Recursive structures in the cohomology of flag varieties

by
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Abstract

EDWARD RICHMOND: Recursive structures in the cohomology of flag varieties

(Under the direction of Prakash Belkale)

Let G be a semisimple complex algebraic group and P be a parabolic subgroup of G and consider the flag variety G/P. The ring $H^*(G/P)$ has interesting combinatorial structures with respect to the additive basis of Schubert classes. For example, if $G = SL(n)$ and P is a maximal parabolic, then G/P is a Grassmannian and the structure constants of $H^*(G/P)$ with respect to the Schubert classes are governed by Schur polynomials and the Littlewood-Richardson rule. We consider the flag varieties associated to the groups $G = SL(n)$ and $Sp(2n)$ and take P to be any parabolic subgroup (not necessarily maximal). We find that $H^*(G/P)$ exhibits Horn recursion on a certain deformation of the cup product. Horn recursion is a term used to describe when non-vanishing products of Schubert classes in $H^*(G/P)$ are characterized by inequalities parameterized by similar non-vanishing products in the cohomology of "smaller" flag varieties. We also find that if a product of Schubert classes is non-vanishing on this deformation, then the associated structure constant can be written as a product of structure constants coming from induced maximal flag varieties.

We also look at a generalization of Horn recursion to the "branching" Schubert calculus setting. Consider a semisimple subgroup $\tilde{G} \subset G$ and an induced embedding of flag varieties $\tilde{G}/\tilde{P} \hookrightarrow G/P$. We construct a list of necessary Horn conditions to determine when the pullback of a Schubert class is nonzero in $H^*(\tilde{G}/\tilde{P})$.

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Contents

CHAPTER 1

Introduction

Let G be a connected semisimple complex algebraic group and let P be a parabolic subgroup containing some fixed Borel subgroup B and consider the homogeneous space G/P. If W is the Weyl group of G, let W^P be the set of minimal length representatives of the coset space W/W_P where W_P denotes the Weyl group of P. For any $w \in W^P$, define $[\Lambda_w]$ to be the cohomology class of the Schubert variety $\bar{\Lambda}_w := \overline{BwP} \subseteq G/P$. It is well known that the elements of W^P parameterize an additive basis of the cohomology ring $H^*(G/P) = H^*(G/P, \mathbb{Z})$. The general problem we address is to determine the product structure with respect to this basis. Let $w^1, w^2, \ldots, w^s \in W^P$ and assume that $\sum_{k=1}^{s} \operatorname{codim} \Lambda_{w^k} = \dim G/P$. Then $\prod_{k=1}^{s} [\Lambda_{w^k}] = c[pt]$ for some structure constant $c \in \mathbb{Z}_{\geq 0}$. The two questions we ask are: Under what conditions is $c \neq 0$ and more specifically, can we explicitly compute c? Equivalently, we can ask: When is the intersection of general translates

$$g_1\Lambda_{w^1} \cap g_2\Lambda_{w^2} \cap \cdots \cap g_s\Lambda_{w^s}$$

nonempty? and if so, how many points are in the intersection?

Horn's conjecture [8] on the Hermitian eigenvalue problem provides an interesting answer to the first question in the case where $G = SL(n)$ and P is a maximal parabolic. In this case, G/P is the standard Grassmannian $\mathrm{Gr}(r, n)$. The result is that a product of such classes in $H^*(\mathrm{Gr}(r, n))$ is non-vanishing if and only if the s-tuple $(w^1, w^2, \ldots, w^s)$ satisfies a certain list of linear inequalities called Horn's inequalities. Remarkably, Horn's inequalities themselves are indexed by the non-vanishing of similar products in smaller Grassmannians (Horn recursion). Hence s-tuples with nonvanishing structure constants can be determined by a "cohomology free" algorithm. Horn's conjecture was proved by the work of Klyachko [11] and the saturation theorem of Knutson-Tao [12]. Belkale [2] later gave a geometric proof of Horn's conjecture set in the context of intersection theory. Purbhoo-Sottile [15] have shown that any G/P where P is cominuscule (i.e. the associated simple root to the maximal parabolic P occurs with coefficient 1 in the highest root) exhibits Horn type recursion as well. A general discussion about the connection of Horn's inequalities to other topics can be found in [6, 7].

The second question on computing structure coefficients has also been studied. Classically, structure coefficients for the Grassmannian can be realized as Littlewood-Richardson numbers [13, 17] which have nice combinatorial formulas. There has been much work to produce analogues of Littlewood-Richardson rules for other flag varieties G/P. In particular, a rule for isotropic Grassmannians (with respect to a symplectic or orthogonal form) was proved by Stembridge in [20] and for general (co)minuscule flag varieties by Thomas-Yong in [21].

In the first part this thesis we consider partial flag varieties G/P for the groups $G = SL(n)$ and $Sp(2n)$. We find that G/P exhibits Horn recursion on a certain deformation of the cup product and that the structure coefficients are a product of structure coefficients coming from induced maximal flag varieties. This deformation can be described by the notion of *Levi-movability* initially defined in [3]. Let L be the Levi subgroup of $P \subseteq G$. We say $(w^1, w^2, \ldots, w^s)$ is Levi-movable (or L-movable) if $\sum_{k=1}^{s} \operatorname{codim} \Lambda_{w^k} = \dim G/P$ and for generic $(l_1, l_2, \ldots, l_s) \in L^s$ the intersection $\bigcap_{i=1}^{s} l_i (w^i)^{-1} \Lambda_{w^i}$ meets transversally at the point $eP \in G/P$. If $(w^1, w^2, \ldots, w^s)$ is L-movable, then the associated structure coefficient $c \neq 0$, however the converse is generally not true. The notion of L-movability defines a new product on the cohomology of G/P denoted $(H^*(G/P), \odot_0)$. This new product is commutative, associative and satisfies Poincaré duality. In the case of cominuscule flag varieties, the usual product and new product coincide (see Section 2.1.2 for more details). We also briefly consider the entire cohomology ring $H^*(SL(n)/P)$ and construct some necessary criteria for all non-vanishing products of Schubert classes.

In the second part of this thesis, we generalize Horn recursion and Levi-movability to *branching Schubert calculus*. Consider the following restatement of the original question found the beginning of this chapter. Let G be a connected semisimple complex algebraic group and consider the diagonal embedding of $G \hookrightarrow G^s$. For any parabolic subgroup $P \subseteq G$, we have the induced embedding $\phi : G/P \hookrightarrow (G/P)^s$. If $\bar{\Lambda}_w := (\bar{\Lambda}_{w^1}, \bar{\Lambda}_{w^2}, \ldots, \bar{\Lambda}_{w^s})$ is a Schubert variety of $(G/P)^s$, we can ask: Under what conditions is

$$\phi^*([\Lambda_w]) \neq 0?$$

This question is equivalent the original question: When is $\prod_{k=1}^{s}[\Lambda_{w^k}] \neq 0$? Instead of the diagonal embedding, consider the setting where $\tilde{G}$ is any semisimple algebraic subgroup of G. Choose parabolic subgroups $\tilde{P}$ and P such that $\phi : \tilde{G}/\tilde{P} \hookrightarrow G/P$. We can now ask the same question: If $w \in W^P$, under what conditions is $\phi^*([\Lambda_w]) \neq 0$? The mathematics addressing this more general setting is called "branching Schubert calculus". In this thesis, there are three main results on this topic. First, we construct a list of necessary inequalities which generalizes Horn's inequalities. Second, we generalize the notion of Levi-movability and give a numerical condition to determine when $w \in W^P$ is Levi-movable given that $\phi^*([\Lambda_w]) \neq 0$. Finally, we construct a second list of necessary Horn inequalities which generalizes the Levi-Horn recursion established in the first part of this thesis. Note that these Horn inequalities are only necessary conditions. It is an interesting question to ask in what cases are they sufficient.

1.1. Results for type A flag varieties

The homogeneous space $SL(n)/P$ corresponds to a partial flag variety $\mathrm{F}\ell(a,n)$ for some set of integers $a := \{0 < a_1 < a_2 < \cdots < a_r < n\}$ and the set W^P can be identified with

$$S_n(a) := \{(w(1), w(2), \ldots, w(n)) \in S_n \mid w(i) < w(i+1) \ \forall\ i \notin a\}$$

where S_n denotes the permutation group on $[n] := \{1, 2, \ldots, n\}$. We denote type A Schubert cells by $X_w := \Lambda_w$.

1.1.1. Horn recursion for $\mathrm{F}\ell(a,n)$

Assume that the associated structure constant to $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ is nonzero and for any $i \in [r]$, consider the projection $f_i : \mathrm{F}\ell(a,n) \to \mathrm{Gr}(a_i, n)$. The expected dimension of the intersection $\bigcap_{k=1}^{s} f_i(X_{w^k})$ in $\mathrm{Gr}(a_i,n)$ is nonnegative and hence for any $i \in [r]$, we have

$$\sum_{k=1}^{s}\sum_{j=1}^{a_i}\Big(n - a_i + j - w^k(j)\Big) \le a_i(n - a_i). \tag{1.1}$$

We find that $\mathrm{F}\ell(a,n)$ exhibits Horn recursion on the boundary where the inequalities (1.1) are equalities. Before we state the first main result, we need the following definition of induced permutations:

Definition 1.1.1. *For any permutation $w \in S_n$, we define the associated permutation to any ordered subset $A = \{\dot{a}_1 < \dot{a}_2 < \cdots < \dot{a}_d\} \subseteq [n]$ of cardinality d by the following: Let $w_A(k) = \#\{\dot{a} \in A \mid w(\dot{a}) \le w(\dot{a}_k)\}$ and define the permutation*

$$w_A := (w_A(1), w_A(2), \ldots, w_A(d)) \in S_d.$$

Set $a_0 = 0$ and $a_{r+1} = n$ and let $b_i := a_i - a_{i-1}$. Consider the subset $A_i := \{a_{i-1}+1 < a_{i-1}+2 < \cdots < a_i\}$. For any $w \in S_n(a)$ and $\{i < j\} \subseteq [r+1]$, define $w_{i,j} := w_{A_i \cup A_j} \in S_{b_i+b_j}(b_i)$.

