Wolper (1989), Lakshmibai-Sandhya (1990)

If w avoids 4231 and 3412 (i.e. $X(w)$ is smooth), then $X(w)$ has a complete parabolic bundle structure.

Observations:

- The converse is FALSE. In particular,

$$X(4231) = \{V_\bullet \mid \dim(E_2 \cap V_2) \geq 1\}$$

has a complete parabolic bundle structure via $\{2\} \subset \{1,2\} \subset \{1,2,3\}$:

$$(V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^4) \mapsto (V_1 \subset V_2 \subset \cancel{V_3} \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset V_2 \subset \cancel{V_3} \subset \mathbb{C}^4).$$

- However

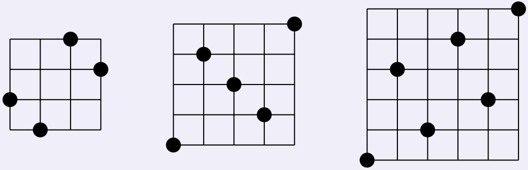
$$X(3412) = \{V_\bullet \mid V_1 \subset E_3,\ E_1 \subset V_3\}$$

has no parabolic bundle structure.

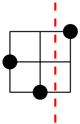
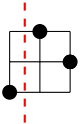
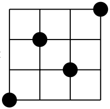
Theorem 2: Alland-R (arXiv 2016)

The following are equivalent:

- $X(w)$ has a complete parabolic bundle structure.
- w avoids 3412, 52341, 635241.



Observations:

- The patterns above contain either  or  at every position.
- The permutation 4231=  avoids these split patterns at $r = 2$.

Key proposition in the proof of Theorem 2:

Proposition: Alland-R (arXiv 2016)

If w avoids 3412, 52341, 635241, then there exists a position r for which w avoids the split patterns $23|1$ and $3|12$.

Proof of Theorem 2: Apply Theorem 1 to the Proposition.

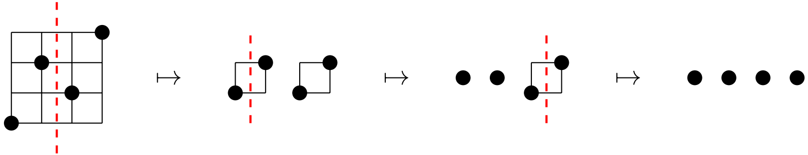
Example: Recall that

$$X(4231) = \{V_\bullet \mid \dim(E_2 \cap V_2) \geq 1\}$$

has a complete parabolic bundle structure via $\emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\}$:

$$(V_1 \subset V_2 \subset \cancel{V_3} \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset V_2 \subset \mathbb{C}^4) \mapsto (\cancel{V_2} \subset \mathbb{C}^4) \mapsto (\mathbb{C}^4).$$

Example (con't): By the proposition, we can find a sequence of positions on $w = 4231$ where "w" always avoids $23|1$ and $3|12$.



Here we have $\emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\}$.

Lemma (R-Slofstra 2016): Such sequences like

$$\emptyset \subset \{2\} \subset \{1,2\} \subset \{1,2,3\}$$

will correspond to a complete parabolic bundle structures on $X(w)$.

$$(V_1 \subset V_2 \subset \cancel{V_3} \subset \mathbb{C}^4) \mapsto (\cancel{V_1} \subset V_2 \subset \mathbb{C}^4) \mapsto (\cancel{V_2} \subset \mathbb{C}^4) \mapsto (\mathbb{C}^4).$$

Possible application to generating functions:
Let a_n denote the number of permutations of size n avoiding 3412 and 4231 (or equivalently, number of smooth Schubert varieties in $\mathrm{Fl}(n)$) and define

$$V(t) := \sum a_n\, t^n.$$

Haiman (unpublished-1990s), Bousquet-Mélou-Butler (2007)

$$V(t) = \frac{1 - 5t + 3t^2 + t^2\sqrt{1-4t}}{1 - 6t + 8t^2 - 4t^3}$$

Remark: Haiman's proof uses complete parabolic bundle structures of smooth Schubert varieties.

Open Question: Can we find a generating function for permutations avoiding 3412, 52341, 635241 using parabolic bundle structures as well?

Thank you!