Fibre bundle structures of Schubert varieties

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Let G be a semi-simple Lie group over k (alg. closed).

Let P be parabolic subgroup of G and consider the projection between flag varieties

$$\pi : G/B \twoheadrightarrow G/P.$$

The projection π induces a P/B-fibre bundle structure on G/B.

Example: Type A

$$G/B = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset k^n \mid \dim(V_i) = i\}.$$

$$G/P = \{V_r \subset k^n \mid \dim(V_r) = r\}$$

$$\pi(V_\bullet) = V_r$$

with fibers

$$\pi^{-1}(V_r) = \{V_1 \subset \cdots \subset V_r\} \times \{V_{r+1}/V_r \subset \cdots \subset k^n/V_r\}$$

Let W be the Weyl group G.

For any $w \in W$, we have the Schubert variety

$$X_w := \overline{BwB}/B \subseteq G/B.$$

Question: Is π restricted to X_w a fiber bundle?

Sometimes yes and sometimes no

Example:

$$\pi : (V_1 \subset V_2 \subset V_3 \subset k^4) \mapsto (\cancel{V_1} \subset \cancel{V_2} \subset V_3 \subset k^4)$$

Consider

$$X_w = \{V_\bullet \mid V_2 \subset E_3\}$$

where E_3 is a fixed 3-dim subspace.

We get

$$\pi : X(w) \twoheadrightarrow (V_3 \subset k^4).$$

with fibers

$$\begin{aligned} \pi^{-1}(V_3) &= \{(V_1 \subset V_2 \subset V_3) \mid V_1 \subset V_2 \subseteq V_3 \cap E_3\} \\ &\cong \begin{cases} (V_1 \subset V_3 \cap E_3) & \text{if } \dim(V_3 \cap E_3) = 2 \\ (V_1 \subset V_2 \subset V_3) & \text{if } V_3 = E_3 \end{cases} \end{aligned}$$

The Weyl group W is a Coxeter group with a generating set S.

- Length function $\ell : W \to \mathbb{Z}_{\geq 0}$
- Bruhat partial order $\leq$.

Let W_J denote the Coxeter subgroup generated by $J \subseteq S$. Let $W^J \simeq W/W_J$ denote the minimal length coset representatives.

For any $w \in W$ we have

- The support $S(w) := \{s \in S \mid s \leq w\}$
- Left descents $D_L(w) := \{s \in S \mid \ell(sw) \leq \ell(w)\}$
- Poincaré polynomial $P_w^J(q) := \displaystyle\sum_{x \in [e,w] \cap W^J} q^{\ell(x)}$

For every $J \subset S$ and $w \in W$ there is a unique parabolic decomposition

$$w = vu$$

where $v \in W^J$ and $u \in W_J$.

If W_J is the Weyl group of P, then geometrically, we have that the restricted projection

$$\pi : X_w \twoheadrightarrow X_v^P := \overline{BvP}/P \subseteq G/P$$

has generic fiber isomorphic to X_u.

Remark: In general, not all fibres are isomorphic to X_u

Theorem: R-Slofstra (14)

Let W_J be the Weyl group of P. Let $w = vu$ be a parabolic decomposition with respect to J. The projection

$$\pi : X_w \to X_v^P$$

is an algebraic fibre bundle with fibre isomorphic to X_u if and only if any of the following equivalent statements are true

1. u is the maximal length element in $[e, w] \cap W_J$.
2. The Poincaré polynomials factor $P_w(q) = P_v^J(q) \cdot P_u(q)$.
3. $S(v) \cap J \subseteq D_L(u)$.

Remarks:

- The theorem is true for Schubert varieties for $\pi : G/P \to G/Q$.
- The theorem is true for Kac-Moody Schubert varieties.

Definition: A parabolic decomposition $w = vu,\ v \in W^J, u \in W_J$ is a BP (Billey-Postnikov) decomposition if u is the maximal length element in $[e, w] \cap W_J$.

Question: Is there a $J \subseteq S$ for which $w = vu$ is a BP decomposition?

We say a parabolic decomposition $w = vu$ is maximal if $|S(w)| = |S(u)| + 1$. Hence $J = S(u)$.

Question: Is there a $J \subseteq S$ for which $w = vu$ is a maximal BP decomposition?

Theorem: Ryan (87), Wolper (89)

Let G/B be a type A flag variety. If X_w is a smooth Schubert variety, then w has a maximal BP decomposition ($w = vu$).

Moreover, X_v^P and X_u are smooth Schubert varieties and the projection

$$\pi : X_w \to X_v^P$$

is an X_u-fiber bundle over X_v^P.

Corollary: Smooth Schubert varieties are iterated fiber bundles of Grassmannians.

$$\begin{tikzcd}
\{pt\} \arrow[r, "f"] & X_{u_k} \arrow[r, "f"] \arrow[d, "\pi_k"] & \cdots \arrow[r, "f"] & X_{u_1} \arrow[r, "f"] \arrow[d, "\pi_1"] & X_w \arrow[d, "\pi"] \\
& X_{v_k}^{P_k} & & X_{v_1}^{P_1} & X_{v_0}^{P_0}
\end{tikzcd}$$

Question: What about other types?

