Eigenvalues of sums of hermitian matrices and the cohomology of Grassmannians

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The eigenvalue problem on hermitian matrices:

Consider the sequences of real numbers

$$\alpha := (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n)$$
$$\beta := (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n)$$
$$\gamma := (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n).$$

Question:

For which triples (α, β, γ) do there exist $n \times n$ hermitian matrices A, B, C with respective eigenvalues α, β, γ and

$$A + B = C\,?$$

Motivation:

- Functional analysis: decomposing self-adjoint operators on Hilbert spaces
- Frame theory (sensor networks, coding theory, and compressed sensing)
- Invariant theory of representations

Example: Let $n = 2$.

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix} = \begin{pmatrix} a+\alpha_1 & c \\ \bar{c} & b+\alpha_2 \end{pmatrix}$$

Then

$$\beta = \frac{a+b \pm \sqrt{(a-b)^2+|c|^2}}{2}$$

$$\gamma = \frac{\alpha_1+\alpha_2+a+b \pm \sqrt{(a-b+\alpha_1-\alpha_2)^2+|c|^2}}{2}$$

Solution for the $n = 2$ case

We get that matrices $A + B = C$ exist if and only if the triple (α, β, γ) satisfies

$$\alpha_1+\alpha_2+\beta_1+\beta_2 = \gamma_1+\gamma_2 \quad \text{and} \quad \begin{array}{c} \alpha_1+\beta_1 \geq \gamma_1 \\ \alpha_1+\beta_2 \geq \gamma_2 \\ \alpha_2+\beta_1 \geq \gamma_2. \end{array}$$

What about $n = 3$?

Theorem: Horn '62

There exist 3×3 matrices $A + B = C$ with eigenvalues (α, β, γ) if and only if (α, β, γ) satisfies

$$\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 = \gamma_1 + \gamma_2 + \gamma_3$$

and up to $\alpha \leftrightarrow \beta$ symmetry

$$\begin{array}{ll}
\alpha_1 + \beta_1 \geq \gamma_1 & \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \geq \gamma_1 + \gamma_2 \\
\alpha_1 + \beta_2 \geq \gamma_2 & \alpha_1 + \alpha_2 + \beta_1 + \beta_3 \geq \gamma_1 + \gamma_3 \\
\alpha_1 + \beta_3 \geq \gamma_3 & \alpha_1 + \alpha_2 + \beta_2 + \beta_3 \geq \gamma_2 + \gamma_3 \\
\alpha_2 + \beta_2 \geq \gamma_2 & \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \geq \gamma_2 + \gamma_3.
\end{array}$$

What about the general case?

Conjecture: Horn '62

There exist $n \times n$ matrices $A + B = C$ with eigenvalues (α, β, γ) if and only if (α, β, γ) satisfies

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{n} \gamma_k$$

and for every $r < n$ we have a "certain" collection of subsets $I, J, K \subset [n] := \{1, 2, \ldots, n\}$ of size r where

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k.$$

Example: If $n = 4$ and $r = 2$, then $(I, J, K) = (\{1, 3\}, \{2, 4\}, \{3, 4\})$ corresponds to the linear inequality

$$\alpha_1 + \alpha_3 + \beta_2 + \beta_4 \geq \gamma_3 + \gamma_4.$$

Schubert calculus of the Grassmannian:

Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ and consider the Grassmannian

$$\mathrm{Gr}(r,n) := \{V \subseteq \mathbb{C}^n \mid \dim V = r\}.$$

For any partition $\lambda := (\lambda_1 \geq \cdots \geq \lambda_r)$ where $\lambda_1 \leq n - r$ define the Schubert subvariety

$$X_\lambda := \{V \in \mathrm{Gr}(r,n) \mid \dim(V \cap E_{n-r+i-\lambda_i}) \geq i \quad \forall\, i \leq r\}$$

where $E_i := \mathrm{Span}\{e_1, \ldots, e_i\}$.

For example

- $X_\varnothing = \mathrm{Gr}(r,n)$
- $X_\Lambda = \{E_r\}$ where $\Lambda := (n-r, \ldots, n-r)$

In general, $\mathrm{codim}_\mathbb{C}(X_\lambda) = |\lambda| := \sum_i \lambda_i$.

Denote the cohomology class of X_λ by

$$\sigma_\lambda \in H^{2|\lambda|}(\mathrm{Gr}(r,n),\mathbb{C}).$$

Additively, we have that Schubert classes σ_λ form a basis of

$$H^*(\mathrm{Gr}(r,n)) \simeq \bigoplus_{\lambda\subseteq\Lambda} \mathbb{C}\,\sigma_\lambda.$$

Define the Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$ as the structure constants

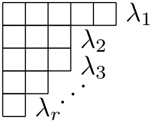
$$\sigma_\lambda\cdot\sigma_\mu = \sum_{\nu\subseteq\Lambda} c^\nu_{\lambda,\mu}\,\sigma_\nu.$$

Facts:

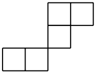
- $c^\nu_{\lambda,\mu}\in\mathbb{Z}_{\geq 0}$.
- If $c^\nu_{\lambda,\mu}>0$, then $|\lambda|+|\mu|=|\nu|$.