Theorem 1.1.2. *Let $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ be such that*

$$\sum_{k=1}^{s} \operatorname{codim} X_{w^k} = \dim \mathrm{F}\ell(a, n). \tag{1.2}$$

Then the following are equivalent:

(i) $\prod_{k=1}^{s}[X_{w^k}] =$ *a nonzero multiple of a class of a point in* $H^*(\mathrm{F}\ell(a,n))$ *and*

$$\sum_{k=1}^{s}\sum_{j=1}^{a_i}\Big(n - a_i + j - w^k(j)\Big) = a_i(n - a_i)\ \forall\ i \in [r]. \tag{1.3}$$

(ii) The s-tuple $(w^1, w^2, \ldots, w^s)$ is L-movable.

(iii) $\prod_{k=1}^{s}[X_{w^k_{i,j}}] =$ *a nonzero multiple of a class of a point in* $H^*(\mathrm{Gr}(b_i, b_i + b_j))\ \forall\ \{i < j\} \subseteq [r+1]$.

(iv) For any $\{i < j\} \subseteq [r+1]$ the following are true:

(iva) The sum $\sum_{k=1}^{s} \operatorname{codim} X_{w^k_{i,j}} = \dim \mathrm{Gr}(b_i, b_i + b_j)$.

(ivb) For any $1 \le d < b_i$ and any choice $(u^1, u^2, \ldots, u^s) \in S_d(b_i)^s$ such that $\prod_{k=1}^{s}[X_{u^k}] =$ a nonzero multiple of a class of a point in $H^*(\mathrm{Gr}(d, b_i))$, the following inequality is valid:

$$\sum_{k=1}^{s}\sum_{l=1}^{d}\Big(b_j + u^k(l) - w^k_{i,j}(u^k(l))\Big) \le d b_j. \tag{1.4}$$

Note that $(iii) \Leftrightarrow (iv)$ is immediate by Horn's conjecture applied to the Grassmannians $\mathrm{Gr}(b_i, b_i + b_j)$. Hence we will prove Theorem 1.1.2 by focusing on $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$. The main object we use in the proof is the tangent space $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ at the standard partial flag $\tilde{E}$. We find that the tangent subspaces of Schubert cells corresponding to a L-movable

s-tuple decompose nicely with respect to a certain decomposition of the tangent space of $\mathrm{F}\ell(a,n)$. Fix a splitting $Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{r+1}$ of $\mathbb{C}^n$ such that $E_{a_k} = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_k$. Then there exists a natural decomposition

$$T_{\tilde{E}}\mathrm{F}\ell(a,n) \;\simeq\; \bigoplus_{i<j}^{r+1} \mathrm{Hom}(Q_i, Q_j).$$

In Propositions 3.2.5 and 3.2.7, we show that for any $l \in L$ and $w \in S_n(a)$,

$$T_{\tilde{E}}(lw^{-1}X_w) \;\simeq\; \bigoplus_{i<j}^{r+1} \big(\mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}_\bullet}(lw^{-1}X_w)\big).$$

Hence $(w^1, w^2, \ldots, w^s)$ is L-movable if and only if $\bigcap_{i=1}^{s} T_{\tilde{E}}(l_i(w^i)^{-1}X_{w^i})$ is transversal in each summand $\mathrm{Hom}(Q_i, Q_j) \subseteq T_{\tilde{E}}\mathrm{F}\ell(a,n)$ for generic $(l_1, l_2, \ldots, l_s) \in L^s$.

1.1.2. Necessary conditions for the standard product on $H^*(\mathrm{F}\ell(a,n))$

A natural question to ask is: Does the full product structure on $H^*(\mathrm{F}\ell(a,n))$ exhibit Horn recursion? The answer to this question at this point is unclear. We begin a discussion on this question by constructing a new set of inequalities in which the inequality (1.1) is a special case. The techniques used to produce these necessary inequalities are inspired by the work of Purbhoo-Sottile in [15]. Consider the projection of $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ onto any vector space V. Indeed, if $T_{\tilde{E}}\mathrm{F}\ell(a,n) \twoheadrightarrow V$ is a surjection of vector spaces, any transversal intersection of subspaces in $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ will be projected to a transversal intersection in V. We apply this technique to the natural geometry of $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ and construct a set of necessary conditions. For any $w \in S_n(a)$, let $A_w^j := \{w(a_j+1), w(a_j+2), \ldots, w(n)\}$ and define

$$p_w^{i,j}(l) := \#\{p \in A_w^j \mid p \le w_i(a_i + w_i(l) - l)\}$$

where w_i is the image of w under the map $S_n(a) \to S_n(a_i)$.

Theorem 1.1.3. *For any* $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ *such that* $\prod_{k=1}^s [X_{w^k}] \neq 0$, *we have that*

$$\sum_{k=1}^{s} \sum_{l=1}^{a_i} (n - a_j - p_{w^k}^{i,j}(l)) \leq a_i(n - a_j) \quad \forall \; \{i \leq j\} \subseteq [r]. \tag{1.5}$$

Note that if $i = j$, then $p_w^{i,j}(l) = w_i(l) - l$ and hence the inequalities (1.5) are exactly the inequalities (1.1).

1.1.3. Computing structure coefficients for $\mathrm{F}\ell(a, n)$

Let $c_{w,u}^v$ be the structure coefficient defined by the product

$$[X_w] \cdot [X_u] = \sum_{v \in S_n(a)} c_{w,u}^v [X_v].$$

In the case of the Grassmannian, the coefficients $c_{w,u}^v$ are Littlewood-Richardson numbers which have several nice combinatorial formulas. However computing these coefficients for $\mathrm{F}\ell(a, n)$ is much more difficult. We find that if $c_{w,u}^v$ is associated to an L-movable 3-tuple, then $c_{w,u}^v$ is a product of Littlewood-Richardson numbers. It is well known that the coefficient $c_{w,u}^v$ is the number of points in a generic intersection of the associated Schubert varieties if the intersection is finite. Consider the projection of $f_1 : \mathrm{F}\ell(a, n) \twoheadrightarrow \mathrm{Gr}(a_1, n)$. For any Schubert cell $X_w \subseteq \mathrm{F}\ell(a, n)$ we have the induced Schubert cells $X_{w_1} := f_1(X_w) \subseteq \mathrm{Gr}(a_1, n)$ and $X_{w_\gamma} := X_w \cap f_1^{-1}(V)$ in the fiber over any point $V \in X_{w_1}$. The Weyl group element w_1 is the image of w under the map $S_n(a) \to S_n(a_1)$. The fiber $f_1^{-1}(V) \simeq \mathrm{F}\ell(a_\gamma, n - a_1)$ where $a_\gamma = \{0 < a_2 - a_1 < \cdots < a_r - a_1 < n - a_1\}$ and $w_\gamma = w_A$ where $A = \bigcup_{i=2}^{r+1} A_i$ under Definition 1.1.1. We now state the main result on type A structure coefficients.

Theorem 1.1.4. *Let $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ be L-movable. If c, c_1 and c_γ are the structure coefficients:*

$$\prod_{k=1}^{s}[X_{w^k}] = c[X_e], \qquad \prod_{k=1}^{s}[X_{w_1^k}] = c_1[X_e], \qquad \prod_{k=1}^{s}[X_{w_\gamma^k}] = c_\gamma[X_e],$$

in $H^*(\mathrm{F}\ell(a,n)), H^*(\mathrm{Gr}(a_1,n))$ and $H^*(\mathrm{F}\ell(a_\gamma, n-a_1))$ respectively, then $c = c_1 \cdot c_\gamma$.

In Proposition 4.2.3, we show that if the s-tuple $(w^1, w^2, \ldots, w^s)$ is Levi-movable, then so is $(w_\gamma^1, w_\gamma^2, \ldots, w_\gamma^s)$. Hence we can once again use Theorem 1.1.4 to decompose c_γ. Thus computing L-movable structure coefficients completely reduces to computing these coefficients in the Grassmannians $\mathrm{Gr}(b_i, n-a_{i-1})$ for $i \in [r]$. Note that these are not the same Grassmannians found in Theorem 1.1.2 (*iii*).

1.2. Results for type C flag varieties

Fix $\mathbb{C}^{2n}$ a $2n$ dimensional complex vector space with a nondegenerate skew-symmetric form. The homogeneous space $Sp(2n)/P$ corresponds to an isotropic partial flag variety $\mathrm{IF}(a, 2n)$ for some set of integers $a = \{a_1 < a_2 < \cdots < a_r\}$. If a consists of a single integer $\{r\}$, we will denote $\mathrm{IF}(a, 2n)$ by the isotropic Grassmannian $\mathrm{IG}(r, 2n)$ and if $r = n$, the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$. The set W^P can be identified with

$$S_{2n}^C(a) := \{w \in S_{2n} \mid w(2n+1-i) = 2n+1-w(i)\ \forall i \in [n] \text{ and } w(i) < w(i+1)\ \forall\ i \notin a\}.$$

We denote type C Schubert cells by $\Phi_w := \Lambda_w$.