Theorem: Billey (98), Billey-Postnikov (05), Oh-Yoo (10)

Let W be a Weyl group of finite type and $w \in W$. If X_w is rationally smooth, then either w or w^{-1} has maximal BP decomposition.
Moreover, we can choose $J = S(w) \setminus \{s\}$, where s is some leaf of the Dynkin diagram of $S(w)$.

Theorem: Billey-Crites (12)

Let W be a Weyl group of affine type A and $w \in W$. If w avoids the patterns 4231 and 3412, then either w or w^{-1} has maximal BP decomposition.

Issue: The Schubert varieties X_w and $X_{w^{-1}}$ are not isomorphic.

Theorem: R-Slofstra (14)

- Let G/B be a flag variety of finite type. If X_w is a rationally smooth Schubert variety, then w has a maximal BP decomposition (not necessarily with respect to a leaf).
- Let G/B be a flag variety of affine type A. If w avoids the patterns 4231 and 3412, then w has a maximal BP decomposition.

Corollary: Rationally smooth Schubert varieties are iterated fiber bundles of rationally smooth Schubert varieties of generalized Grassmannians (G/P where P is maximal).

$$\begin{tikzcd}
\{pt\} \arrow[r, "f"] & X_{u_k} \arrow[r, "f"] \arrow[d, "\pi_k"] & \cdots \arrow[r, "f"] & X_{u_1} \arrow[r, "f"] \arrow[d, "\pi_1"] & X_w \arrow[d, "\pi"] \\
& X^{P_k}_{v_k} & & X^{P_1}_{v_1} & X^{P_0}_{v_0}
\end{tikzcd}$$

Remark: Rationally smooth Schubert varieties of generalized Grassmannians are classified.

Theorem:Lakshmibai-Weyman (90), Brion-Polo (99), Robles (12), Hong-Mok (13)

Let G/P be a generalized Grassmannian with $W_P = W_J$ and $J = S \setminus \{s\}$.

Then X_w^P is rationally smooth if and only if w is maximal in $W^P \cap W_{S(w)}$, or w is one of the following elements:

W	s	w	index set	smooth
B_n	s_n	$s_1 \dots s_n$	$n \geq 2$	yes
B_n	s_1	$s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$	$1 < k \leq n$	no
B_n	s_k	$u_{n,k+1} s_1 \cdots s_k$	$1 < k < n$	no
C_n	s_n	$s_1 \dots s_n$	$n \geq 2$	no
C_n	s_1	$s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$	$1 < k \leq n$	yes
C_n	s_k	$u_{n,k+1} s_1 \cdots s_k$	$1 < k < n$	yes
F_4	s_1	$s_4 s_3 s_2 s_1$	n/a	no
F_4	s_2	$s_3 s_2 s_4 s_3 s_4 s_2 s_3 s_1 s_2$	n/a	no
F_4	s_4	$s_1 s_2 s_3 s_4$	n/a	yes
F_4	s_3	$s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_4 s_3$	n/a	yes
G_2	s_1	$s_2 s_1,\ s_1 s_2 s_1,\ s_2 s_1 s_2 s_1$	n/a	no
G_2	s_2	$s_1 s_2$	n/a	yes
G_2	s_2	$s_2 s_1 s_2,\ s_1 s_2 s_1 s_2$	n/a	no

Here $u_{n,k}$ denotes the maximal element in $W^{S \setminus \{s_1, s_k\}} \cap W_{S \setminus \{s_1\}}$ when W has type B_n or C_n.

If w is a maximal element of $W^P \cap W_{S(w)}$, then X_w^P is smooth.

Applications

Corollary (Peterson's ADE Theorem): If G is simply laced, then a Schubert variety X_w in G/B is smooth if and only if it is rationally smooth.

Proof: All rationally smooth Schubert varieties in generalized Grassmannians of type ADE isomorphic to flag varieties and hence smooth.

Applications

Theorem: Billey-Crites (12)

Let G/B be a flag variety of affine type A.

- If the Schubert variety X_w is smooth, then w avoids the patterns 4231 and 3412.
- If w avoids the patterns 4231 and 3412, then the Schubert variety X_w is rationally smooth.

Conjecture: X_w is smooth.

Corollary: The conjecture is true.

Proof: The element w has a BP decomposition $w = vu$ where both v and u have proper support in $S(w)$. Thus X_u and X_v^P are rationally smooth Schubert varieties of finite type A and hence smooth.

Applications (in progress)

We can use previous theorems to enumerate smooth and rationally smooth Schubert varieties in the complete flag variety G/B in classical types.

n	A	B (smooth)	C (smooth)	B/C (r.s.)	D
4	88	116	114	142	108
5	366	490	472	596	490
6	1552	2094	1988	2530	2164
7	6652	9014	8480	10842	9474
8	28696	38988	36474	46766	41374
9	124310	169184	157720	202594	180614
10	540040	735846	684404	880210	788676
11	2350820	3205830	2976994	3832004	3445462

The generating series for type A is due to Haiman.