The LR-rule for computing $c_{\lambda,\mu}^{\nu}$:

For any partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r)$, we have the associated Young diagram



If $\lambda \subseteq \nu$, then we can define the skew diagram ν/λ by removing the boxes of the Young diagram of λ from the Young diagram of ν.

$$(4,3,2)/(2,2) =$$


We have

$$c_{\lambda,\mu}^{\nu} = \#\{\text{LR skew tableaux of shape } \nu/\lambda \text{ with content } \mu\}.$$

Appearances of Littlewood-Richardson coefficients $c_{\lambda,\mu}^{\nu}$:

The number of points in a finite intersection of translated Schubert varieties

$$|g_1 X_\lambda \cap g_2 X_\mu \cap g_3 X_{\nu^\vee}| = c_{\lambda,\mu}^{\nu}.$$

Let V_λ be the irreducible, finite-dimensional representation of $\mathrm{GL}_r(\mathbb{C})$ of highest weight λ. Then

$$V_\lambda \otimes V_\mu \simeq \bigoplus_{\nu} V_\nu^{\oplus\, c_{\lambda,\mu}^{\nu}}.$$

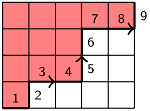
Let s_λ denote the Schur function indexed by λ. Then

$$s_\lambda \cdot s_\mu = \sum_{\nu} c_{\lambda,\mu}^{\nu}\ s_\nu.$$

Let $\Lambda = \underbrace{(n-r,\ldots,n-r)}_{r}$. There is a bijection between

$$\{\text{Partitions } \lambda \subseteq \Lambda\} \Leftrightarrow \{\text{Subsets of } [n] \text{ of size } r\}.$$

Consider the Young diagram of $\lambda \subseteq \Lambda$.



Example: $n = 9$ and $r = 4$ with $\lambda = (5,3,3,1)$.

We identify λ with vertical labels on the boundary path.

$$(5,3,3,1) \leftrightarrow \{2,5,6,9\}.$$

For any subset $I = \{i_1 < \cdots < i_r\} \subseteq [n]$, let

$$\phi(I) := \{i_r - r \geq \cdots \geq i_1 - 1\}$$

denote the corresponding partition.

Solution to the eigenvalue problem:

Theorem: Klyachko '98

Let (α, β, γ) be weakly decreasing sequences of real numbers such that

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{n} \gamma_k.$$

Then the following are equivalent.

- There exist $n \times n$ matrices $A + B = C$ with eigenvalues (α, β, γ).
- For every $r < n$ and every triple (I, J, K) of subsets of $[n]$ of size r such that the Littlewood-Richardson coefficient $c_{\phi(I),\phi(J)}^{\phi(K)} > 0$, we have

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k.$$

Question: Is Klyachko's solution the same as Horn's conjectured solution?

Theorem: Knutson-Tao '99

Klyachko's solution is the same to Horn's solution.

Let $\lambda, \mu, \nu \subseteq \Lambda$ be partitions such that

$$|\lambda| + |\mu| = |\nu|.$$

The following are equivalent:

- The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu} > 0$.
- There exist $r \times r$ matrices $A + B = C$ with respective eigenvalues λ, μ, ν.

Example: Consider the Grassmannian $\mathrm{Gr}(2,4)$. $\ydiagram{2,2}$

We have $(\sigma_{\ydiagram{1}})^2 = \sigma_{\ydiagram{1,1}} + \sigma_{\ydiagram{2}}$.

For $c^{\ydiagram{2}}_{\ydiagram{1},\ydiagram{1}} > 0$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $c^{\ydiagram{1,1}}_{\ydiagram{1},\ydiagram{1}} > 0$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But for $c^{\ydiagram{2,2}}_{\ydiagram{1,1},\ydiagram{2}} = 0$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & c \\ \bar{c} & b \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Not possible with eigenvalues $(2,0)$!

Generalizations of the eigenvalue problem:

- Lie groups of other types (Berenstein-Sjamaar '00):
  Let O_λ denote an orbit of G acting on the dual Lie algebra $\mathfrak{g}^*$.
  Question: For which (λ, μ, ν) is

$$(O_\lambda + O_\mu) \cap O_\nu \neq \varnothing?$$

- Find a minimal list of inequalities (Knuston-Tao-Woodward '04).

$$c_{\lambda,\mu}^{\nu} > 0 \text{ (redundant list) } \Rightarrow c_{\lambda,\mu}^{\nu} = 1 \text{ (minimal list)}$$

  Find a minimal list in any type (Belkale-Kumar '06, Ressayre '10).