1.2.1. Horn recursion for $\mathrm{IF}(a, 2n)$

In this section we state the analogue of Theorem 1.1.2 for $\mathrm{IF}(a, 2n)$. In this case, the set a induces a natural partition of $[2n]$. Let $\bar{a}_i := 2n+1-a_i$ and set $a_0 = 0$ and $\bar{a}_0 = 2n+1$. For any $k \in [r]$, let

$$\begin{aligned} I_k &:= \{a_{k-1}+1, a_{k-1}+2, \ldots, a_k\} \\ \bar{I}_k &:= \{\bar{a}_k, \bar{a}_k+1, \ldots, \bar{a}_{k-1}-1\} \\ \tilde{I} &:= \{a_r+1, a_r+2, \ldots, \bar{a}_r\}. \end{aligned}$$

Clearly $[2n] = I_1 \sqcup \cdots \sqcup I_r \sqcup \tilde{I} \sqcup \bar{I}_r \sqcup \cdots \sqcup \bar{I}_1$. Define the unions

$$I^k := \bigcup_{i=1}^{k} I_i \quad \text{and} \quad \bar{I}^k := \bigcup_{i=1}^{k} \bar{I}_i \quad \text{and} \quad \tilde{I}^k := [2n] \backslash (I^k \cup \bar{I}^k).$$

For $w \in S_{2n}$ and any subset $I \subseteq [2n]$, let $w(I) := \{w(i) \mid i \in I\} \subseteq [2n]$. Also for any two sets $I, J \subseteq [2n]$, define

$$|I > J| := \#\{(i,j) \in I \times J \mid i > j\}.$$

As in Section 1.1.1, let $b_i := a_i - a_{i-1}$ (note that we still take $a_{r+1} = n$). For $w \in S^C_{2n}(a)$ consider the following induced permutations using Definition 1.1.1:

(i) For any $\{i < j\} \subseteq [r]$, denote $w_{i,j} := w_{I_i \cup I_j} \in S_{b_i+b_j}(b_i)$

(ii) For any $\{i < j\} \subseteq [r]$, denote $\bar{w}_{i,j} := w_{I_i \cup \bar{I}_j} \in S_{b_i+b_j}(b_i)$

(iii) For any $i \in [r]$, denote $\tilde{w}_i := w_{I_i \cup \tilde{I}} \in S_{b_i+2b_{r+1}}(b_i)$

(iv) For any $i \in [r]$, denote $\bar{w}_i := w_A \in S_{2a_i}(a_i)$ where $A = I^k \cup \bar{I}^k$.

Similar to (1.1), we consider a necessary numerical condition where $\mathrm{IF}(a, 2n)$ exhibits Horn recursion on the boundary of this condition. Let $(w^1, w^2, \ldots, w^s) \in (S_{2n}^C(a))^s$ be such that the associated structure constant is nonzero. Let w_i denote the projection of w in $S_{2n}^C(a) \to S_{2n}^C(a_i)$. For any $i \in [r]$, we have that

$$\sum_{k=1}^{s} \mathrm{codim}\Phi_{w_i^k} + \mathrm{codim}\Phi_{\bar{w}_i^k} \leq \dim \mathrm{IG}(a_i, 2n) + \dim \mathrm{LG}(a_i, 2a_i) \tag{1.6}$$

Note that the geometric interpretation of (1.6) is different compared to (1.1).

Theorem 1.2.1. *Let $(w^1, w^2, \ldots, w^s) \in (S_{2n}^C(a))^s$ be such that*

$$\sum_{k=1}^{s} \mathrm{codim}(\Phi_{w^k}) = \dim \mathrm{IF}(a, 2n). \tag{1.7}$$

Then the following are equivalent:

(i) $\prod_{k=1}^{s}[\Phi_{w^k}] =$ *a nonzero multiple of a class of a point in* $H^*(\mathrm{IF}(a, 2n))$ *and*

$$\sum_{k=1}^{s} |w^k(\tilde{I}^i \cup \bar{I}^i) > w^k(I^i)| + |w^k(\bar{I}^i) > n| = a_i(2n - a_i + 1) \ \forall\ i \in [r]. \tag{1.8}$$

(ii) The s-tuple $(w^1, w^2, \ldots, w^s)$ *is L-movable.*

(iii) The following are true:

(iiia) The products $\prod_{k=1}^{s}[X_{w_{i,j}^k}]$ *and* $\prod_{k=1}^{s}[X_{\bar{w}_{i,j}^k}]$ *are nonzero multiples of a class of a point in* $H^*(\mathrm{Gr}(b_i, b_i + b_j))$, $\forall\ \{i < j\} \subseteq [r]$.

(iiib) The products $\prod_{k=1}^{s}[X_{\bar{w}_i^k}]$ *and* $\prod_{k=1}^{s}[\Phi_{\bar{w}_{i,i}^k}]$ *are nonzero multiples of a class of a point in* $H^*(\mathrm{Gr}(b_i, b_i + 2b_{r+1}))$ *and* $H^*(\mathrm{LG}(b_i, 2b_i))$ *respectively,* $\forall\ i \in [r]$.

Note that the LHS and RHS of numerical condition (1.8) are equal to the LHS and RHS of equation (1.6) respectively. As in Theorem 1.1.2 (*iv*), we can apply Horn recursion to the flag varieties in Theorem 1.2.1 (*iii*). Note that for $(\bar{w}^1_{i,i}, \bar{w}^2_{i,i}, \ldots, \bar{w}^s_{i,i})$ in the second part of (*iiib*), we need to apply Purbhoo-Sottile's cominuscule recursion found in [15]. The proof of Theorem 1.2.1 follows the same outline as the proof of Theorem 1.1.2 in that we consider the tangent space $T_{\tilde{E}}\mathrm{IF}(a,2n)$ with respect to a certain splitting of $\mathbb{C}^{2n}$. The proof also relies on an important result of Belkale-Kumar from [4] in which they show that maximal type A Schubert cells (Schubert cells in the regular Grassmannian) can be moved to intersect properly by generic elements of $Sp(2n) \subseteq SL(2n)$. In [19], Sottile shows that this intersection can actually be made transverse. See Lemma 3.3.8 for a precise statement of what is needed from their work to prove Theorem 1.2.1.

1.2.2. Computing structure coefficients for $\mathrm{IF}(a,2n)$

In this section, we give a formula to compute structure coefficients for $\mathrm{IF}(a,2n)$. Instead of projecting $\mathrm{IF}(a,2n)$ onto the first factor as in Section 1.1.3, consider the projection $f_r : \mathrm{IF}(a,2n) \twoheadrightarrow \mathrm{IG}(a_r,2n)$ onto the last factor. In this case, the fiber over any point $V \in \mathrm{IG}(a_r,2n)$ is isomorphic to the type A flag variety $\mathrm{F}\ell(a^C_\gamma, n)$ where $a^C_\gamma := \{a_1, a_2, \ldots, a_{r-1}\}$. As in Section 1.1.3, we find that if $(w^1, w^2, \ldots, w^s)$ is L-movable, then the associated structure constant is a product of the induced structure constants coming from the projection and fiber. For any $w \in S^C_{2n}(a)$, let w_r denote the projection of w in $S^C_{2n}(a) \to S^C_{2n}(a_r)$ and let $w_\gamma := w_{I^r}$ using Definition 1.1.1.

Theorem 1.2.2. *Let $(w^1, w^2, \ldots, w^s)$ be L-movable. If c, c_r and c_γ are the structure coefficients:*

$$\prod_{k=1}^{s}[\Phi_{w^k}] = c[\Phi_e], \qquad \prod_{k=1}^{s}[\Phi_{w^k_r}] = c_r[\Phi_e], \qquad \prod_{k=1}^{s}[X_{w^k_\gamma}] = c_\gamma[X_e],$$

in $H^*(\mathrm{IF}(a,2n)), H^*(\mathrm{IG}(a_r,2n))$ and $H^*(\mathrm{F}\ell(a^C_\gamma, a_r))$ respectively, then $c = c_r \cdot c_\gamma$.

We show in Proposition 4.3.1 that if $(w^1, w^2, \ldots, w^s)$ is Levi-movable, then so is $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$.

Hence we can apply Theorem 1.1.4 to write the constant c as a product of coefficients coming from the isotropic Grassmannian $\mathrm{IG}(a_r, 2n)$ and the regular Grassmannians $\mathrm{Gr}(b_i, a_r - a_{i-1})$.

1.3. Remarks on Levi-movability

We remark that L-movability has been used outside the context of determining cohomology products. In particular, Belkale-Kumar [3] use L-movability to give a refined solution to the generalized eigenvalue problem which was initially solved by Kapovich-Leeb-Millson [9] following works of Klyachko [11], Belkale [1] and Berenstein-Sjamaar [5]. Let $K \subseteq G$ be a maximal compact subgroup and let $\mathfrak{k}$ denote the Lie algebra of K (note that K is a real Lie group and $\mathfrak{k}$ is a real Lie algebra). Consider the space $\mathfrak{k}/K$ where K acts on $\mathfrak{k}$ by the adjoint representation. The goal of the "eigenvalue problem" is to determine which $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_s \in \mathfrak{k}/K$ satisfy the condition that $\sum_{i=1}^{s} k_i = 0$. Kapovich-Leeb-Millson [9] proved that $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_s$ satisfy this condition if and only if they satisfy a certain list of inequalities indexed by s-tuples of $(W^P)^s$ with associated structure constant $c = 1$ where P runs over all standard maximal parabolic subgroups of G. This initial list is known to have redundancies for some groups G. Belkale-Kumar showed that it is sufficient to only consider the subset of these inequalities indexed by L-movable s-tuples. In [16], Ressayre shows that this list is in fact irredundant.

1.4. Branching Schubert Calculus and Horn recursion

Let $\tilde{G}$ be a semisimple algebraic subgroup of G and choose parabolic subgroups $\tilde{P}$ and P such that $\phi : \tilde{G}/\tilde{P} \hookrightarrow G/P$. We ask the question: For which $w \in W^P$ is $\phi^*([\Lambda_w]) \neq 0$? In this thesis, we consider a special choice of $\tilde{P}$ and P based on a fixed one parameter subgroup of $\tilde{G}$ (ie. an algebraic homomorphism $\tau : \mathbb{C}^* \to \tilde{G}$, OPS for short). Fix a maximal torus

$\tilde{H} \subseteq \tilde{G}$ such that $\tilde{H} = \tilde{G} \cap H$. Choose positive Weyl chambers $\mathfrak{h}_+$, $\tilde{\mathfrak{h}}_+$ in $\mathfrak{h}$, $\tilde{\mathfrak{h}}$ respectively where $\mathfrak{h}$, $\tilde{\mathfrak{h}}$ denote the Lie algebras of H, $\tilde{H}$. Let B, $\tilde{B}$ be the Borel subgroups of G, $\tilde{G}$ corresponding to this choice of positive Weyl chambers.

Definition 1.4.1. *For any one parameter subgroup τ of G we have the associated parabolic subgroup of G defined*

$$P^G(\tau) := \{g \in G \mid \lim_{t \to 0} \tau(t) g \tau(t)^{-1} \textit{ exists in } G\}.$$

We say τ is a dominant OPS if $B \subseteq P^G(\tau)$. Fix τ to be a dominant OPS of $\tilde{G}$. Clearly τ is also an OPS of G, however it may not dominant with respect to G. Choose $v \in W$ such that $\tau_v := v^{-1}\tau(t)v$ is dominant with respect to G. We will later see that we do not need to consider all of W to make such a choice. In [5], Berenstein-Sjamaar define a certain subset of W denoted W_{rel} which depends only on the choice of positive Weyl chambers $\mathfrak{h}_+$, $\tilde{\mathfrak{h}}_+$ (see Definition 5.2.4). It is sufficient to only consider this subset when conjugating τ to τ_v. Define the map

$$\phi_{\tau,v} : \tilde{G}/P^{\tilde{G}}(\tau) \hookrightarrow G/P^G(\tau_v)$$

by $\phi_{\tau,v}(gP^{\tilde{G}}(\tau)) := gvP^G(\tau_v)$. We will denote $\phi_{\tau,v}$, $P^G(\tau_v)$, $P^{\tilde{G}}(\tau)$ by ϕ, P, $\tilde{P}$ when the choice of τ and v are clear. Let

$$\phi^* : H^*(G/P, \mathbb{Z}) \to H^*(\tilde{G}/\tilde{P}, \mathbb{Z})$$

be the induced map on singular cohomology. In this thesis, we construct a list of necessary conditions for any (τ, v, w) that satisfies $\phi^*([\Lambda_w]) \neq 0$ which are generalizations of the Horn conditions found in the case of the diagonal embedding. Abusing notation, let $v \in N_G(H)$ be a representative of $v \in W_{\text{rel}}$ and consider the twisted embedding of $\tilde{L} \hookrightarrow L$ given by $l \mapsto v^{-1}lv \in L$ where $\tilde{L}$ and L denote the Levi subgroups of $\tilde{P}$ and P respectively.

The necessary conditions come in the form of inequalities indexed by $(\lambda, \hat{v}, \hat{w}) \in OPS(\tilde{L}) \times (W_L)_{\mathrm{rel}} \times W_L$ which satisfy a similar non-vanishing condition in $H^*(\tilde{L}/\tilde{Q}(\lambda))$ associated this twisted embedding of Levi factors. Here $\tilde{Q}(\lambda) := P^{\tilde{L}}(\lambda)$ denotes the parabolic subgroup of L associated to λ. This result is stated in Theorem 5.3.8. We remark that this result is independent of the choice of representative of $v \in W_{\mathrm{rel}}$.

The statement and proof of Theorem 5.3.8 is a generalization of work by Belkale-Kumar in [3, Theorem 29], in which they construct these inequalities in the case of the diagonal embedding. For the diagonal embedding, every dominant OPS of $\tilde{G}$ is also dominant for $\tilde{G}^s$. Therefore $v \in W_{\mathrm{rel}}$ can be taken to be the identity; in fact, $W_{\mathrm{rel}} = \{1\}$.

1.4.1. Levi-movability and branching Schubert calculus

We generalize ideas of Levi-movability to branching Schubert Calculus. We say $w \in W^P$ is (L, ϕ)-movable if $\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}$ and the point $e\tilde{P}$ is scheme theoretically isolated in $\phi^{-1}(vlw^{-1}\Lambda_w)$ for generic $l \in L$. As with usual Levi-movability, this condition implies that $\phi^*([\Lambda_w]) \neq 0$, however the converse is generally not true. There are two main results on this topic. The first is a generalization of the numerical condition in Theorems 1.1.2 and 1.2.1 $(i) \Leftrightarrow (ii)$ and is stated in Theorem 5.6.3. The second is a generalization of Theorem 1.1.2 $(ii) \Rightarrow (iv)$ and is stated in Theorem 5.6.7. We remark that Belkale-Kumar in [2], establish these results for general G/P in the case of diagonal embedding. Their results are stated in this thesis in Proposition 2.1.5 and Theorem 5.6.3.

1.4.2. Connections to representation theory

The study of branching Schubert calculus is motivated by its connections to representation theory. Let $\mu \in \mathfrak{h}_+^*$ be an integral dominant weight such that the irreducible representation (of highest weight μ) V_μ of G contains a nonzero $\tilde{G}$-invariant vector. The set of all such dominant weights generate a convex cone $\mathcal{C}$ in a certain real subspace of $\mathfrak{h}^*$. In [5], Berenstein-Sjamaar describe $\mathcal{C}$ by a system of linear inequalities indexed by $(\tau, v, w) \in OPS(\tilde{G}) \times W_{\text{rel}} \times W^P$ which satisfy $\phi^*_{\tau,v}([\Lambda_w]) \neq 0$. Part of this condition is that τ is dominant with respect to $\tilde{G}$ and τ_v is dominant with respect to G. This result is stated in Theorem 5.3.2. For the diagonal embedding, the corresponding picture is to characterize the convex cone $\mathcal{C}$ generated by s-tuples of integral dominant weights $(\mu_1, \ldots, \mu_s)$ of $\tilde{G}$ for which $V_{\mu_1} \otimes \cdots \otimes V_{\mu_s}$ contains a nonzero $\tilde{G}$-invariant vector.

CHAPTER 2

Preliminaries on partial flag varieties

Let G be a connected semisimple complex algebraic group. Fix a Borel subgroup B and a maximal torus $H \subseteq B$. Let $W := N_G(H)/H$ denote the Weyl group of G where $N_G(H)$ is the normalizer of H in G. Let $P \subseteq G$ be a standard parabolic subgroup (P contains B). Let L be the Levi (maximal reductive) subgroup of P. Denote the Lie algebras of G, H, B, P, L by the corresponding frankfurt letters $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \mathfrak{l}$. Let $R \subseteq \mathfrak{h}^*$ be the set of roots and let $R^+ \subseteq R$ denote the set of positive roots. Let $R_{\mathfrak{l}}$ denote the set of roots corresponding to $\mathfrak{l}$ and let $R_{\mathfrak{l}}^+$ denote the set of positive roots with respect to $B_L := B \cap L$. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+$ be the set of simple roots and let $\{s_1, s_2, \ldots, s_n\} \subseteq W$ denote the corresponding simple reflections. Note that the set Δ forms a basis for $\mathfrak{h}^*$ and let $\{x_1, x_2, \ldots, x_n\}$ be the dual basis to Δ (i.e. $\alpha_i(x_j) = \delta_{i,j}$). Let $\Delta(P) \subset \Delta$ denote the simple roots associated to P (the simple roots that generate $R_{\mathfrak{l}}^+$). Let P_i denote the maximal parabolic subgroup associated to the root $\{\alpha_i\} = \Delta \backslash \Delta(P_i)$ and let $R_{\mathfrak{l}_i}^+$ denote the set of positive roots generated by $\Delta(P_i)$.

Let W^P be the set of minimal length representatives of the coset space W/W_P where W_P is the subgroup of W generated by $\{s_i \mid \alpha_i \in \Delta(P)\}$. For any $w \in W^P$, define the Schubert cell

$$\Lambda_w := BwP/P \subseteq G/P.$$

The Schubert cells make up the Bruhat decomposition

$$G/P = \bigsqcup_{w \in W^P} \Lambda_w.$$

We denote the cohomology class of the closure $\bar{\Lambda}_w$ by $[\Lambda_w] \in H^*(G/P,\mathbb{Z})$. The set $\{[\Lambda_w] \mid w \in W^P\}$ forms an additive basis for $H^*(G/P)$.

2.1. Levi-movability

The goal of this section is to reduce the problem of determining nonvanishing products in $H^*(G/P)$ to problems of determining tranversality on the tangent space of G/P. Let $X_1, X_2, \ldots, X_s$ be smooth connected subvarieties of smooth variety X. We say an intersection is **transversal** at a point $x \in \bigcap_{i=1}^s X_i$, if

$$\dim\Big(\bigcap_{i=1}^{s} T_x X_i\Big) = \dim(T_x X) - \sum_{i=1}^{r} \operatorname{codim}(T_x X_i).$$

Our analysis begins with the following important proposition on transversal intersections on varieties with a transitive group action (see Kleiman [10]).