- Replace $A + B = C$ with majorized sums $A + B \geq C$
  (Friedland '00, Fulton '00).

Definition: We say a matrix $A \geq B$ (A majorizes B) if $A - B$ is semi-positive definite (i.e. $A - B$ has nonnegative eigenvalues).

Consider the sequences of real numbers

$$\alpha := (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n)$$
$$\beta := (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n)$$
$$\gamma := (\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n).$$

Question:

For which triples (α, β, γ) do there exist $n \times n$ hermitian matrices A, B, C with respective eigenvalues α, β, γ and

$$A + B \geq C\,?$$

Solution to the majorized eigenvalue problem:

Theorem: Friedland '00, Fulton '00

Let (α, β, γ) be weakly decreasing sequences of real numbers such that

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j \geq \sum_{k=1}^{n} \gamma_k.$$

Then the following are equivalent.

- There exist $n \times n$ matrices $A + B \geq C$ with eigenvalues (α, β, γ).
- For every $r < n$ and every triple (I, J, K) of subsets of $[n]$ of size r such that the Littlewood-Richardson coefficient $c^{\phi(K)}_{\phi(I),\phi(J)} > 0$, we have

  $$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k$$

  ~~and (α, β, γ) satisfies other inequalities.~~

Question: Is there an analogue of the Knutson-Tao equivalence for majorized sums?

Theorem: Knutson-Tao '99

Let $\lambda, \mu, \nu \subseteq \Lambda$ be partitions such that

$$|\lambda| + |\mu| = |\nu|.$$

The following are equivalent:

- The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu} > 0$.
- There exist $r \times r$ matrices $A + B = C$ with respective eigenvalues λ, μ, ν.

Answer: YES! Use torus-equivariant cohomology.

Equivariant Schubert calculus of the Grassmannian:

Let $T = (\mathbb{C}^*)^n$ denote the standard torus acting on $\mathbb{C}^n$.

This induces a T-action on the Grassmannian $\mathrm{Gr}(r, n)$. Let $H_T^*(\mathrm{Gr}(r, n))$ be the T-equivariant cohomology ring of $\mathrm{Gr}(r, n)$.

Since the Schubert variety X_λ is T-stable, it determines an equivariant Schubert class denoted by

$$\Sigma_\lambda \in H_T^{2|\lambda|}(\mathrm{Gr}(r, n)).$$

Additively, we have that Schubert classes Σ_λ form a basis of

$$H_T^*(\mathrm{Gr}(r, n)) \simeq \bigoplus_{\lambda \subseteq \Lambda} \mathbb{C}[t_1, \ldots, t_n] \; \Sigma_\lambda$$

over the polynomial ring $H_T^*(\mathrm{pt}) = \mathbb{C}[t_1, \ldots, t_n]$.

Define the equivariant structure coefficients $C_{\lambda,\mu}^{\nu} \in \mathbb{C}[t_1, \ldots, t_n]$ as the structure constants

$$\Sigma_\lambda \cdot \Sigma_\mu = \sum_{\nu \subseteq \Lambda} C_{\lambda,\mu}^{\nu} \; \Sigma_\nu .$$

Facts:

- $C_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}[t_1 - t_2, t_2 - t_3, \ldots, t_{n-1} - t_n]$.
- If $C_{\lambda,\mu}^{\nu} \neq 0$, then $|\lambda| + |\mu| \geq |\nu|$ and $\lambda, \mu \subseteq \nu$.
  In particular, $C_{\lambda,\mu}^{\nu}$ is a homogeneous polynomial of degree $|\lambda| + |\mu| - |\nu|$.
- If $|\lambda| + |\mu| = |\nu|$, then $C_{\lambda,\mu}^{\nu} = c_{\lambda,\mu}^{\nu}$.
- Let S_λ denote the factorial Schur function indexed by λ. Then

$$S_\lambda \cdot S_\mu = \sum_{\nu} C_{\lambda,\mu}^{\nu} \; S_\nu .$$

  (character theory of $\mathrm{GL}_r(\mathbb{C}) \times T$)

Example: $(\Sigma_{\square})^2 = (t_1 - t_2)\,\Sigma_{\square} + \Sigma_{\substack{\square \\ \square}} + \Sigma_{\square\square}$

Theorem: Anderson-R.-Yong '12

Let $\lambda, \mu, \nu \subseteq \Lambda$ be partitions such that

$$|\lambda| + |\mu| \geq |\nu| \quad \text{and} \quad \lambda, \mu \subseteq \nu.$$