Proposition 2.1.1. *Let G be a complex connected algebraic group that acts transitively on complex variety X. Let $X_1, X_2, \ldots, X_r$ be smooth irreducible (not necessarily closed) subvarieties of X. Then there exists a nonempty open subset of $U \subseteq G^s$ such that for $(g_1, g_2, \ldots, g_s) \in U$, the intersection $\bigcap_{i=1}^s g_i X_i$ (possibly empty) is transverse at every point in the intersection, and $\bigcap_{i=1}^s g_i X_i$ is dense in $\bigcap_{i=1}^s g_i \bar{X}_i$.*

The following proposition is a basic fact on transversality and Schubert varieties. For a proof see [3].

Proposition 2.1.2. *Let $(w^1, w^2, \ldots, w^s) \in (W^P)^s$. The following are equivalent.*

(i) $\prod_{k=1}^s [\Lambda_{w^k}] \neq 0$ *in $H^*(G/P)$.*

(ii) There exist $(g_1, g_2, \ldots, g_s) \in G^s$ such that the intersection $\bigcap_{k=1}^s g_k \Lambda_{w^k}$ is transverse and nonempty.

(iii) *For generic* $(g_1, g_2, \ldots, g_s) \in G^s$*, the intersection* $\bigcap_{k=1}^{s} g_k \Lambda_{w^k}$ *is transverse and nonempty.*

(iv) *For generic* $(p_1, p_2, \ldots, p_s) \in P^s$*, the intersection* $\bigcap_{k=1}^{s} p_k (w^k)^{-1} \Lambda_{w^k}$ *is transverse at* eP*.*

Part *(iv)* of Proposition 2.1.2 gives motivation for the following definition found in [3].

Definition 2.1.3. *The s-tuple* $(w^1, w^2, \ldots, w^s) \in (W^P)^s$ *is* ***Levi-movable*** *or* ***L-movable*** *if*

$$\sum_{k=1}^{s} \operatorname{codim} \Lambda_{w^k} = \dim G/P \tag{2.1}$$

and for generic $(l_1, l_2, \ldots, l_s) \in L^s$*, the intersection* $\bigcap_{k=1}^{s} l_k (w^k)^{-1} \Lambda_{w^k}$ *is transverse at the point* eP*.*

Note that if $(w^1, w^2, \ldots, w^s)$ is L-movable, then $\prod_{k=1}^{s} [\Lambda_{w^k}] \neq 0$. The converse of this statement is generally not true.

2.1.1. A numerical condition for L-movability

The following character is used in [3] to give a numerical condition to determine when $(w^1, w^2, \ldots, w^s)$ is L-movable.

Definition 2.1.4. *For any standard parabolic subgroup* P *and* $w \in W^P$*, define the character*

$$\chi_w = \sum_{\beta \in R^+ \setminus R_{\mathfrak{l}}^+ \cap w^{-1} R^+} \beta.$$

Proposition 2.1.5. *For any* $(w^1, w^2, \ldots, w^s) \in (W^P)^s$*, the following are equivalent.*

(a) $(w^1, w^2, \ldots, w^s)$ *is L-movable.*

(b) $\prod_{k=1}^{s} [\Lambda_{w^k}] =$ *a nonzero multiple of a class of a point in* $H^*(G/P)$ *under the usual*

cohomology product, and for every $\alpha_i \in \Delta\backslash\Delta(P)$, we have

$$\big((\sum_{k=1}^{s} \chi_{w^k}) - \chi_1\big)(x_i) = 0. \tag{2.2}$$

For the proof see [3, Theorem 15] (We remark that Theorem 5.6.3 in Chapter 5 also implies Proposition 2.1.5). Let $R^- = \{-\alpha \mid \alpha \in R^+\}$ denote the set of negative roots in R. The following is an important lemma connecting the character χ_w to the geometry of G/P.

Lemma 2.1.6. *For any $w \in W^P$,* $\operatorname{codim}(\Lambda_w) = |R^+\backslash R_{\mathfrak{l}}^+ \cap w^{-1}R^+|$.

Proof. Consider the following Cartan decompositions

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{h} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha} \\
\mathfrak{p} &= \mathfrak{h} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in R_{\mathfrak{l}}^+} \mathfrak{g}_{-\alpha} \\
\mathfrak{b} &= \mathfrak{h} \oplus \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha
\end{aligned}$$

Using these decompositions, we identify the tangent space of G/P at eP with

$$T_{eP}(G/P) = \bigoplus_{\alpha\in R^+\backslash R_{\mathfrak{l}}^+} \mathfrak{g}_{-\alpha}.$$

Let $w \in W^P$ and consider the subgroup $w^{-1}Bw \subseteq G$. Since the corresponding Lie algebra is

$$w^{-1}\mathfrak{b}w = \mathfrak{h} \oplus \bigoplus_{\alpha\in w^{-1}R^+} \mathfrak{g}_\alpha,$$

the tangent space of $w^{-1}BwP \subseteq G/P$ can be identified with

$$T_{eP}(w^{-1}BwP) = \bigoplus_{\alpha\in R^+\backslash R_{\mathfrak{l}}^+ \cap w^{-1}R^-} \mathfrak{g}_{-\alpha}.$$

Since $w^{-1}R = w^{-1}R^+ \sqcup w^{-1}R^- = R$, we have that

$$T_{eP}(G/P)/T_{eP}(w^{-1}BwP) = \bigoplus_{\alpha \in R^+ \setminus R_{\mathfrak{l}}^+ \cap w^{-1}R^+} \mathfrak{g}_{-\alpha}.$$

$\square$

2.1.2. Deformation of the cup product

In [3], Belkale-Kumar use the notion of L-movability to define a new cohomology product $\odot_0$ on $H^*(G/P, \mathbb{Z})$. We state the definition and some basic facts on the new product. For any $u, v, w \in W^P$, let $c_{u,v}^w$ be the structure coefficient defined by the product

$$[\Lambda_u] \cdot [\Lambda_v] = \sum_{w \in W^P} c_{u,v}^w \, [\Lambda_w].$$

For each $\alpha_i \in \Delta/\Delta(P)$, consider an indeterminate τ_i and define

$$[\Lambda_u] \odot [\Lambda_v] := \sum_{w \in W^P} \Bigg(\prod_{\alpha_i \in \Delta/\Delta(P)} \tau_i^{(\chi_w - \chi_v - \chi_u)(x_i)} \Bigg) c_{u,v}^w \, [\Lambda_w].$$

Extend this operation to a $\mathbb{Z}[\tau_i]_{a_i \in \Delta/\Delta(P)}$-linear product structure on $H^*(G/P, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau]$ where $\mathbb{Z}[\tau] = \mathbb{Z}[\tau_1, \tau_2, \ldots, \tau_r]$. The following are some properties of the product $\odot$. Proofs for these properties can be found in [3, Section 6].

Proposition 2.1.7. *The following are true:*

(i) For any $w, u, v \in W^P$ such that $c_{u,v}^w \neq 0$, $(\chi_w - \chi_v - \chi_u)(x_i) \geq 0$ for each $\alpha_i \in \Delta/\Delta(P)$.

(ii) The product $\odot$ in $H^*(G/P, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_i]$ is well defined, associative and commutative.

(iii) For any $(w^1, w^2, \ldots, w^s) \in (W^P)^s$*, the coefficient of* $[\Lambda_w]$ *in* $[\Lambda_{w^1}] \odot \cdots \odot [\Lambda_{w^s}]$ *is*

$$\prod_{\alpha_i \in \Delta/\Delta(P)} \tau_i^{(\chi_w - \sum_{j=1}^{s} \chi_{w^j})(x_i)}$$

times the coefficient of $[\Lambda_w]$ *in the usual cohomology product* $\prod_{k=1}^{s} [\Lambda_{w^k}]$*.*

Definition 2.1.8. *Define the product* $(H^*(G/P, \mathbb{Z}), \odot_0)$ *by*

$$[\Lambda_u] \odot_0 [\Lambda_v] := [\Lambda_u] \odot [\Lambda_v]\Big|_{\tau_1 = \tau_2 = \cdots = \tau_r = 0}$$

Note that as $\mathbb{Z}$-modules, $(H^*(G/P, \mathbb{Z}), \odot_0) = H^*(G/P, \mathbb{Z})$. By Proposition 2.1.5, the new product $\odot_0$ has standard product structure with the additional condition of setting all non L-movable structure constants to zero. We would also like to remark that $\odot_0$ satisfies Poincaré duality of the standard product structure.

Proposition 2.1.9. *If* P *is cominuscule in* G*, then product structure of* $(H^*(G/P, \mathbb{Z}), \odot_0)$ *and* $H^*(G/P, \mathbb{Z})$ *is the same.*

For the proof see [3, Lemma 19]. We now consider the groups $SL(n)$ and $Sp(2n)$ in more detail.

2.2. The special linear group $SL(n)$

Let $SL(n)$ be the special linear group on the vector space $\mathbb{C}^n$ with standard basis $\{e_1, e_2, \ldots, e_n\}$. Let $H \subseteq SL(n)$ be the standard maximal torus of diagonal matrices of determinant 1 and let B denote the standard Borel subgroup of upper triangular matrices (of determinant 1). Then the Lie algebra is

$$\mathfrak{h} = \{\mathbf{t} = \operatorname{diag}(t_1, t_2, \ldots, t_n) \mid \sum_{i=1}^{n} t_i = 0\}$$

and the set of roots is $R = \{\varepsilon_i - \varepsilon_j \mid i \neq j\} \subseteq \mathfrak{h}^*$ where $\varepsilon_i(\mathbf{t}) := t_i$. The set of positive roots is $R^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ and the set of simple roots is $\Delta = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1}\}$. It is well known that the Weyl group W can be identified with S_n, the permutation group on $[n]$. The simple reflections corresponding to Δ are given by the simple transpositions $s_i = (i, i+1)$. The action of W on $\mathfrak{h}^*$ is given by $w\varepsilon_i = \varepsilon_{w(i)}$. Let P be a standard parabolic subgroup of $SL(n)$ and let $\Delta \backslash \Delta(P) = \{\alpha_{a_1}, \alpha_{a_2}, \ldots, \alpha_{a_r}\}$. We associate P with the subset $a = \{a_1, a_2, \ldots, a_r\} \subseteq [n-1]$. The homogeneous space $SL(n)/P$ is $SL(n)$-equivariantly isomorphic to the space of partial flags

$$\mathrm{F}\ell(a, n) := \{V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_r \subset \mathbb{C}^n \mid \dim(V_i) = a_i\}.$$

Let $E_\bullet$ denote the standard complete flag where $E_i = \mathrm{Span}\{e_1, \ldots, e_i\}$ and denote the standard partial sub-flag by

$$\tilde{E} := E_{a_1} \subseteq E_{a_2} \subseteq \cdots \subseteq E_{a_r}.$$

It is easy to see that the map between $SL(n)/P$ and $\mathrm{F}\ell(a, n)$ is given by

$$gP \mapsto g\tilde{E} = gE_{a_1} \subseteq gE_{a_2} \subseteq \cdots \subseteq gE_{a_r}.$$

The set W^P is equal to

$$S_n(a) := \{(w(1), w(2), \ldots, w(n)) \in S_n \mid w(i) < w(i+1) \ \forall \ i \notin a\}$$

as subsets of S_n. Note that the length of any $w \in W$ is given by $\ell^A(w) := \#\{i < j \mid w(i) > w(j)\}$. The dimension of $\mathrm{F}\ell(a, n)$ is equal to $\sum_{i=1}^{r} a_i(a_{i+1} - a_i)$ where we set $a_0 = 0$ and $a_{r+1} = n$. If $a = [n-1]$, we denote the set of complete flags by $\mathrm{F}\ell(n)$. In general, if V

is a complex vector space, let $\mathrm{F}\ell(V)$ denote the complete flags on V. We will use script letters $\mathcal{F} := (F^1_\bullet, \ldots, F^s_\bullet) \in \mathrm{F}\ell(n)^s$ to denote s-tuples of complete flags. For any complete flag $F_\bullet = F_1 \subset F_2 \subset \ldots \subset F_n \in \mathrm{F}\ell(n)$ and $w \in S_n(a)$, we define the Schubert cell by

$$X^\circ_w(F_\bullet) := \{V_\bullet \in \mathrm{F}\ell(a,n) \mid \dim(V_i \cap F_j) = \#\{t \leq a_i : w(t) \leq j\} \quad \forall i,j\}.$$

If $F_\bullet = gE_\bullet$ for some $g \in SL(n)$, then $X^\circ_w(F_\bullet) = g\Lambda_w$. To see this, we consider the Bruhat decomposition of $SL(n)/P = \bigsqcup_{w \in S_n(a)} BwP$. If $V_\bullet \in X^\circ_w(E_\bullet)$ and $b \in B$, then

$$\dim(bV_i \cap E_j) = \dim(V_i \cap b^{-1}E_j) = \dim(V_i \cap E_j) \quad \forall\ i,j.$$

Hence $X^\circ_w(E_\bullet)$ is a union of B-orbits in $\mathrm{F}\ell(a,n)$. For each $w \in S_n(a)$, we have that $wP \mapsto w\tilde{E}_\bullet$ and $w\tilde{E}_\bullet \in X^\circ_w(E_\bullet)$. Thus $X^\circ_w(E_\bullet)$ can only consist of the single orbit BwP. It is easy to see that the $SL(n)$-equivariant action on the Schubert cells is exactly $gBwP = g\Lambda_w = X^\circ_w(gE_\bullet)$.

2.2.1. Proof of $(i) \Leftrightarrow (ii)$ in Theorem 1.1.2

In this section we prove $(i) \Leftrightarrow (ii)$ in Theorem 1.1.2 by showing that (2.2) is equivalent to (1.3). For any $w \in S_n(a)$, let w_i be the image of w under the projection $S_n(a) \to S_n(a_i)$.

Lemma 2.2.1. *Let $w, \dot{w} \in S_n(a)$ and $i \in [r]$. If $w_i = \dot{w}_i \in S_n(a_i)$, then the order of the following sets are equal:*

$$|R^+ \backslash R^+_{\mathfrak{l}_i} \cap w^{-1}R^+| = |R^+ \backslash R^+_{\mathfrak{l}_i} \cap \dot{w}^{-1}R^+|.$$

Proof. If $w_i = \dot{w}_i$, then

$$\{w(1), w(2), \ldots, w(a_i)\} = \{\dot{w}(1), \dot{w}(2), \ldots, \dot{w}(a_i)\}$$

and

$$\{w(a_i+1), w(a_i+2), \ldots, w(n)\} = \{\dot{w}(a_i+1), \dot{w}(a_i+2), \ldots, \dot{w}(n)\}$$

as unordered sets. Let $A, B \subseteq [n]$ denote these sets respectively. Then for any $\alpha = \varepsilon_a - \varepsilon_b \in R^+$, we have

$$w^{-1}\alpha \in R^+ \backslash R^+_{\mathfrak{l}_i} \Leftrightarrow a \in A \text{ and } b \in B \Leftrightarrow \dot{w}^{-1}\alpha \in R^+ \backslash R^+_{\mathfrak{l}_i}$$

since $w^{-1}(a), \dot{w}^{-1}(a) \in \{1, 2, \ldots, a_i\}$ and $w^{-1}(b), \dot{w}^{-1}(b) \in \{a_i+1, a_i+2, \ldots, n\}$. This proves the lemma. □

Proposition 2.2.2. *For any $w \in S_n(a)$ and $i \in [r]$, we have that $\chi_w(x_{a_i}) = \operatorname{codim}(X_{w_i})$.*

Proof. For any $\beta \in R^+ \backslash R^+_{\mathfrak{l}}$, the value $\beta(x_{a_i}) = 0$ if $\beta \in R^+_{\mathfrak{l}_i}$ and $\beta(x_{a_i}) = 1$ otherwise. Since $R^+ \backslash R^+_{\mathfrak{l}_i} \subseteq R^+ \backslash R^+_{\mathfrak{l}}$, it suffices to determine the size of $R^+ \backslash R^+_{\mathfrak{l}_i} \cap w^{-1}R^+$. By Lemmas 2.1.6 and 2.2.1, the order of this set is exactly equal to $\operatorname{codim}(X_{w_i})$. □

By Proposition 2.2.2, for any $i \in [r]$ we have

$$\begin{aligned} \big((\sum_{k=1}^{s} \chi_{w^k}) - \chi_1\big)(x_{a_i}) &= \sum_{k=1}^{s} \operatorname{codim}(X_{w_i^k}) - \dim \operatorname{Gr}(a_i, n) \\ &= \Big(\sum_{k=1}^{s} \Big(\sum_{j=1}^{a_i} n - a_i + j - w_i^k(j)\Big)\Big) - a_i(n - a_i) = 0. \end{aligned}$$

Thus Proposition 2.1.5 proves $(i) \Leftrightarrow (ii)$ in Theorem 1.1.2.

2.3. The symplectic group $Sp(2n)$

The same notation will be used to describe objects associated to $Sp(2n)$ (i.e. $H, B, P, \ldots$). If the groups $SL(2n)$ and $Sp(2n)$ are considered in the same context, then we will use superscripts $H^A, B^A, P^A, \ldots$ to denote objects for $SL(2n)$ and $H^C, B^C, P^C, \ldots$ for $Sp(2n)$.

Let $\mathbb{C}^{2n}$ be a $2n$ dimensional complex vector space with basis $\{e_1, e_2, \ldots, e_{2n}\}$. We define a skew-symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{C}^{2n}$ as follows:

$$\langle e_i, e_{2n+1-i}\rangle = 1 \text{ if } i < 2n+1-i \quad \text{and} \quad \langle e_i, e_j\rangle = 0 \text{ if } j \neq 2n+1-i.$$

Define the symplectic group on $\mathbb{C}^{2n}$ to be

$$Sp(2n) := \{A \in SL(2n) \mid A \text{ leaves the form } \langle\ ,\ \rangle \text{ invariant}\}.$$

Let H be the standard maximal torus of diagonal matrices in $Sp(2n)$ and let B be the standard Borel subgroup of upper triangular matrices in $Sp(2n)$. The Lie algebra $\mathfrak{h}$ is equal to

$$\mathfrak{h} = \{\mathbf{t} = \operatorname{diag}(t_1, t_2, \ldots, t_n, -t_n, -t_{n-1}, \ldots, -t_1)\}$$

and the set of roots is

$$R = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid i < j\} \cup \{\pm 2\varepsilon_i \mid i \in [n]\} \subseteq \mathfrak{h}^*$$

with positive roots

$$R^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{2\varepsilon_i \mid i \in [n]\} \subseteq R$$

where $\varepsilon_i(\mathbf{t}) = t_i$. Let $\Delta := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote the set of simple roots, where $\alpha_i :=$