The following are equivalent:

- The structure coefficient $C^{\nu}_{\lambda,\mu} \neq 0$.
- There exist $r \times r$ matrices $A + B \geq C$ with respective eigenvalues λ, μ, ν.
- For every $d < r$ and every triple (I, J, K) of subsets of $[r]$ of size d such that the Littlewood-Richardson coefficient $c^{\phi(K)}_{\phi(I),\phi(J)} > 0$, we have

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k.$$

Corollary in Saturation:

Saturation Theorem: Knutson-Tao '99

Let λ, μ, ν be partitions. Then the Littlewood-Richardson coefficients

$$c^{\nu}_{\lambda,\mu} \neq 0 \qquad \text{if and only if} \qquad c^{N\nu}_{N\lambda,N\mu} \neq 0 \quad \text{for any} \quad N > 0.$$

Equivariant Saturation Theorem: Anderson-R.-Yong '12

Let λ, μ, ν be partitions. Then the equivariant structure coefficients

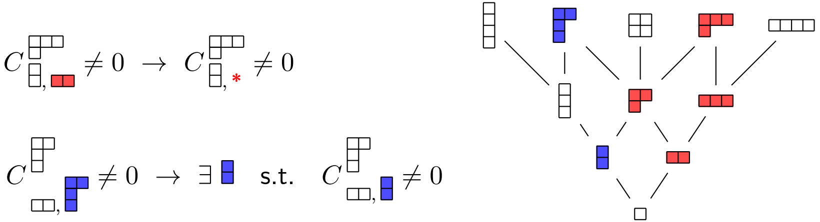
$$C^{\nu}_{\lambda,\mu} \neq 0 \qquad \text{if and only if} \qquad C^{N\nu}_{N\lambda,N\mu} \neq 0 \quad \text{for any} \quad N > 0.$$

Proof: $A + B \geq C$ if and only if $N \cdot A + N \cdot B \geq N \cdot C$ for any $N > 0$.

Key Proposition: Anderson-R.-Yong '12

Assume $C^{\nu}_{\lambda,\mu} \neq 0$. Then:

- $C^{\nu}_{\lambda,\mu^{\uparrow}} \neq 0$ for any $\mu \subset \mu^{\uparrow} \subseteq \nu$;
- If $|\lambda| + |\mu| > |\nu|$, then there is a $\mu^{\downarrow}$ such that $|\lambda| + |\mu^{\downarrow}| = |\nu|$, with $\mu^{\downarrow} \subsetneq \mu$ and $C^{\nu}_{\lambda,\mu^{\downarrow}} \neq 0$.



Proof: Use equivariant tableaux combinatorics of Thomas-Yong '12.

Proof that the inequalities are necessary:

- They are necessary if $|\lambda| + |\mu| = |\nu|$ (Klyachko '98).
- If $C^{\nu}_{\lambda,\mu} \neq 0$ and $|\lambda| + |\mu| > |\nu|$, then there exists $c^{\nu}_{\lambda,\mu^{\downarrow}} \neq 0$ with $\mu^{\downarrow} \subsetneq \mu$ and
$$|\lambda| + |\mu^{\downarrow}| = |\nu|.$$
- For any necessary inequality (I, J, K) we have
$$\sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j \geq \sum_{i\in I} \lambda_i + \sum_{j\in J} \mu^{\downarrow}_j \geq \sum_{k\in K} \nu_k.$$

Proof that the inequalities are sufficient:

- Suppose that $C^{\nu}_{\lambda,\mu} = 0$ and satisfies all the inequalities.
- Show that $C^{\nu}_{\lambda,\mu} \neq 0$, a contradiction.
- Run a double induction on the degree $p := |\lambda| + |\mu| - |\nu|$ and r the maximum number of parts in each partition.
- Proof of the base case: $p = 0$ follows from (Klyachko '98) and $r = 1$ follows from $\mathrm{Gr}(1, n) \simeq \mathbb{CP}^{n-1}$.

Future projects on this topic:

- Prove Horn and Saturation theorems for $H_T^*(G/P)$ where G/P is a (co)minuscule flag variety (following Belkale '06, Sottile-Purbhoo '08).

- Prove Horn and Saturation theorems for other cohomology theories:

$$H^*(\mathrm{Gr}(r,n)) \Leftrightarrow \text{sum of hermitian matrices}$$
$$H_T^*(\mathrm{Gr}(r,n)) \Leftrightarrow \text{majorized sums of hermitian matrices}$$

$$QH^*(\mathrm{Gr}(r,n)) \Leftrightarrow \text{products in } SU(r) \text{ (Belkale '08)}$$
$$QH_T^*(\mathrm{Gr}(r,n)) \Leftrightarrow \text{???}$$

Thank you.