$\varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n = 2\varepsilon_n$. The Weyl group of $Sp(2n)$ can be identified with the following subset of S_{2n}:

$$W^C := \{(w(1), w(2), \dots, w(2n)) \in S_{2n} \mid w(2n+1-i) = 2n+1-w(i)\}.$$

It is easy to see that any $w \in W$ is uniquely determined by the set $\{w(1), w(2), \dots, w(n)\}$ and that under the inclusion $W \subseteq S_{2n}$, the simple reflections corresponding to Δ are $s_i := (i, i+1)(2n-i, 2n-i+1)$ for $i < n$ and $s_n := (n, n+1)$. The action of any $w \in W$ on ε_i is given by

$$w\varepsilon_i = \varepsilon_{w(i)} \text{ if } w(i) \leq n \quad \text{and} \quad w\varepsilon_i = -\varepsilon_{w(2n+1-i)} \text{ if } w(i) > n.$$

Let P be the standard parabolic subgroup of $Sp(2n)$ corresponding to the set of simple roots $\Delta \backslash \Delta(P) := \{\alpha_{a_1}, \alpha_{a_2}, \dots, \alpha_{a_r}\}$ for the set $a := \{a_1 < a_2 < \cdots < a_r\} \subseteq [n]$. The homogeneous space $Sp(2n)/P$ is $Sp(2n)$-equivariantly isomorphic to the isotropic partial flag variety

$$\mathrm{IF}(a, 2n) := \{V_\bullet := V_1 \subseteq V_2 \subseteq \cdots \subseteq V_r \subseteq \mathbb{C}^{2n} \mid \dim V_i = a_i \text{ and } V_r \subseteq V_r^\perp\}$$

where $V^\perp := \{v \in \mathbb{C}^{2n} \mid \langle v, u \rangle = 0 \ \forall \ u \in V\}$. Observe that $V_r \subseteq V_r^\perp$ implies that $V_i \subseteq V_i^\perp$ for all $i \in [r]$. Also note that for any subspace $V \subseteq \mathbb{C}^{2n}$, we have that $\dim V^\perp = 2n - \dim(V)$. Thus the maximum dimension of an isotropic subspace in $\mathbb{C}^{2n}$ is n. The set W^P is equal to

$$S_{2n}^C(a) := \{(w(1), w(2), \dots, w(2n)) \in W^C \mid w(i) < w(i+1) \ \forall \ i \notin a\}.$$

The length of $w \in S_{2n}^C(a)$ is given by $\ell^C(w) := \frac{1}{2}(\ell^A(w) + |w(I^r) > n|)$. For any $i \in [r]$, define $\bar{a}_i := 2n+1-a_i$ and let $\bar{a} := \{\bar{a}_1, \dots, \bar{a}_r\}$. The natural inclusion of $W^C \subseteq S_{2n}$ gives a well defined injection $S_{2n}^C(a) \hookrightarrow S_{2n}^A(a)$ (there also exists a natural injection $S_{2n}^C(a) \hookrightarrow S_{2n}^A(a \cup \bar{a})$). Let

$$\mathrm{IF}(2n) := \{F_\bullet := F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{2n} = \mathbb{C}^{2n} \mid \dim F_i = i \text{ and } F_{2n-i} = F_i^\perp \ \forall\, i \in [n]\}$$

denote the set of complete isotropic flags on $\mathbb{C}^{2n}$. Note that $Sp(2n)$ acts transitively on $\mathrm{IF}(2n)$. For any $w \in S_{2n}^C(a)$ and $F_\bullet \in \mathrm{IF}(2n)$, define the Schubert cell in $\mathrm{IF}(a,n)$ as

$$\Phi_w^\circ(F_\bullet) := \{V_\bullet \in \mathrm{IF}(a,n) \mid \dim(V_i \cap F_j) = \#\{t \le a_i : w(t) \le j\} \quad \forall i,j\}.$$

If $E_\bullet$ is the standard complete isotropic flag on $\mathbb{C}^{2n}$ and $F_\bullet = gE_\bullet$ from some $g \in Sp(2n)$, then

$$\Phi_w^\circ(F_\bullet) = g\Lambda_w.$$

This fact follows from the analogous result for type A Schubert cells since under the natural inclusion $\mathrm{IF}(a,2n) \hookrightarrow \mathrm{F}\ell(a,2n)$, we have the scheme theoretic intersection

$$\Phi_w^\circ(F_\bullet) = X_w^\circ(F_\bullet) \cap \mathrm{IF}(a,2n).$$

2.3.1. Proof of $(i) \Leftrightarrow (ii)$ in Theorem 1.2.1

Similar to Section 2.2.1, we prove $(i) \Leftrightarrow (ii)$ in Theorem 1.2.1 by showing that (2.2) is equivalent to (1.8). For any $w \in S_{2n}^C(a)$, let w_i be the image of w under the projection $S_{2n}^C(a) \to S_{2n}^C(a_i)$.

Lemma 2.3.1. *Let $w, \dot{w} \in W^C \subseteq S_{2n}$ and $i \in [r]$. If $w_i = \dot{w}_i \in S_{2n}^C(a_i)$, then the order of* *the following sets are equal:*

$$|R^+\backslash R^+_{\mathfrak{l}_i} \cap w^{-1}R^+| = |R^+\backslash R^+_{\mathfrak{l}_i} \cap \dot{w}^{-1}R^+|.$$

Proof. If $w_i = \dot{w}_i$, then as unordered sets, we have

$$\begin{aligned}
\{w(1), w(2), \ldots, w(a_i)\} &= \{\dot{w}(1), \dot{w}(2), \ldots, \dot{w}(a_i)\} \\
\{w(a_i+1), w(a_i+2), \ldots, w(\bar{a}_i)\} &= \{\dot{w}(a_i+1), \dot{w}(a_i+2), \ldots, \dot{w}(\bar{a}_i)\} \\
\{w(\bar{a}_i+1), \ldots, w(2n)\} &= \{\dot{w}(\bar{a}_i+1), \ldots, \dot{w}(2n)\}.
\end{aligned}$$

Let $A, B, C \subseteq [2n]$ denote these sets respectively and define

$$A^+ := \{u \in A | \; u \leq n\}, \quad B^+ := \{u \in B | \; u \leq n\}, \quad C^+ := \{u \in C | \; u \leq n\}.$$

The set R^+ consists of roots of the form $\varepsilon_a - \varepsilon_b$ or $\varepsilon_a + \varepsilon_b$. If $\alpha = \varepsilon_a - \varepsilon_b$, then we have $w^{-1}\alpha \in R^+\backslash R^+_{\mathfrak{l}_i}$ if and only if one of the following are true:

- $a \in A^+$ and $b \in B^+ \cup C^+$
- $a \in A^+ \cup B^+$ and $b \in C^+$

If $\alpha = \varepsilon_a + \varepsilon_b$, then we have $w^{-1}\alpha \in R^+\backslash R^+_{\mathfrak{l}_i}$ if and only if one of the following are true:

- $a \in A^+$ and $b \in A^+ \cup B^+$
- $a \in A^+ \cup B^+$ and $b \in A^+$

Since the same conditions hold for $\dot{w}^{-1}\alpha \in R^+\backslash R^+_{\mathfrak{l}_i}$, the lemma is proved. □

Proposition 2.3.2. *For any $w \in S^C_{2n}(a)$ and $i \in [r]$, we have that*

$$\chi_w(x_{a_i}) = \operatorname{codim}(\Phi_{w_i}) + \operatorname{codim}(\Phi_{\bar{w}_i}).$$

Proof. For any $\beta \in R^+ \backslash R^+_{\mathfrak{l}}$, the value $\beta(x_{a_i}) = 0$ if $\beta \in R^+_{\mathfrak{l}_i}$ and $\beta(x_{a_i}) = 1$ or 2 otherwise. Define

$$R_2(a_i, w) := \{\alpha \in R^+ \backslash R^+_{\mathfrak{l}_i} \cap w^{-1}R^+ \mid \alpha(x_{a_i}) = 2\}.$$

By Lemmas 2.1.6 and 2.3.1, we have that

$$\chi_w(x_{a_i}) = \operatorname{codim}(\Phi_{w_i}) + |R_2(a_i, w)|.$$

Observe that $\alpha(x_{a_i}) = 2$ if and only if $\alpha = \varepsilon_a + \varepsilon_b$ where $a \leq b \leq a_i$. Hence we can identify $R_2(a_i, id)$ with the set of positive roots associated to the Lagrangian Grassmannian $Sp(2a_i)/P_{a_i} \simeq \mathrm{LG}(a_i, 2a_i)$ (i.e. the positive roots of $Sp(2a_i)$ where $\alpha(x_{a_i}) \neq 0$). The root $\alpha \in R_2(a_i, w) \subseteq R_2(a_i, id)$ if and only if $w^{-1}(a), w^{-1}(b) \in \{1, 2, \ldots, a_i\}$. Note that it suffices to know the values $\{w(1), \ldots, w(a_i)\}$ in order to determine if $\alpha \in R_2(a_i, w)$. Applying Lemma 2.1.6 to $\mathrm{LG}(a_i, 2a_i)$, we have that $R_2(a_i, w) = \operatorname{codim}(\Phi_{\bar{w}_i})$. □

Proposition 2.3.2 together with Proposition 2.1.5 immediately implies $(i) \Leftrightarrow (ii)$ in Theorem 1.2.1.

CHAPTER 3

Tangent spaces of flag varieties

In this chapter we complete the proofs of Theorems 1.1.2 and 1.2.1 by showing $(ii) \Leftrightarrow (iii)$ in both cases. In the first section, we first give some preliminary facts on the linear algebra of complex vector spaces. In the second and third section, we prove Theorems 1.1.2 and 1.2.1. In Section 3.4, we look at the connections of Theorem 1.1.2 to a question asked by Belkale-Kumar in [3]. Finally, in Section 3.5 we prove Theorem 1.1.3 which provides some necessary criteria for non-vanishing products of Schubert classes in the usual cohomology ring $H^*(\mathrm{F}\ell(a,n))$.

3.1. Some complex linear algebra

Lemmas 3.1.1 and 3.1.2 are basic facts about complex vector spaces.

Lemma 3.1.1. *Let $X = \bigoplus_{i=1}^r X_i$ be a complex vector space. Let $t = \{t_1, t_2, \ldots, t_r\}$ be a set of distinct positive integers. Define the action of t on X by $t(\sum_{i=1}^r x_i) = \sum_{i=1}^r t_i x_i$. Let V be a subspace of X. Then $t(V) = V$ if and only if $V = \bigoplus_{i=1}^r (V \cap X_i)$.*

Proof. Assume $t(V) = V$. Then X can be expressed as the direct sum of t-invariant subspaces $X = V \oplus V^\perp$ with respect to some inner product. Since the action of t on X is diagonal, it is also diagonal on V and $V^\perp$. Therefore V is a direct sum of its eigenspaces and each eigenspace is contained in an eigenspace of X. The reverse implication is trivial. □

Lemma 3.1.2. *Let $X = \bigoplus_{i=1}^r X_i$ be a complex vector space. Let $Y^1, Y^2, \ldots, Y^s$ be vector* *subspaces such that* $Y^k = \bigoplus_{i=1}^r Y_i^k$ *with* $Y_i^k \subseteq X_i$ *for all* $i \in [r]$. *If*

$$\dim\Big(\bigcap_{k=1}^s Y^k\Big) = \dim(X) - \sum_{k=1}^s \operatorname{codim}(Y^k) = 0$$

then

$$\dim\Big(\bigcap_{k=1}^s Y_i^k\Big) = \dim(X_i) - \sum_{k=1}^s \operatorname{codim}(Y_i^k) = 0$$

for each $i \in [r]$.

Proof. By the assumption, we have that

$$\dim\Big(\bigcap_{k=1}^s Y^k\Big) = \dim\Big(\bigcap_{k=1}^s \bigoplus_{i=1}^r Y_i^k\Big) = \sum_{i=1}^r \dim\Big(\bigcap_{k=1}^s Y_i^k\Big) = 0.$$

Hence, for each $i \in [r]$, we have that $\dim\Big(\bigcap_{k=1}^s Y_i^k\Big) = 0$. We also have that

$$\begin{aligned}\dim(X) - \sum_{k=1}^s \operatorname{codim}(Y^k) &= \sum_{k=1}^s \dim(Y^k) - (s-1)\dim(X)\\ &= \sum_{k=1}^s \Big(\sum_{i=1}^r \dim(Y_i^k)\Big) - (s-1)\sum_{i=1}^r \dim(X_i)\\ &= \sum_{i=1}^r \Big(\sum_{k=1}^s \dim(Y_i^k) - (s-1)\dim(X_i)\Big) = 0.\end{aligned}$$

For any vector subspaces, it is always true that

$$\sum_{k=1}^s \dim(Y_i^k) - (s-1)\dim(X_i) \leq \dim\Big(\bigcap_{k=1}^s Y_i^k\Big).$$

Thus $\sum_{k=1}^s \dim(Y_i^k) - (s-1)\dim(X_i) = 0$ and hence, for each $i \in [r]$, we have

$$\dim\Big(\bigcap_{k=1}^s Y_i^k\Big) = \dim(X_i) - \sum_{k=1}^s \operatorname{codim}(Y_i^k) = 0.$$

□

3.2. Tangent space of $\mathrm{F}\ell(a, n)$

Consider the injective map from $\mathrm{F}\ell(a, n)$ to the product $\prod_{i=1}^{r} \mathrm{Gr}(a_i, n)$ given by

$$V_\bullet \mapsto (V_1, V_2, \ldots, V_r).$$

It is well known that the tangent space of $\mathrm{Gr}(a_i, n)$ at the point V_i is canonically isomorphic with $\mathrm{Hom}(V_i, \mathbb{C}^n/V_i)$. This induces an injective map on tangent spaces

$$T_{V_\bullet}\mathrm{F}\ell(a, n) \hookrightarrow \bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^n/V_i).$$

Let $i_j : V_j \hookrightarrow V_{j+1}$ and $\rho_j : \mathbb{C}^n/V_j \twoheadrightarrow \mathbb{C}^n/V_{j+1}$ denote the naturally induced maps from the flag structure of $V_\bullet$ and let $\phi = (\phi_1, \phi_2, \cdots, \phi_r)$ denote an element of $\bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^n/V_i)$. For any $i \in [r+1]$, let $Q_i := \mathrm{Span}\{e_{a_{i-1}+1}, e_{a_{i-1}+2}, \ldots, e_{a_i}\}$. This defines a splitting of

$$\mathbb{C}^n = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{r+1}. \tag{3.1}$$

Proposition 3.2.1. *The tangent space of* $\mathrm{F}\ell(a, n)$ *at the point* $V_\bullet$ *is given by*

$$T_{V_\bullet}\mathrm{F}\ell(a, n) \simeq \{\phi \in \bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^n/V_i) \mid \rho_j \circ \phi_j = \phi_{j+1} \circ i_j \ \forall j \in [r-1]\}. \tag{3.2}$$

In other words, the set of $\phi \in \bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^n/V_i)$ *such that the following diagram commutes:*

$$\begin{array}{ccccccc}
V_1 & \xhookrightarrow{i_1} & V_2 & \xhookrightarrow{i_2} & \cdots & \xhookrightarrow{i_{r-1}} & V_r \\
\downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & & & \downarrow{\scriptstyle \phi_r} \\
\mathbb{C}^n/V_1 & \xrightarrow[\rho_1]{} & \mathbb{C}^n/V_2 & \xrightarrow[\rho_2]{} & \cdots & \xrightarrow[\rho_{r-1}]{} & \mathbb{C}^n/V_r
\end{array}$$

Proof. Since both sides of the equation (3.2) are vector subspaces of $\bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^n/V_i)$

of the same dimension, it suffices to show the $LHS \subseteq RHS$. Since $\mathrm{F}\ell(a,n)$ is homogeneous, without loss of generality we can assume $V_\bullet = \tilde{E}_\bullet$. Fix the splitting of $\mathbb{C}^n$ given in (3.1). We have that

$$\mathrm{Hom}(E_{a_i}, \mathbb{C}^n/E_{a_i}) \simeq \mathrm{Hom}(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_i, Q_{i+1} \oplus Q_{i+2} \oplus \cdots \oplus Q_{r+1}).$$

For any $\phi \in \bigoplus_{i=1}^r \mathrm{Hom}(E_{a_i}, \mathbb{C}^n/E_{a_i})$ and $i \in [r]$ define

$$V_i^\phi := \langle e_1 + \phi_i(e_1), e_2 + \phi_i(e_2), \ldots, e_{a_i} + \phi_i(e_{a_i})\rangle.$$

It is easy to see that up to first order we have that

$$T_{\tilde{E}_\bullet}\mathrm{F}\ell(a,n) \simeq \{\phi \in \bigoplus_{i=1}^{r} \mathrm{Hom}(E_{a_i}, \mathbb{C}^n/E_{a_i}) \mid V_1^\phi \subseteq V_2^\phi \subseteq \cdots \subseteq V_r^\phi\}.$$

Therefore any $\phi \in T_{\tilde{E}_\bullet}\mathrm{F}\ell(a,n)$ must satisfy the commuting conditions given in (3.2). $\square$

If we consider the splitting (3.1) together with the commuting conditions in (3.2), we have the following simplification of the tangent space of $\mathrm{F}\ell(a,n)$ at the point $\tilde{E}$:

$$T_{\tilde{E}}\mathrm{F}\ell(a,n) \simeq \bigoplus_{i=1}^{r} \mathrm{Hom}(Q_i, Q_{i+1} \oplus Q_{i+2} \oplus \cdots \oplus Q_{r+1}) \simeq \bigoplus_{i<j}^{r+1} \mathrm{Hom}(Q_i, Q_j). \tag{3.3}$$

Note that $\mathrm{Hom}(Q_i, Q_j)$ is canonically isomorphic to the tangent space $T_{Q_i}(Gr(b_i, Q_i \oplus Q_j))$. We now describe the tangent space of a Schubert cell $X_w^\circ(F_\bullet) \subseteq \mathrm{F}\ell(a,n)$. To do this we need the notion of induced flags. For any complete flag $F_\bullet \in \mathrm{F}\ell(n)$ and any subspace $V \subseteq \mathbb{C}^n$, we have the induced complete flags on V and $\mathbb{C}^n/V$ given by the intersection of V with $F_\bullet$ and the projection $\mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/V$ of $F_\bullet$. We denote these induced flags by $F_\bullet(V)$ and $F_\bullet(\mathbb{C}^n/V)$ respectively. Consider the following description of the tangent space of a Schubert cell in the Grassmannian. For the proof see [18, Section 2.7].

Lemma 3.2.2. *Let $r < n$ and $w \in S_n(r)$ and let $F_\bullet \in \mathrm{F}\ell(n)$. For any $V \in X_w^\circ(F_\bullet) \subseteq \mathrm{Gr}(r,n)$, the tangent space of the Schubert cell at the point V is given by*

$$T_V X_w^\circ(F_\bullet) = \{\phi \in \mathrm{Hom}(V, \mathbb{C}^n/V) \mid \phi(F_j(V)) \subseteq F_{w(j)-j}(\mathbb{C}^n/V) \quad \forall\, j \in [r]\}.$$

We generalize this description to Schubert cells on the partial flag variety $\mathrm{F}\ell(a,n)$. For any $i \in [r]$ and $w \in S_n(a)$, let w_i denote the image of w under the projection $S_n(a) \twoheadrightarrow S_n(a_i)$.

Proposition 3.2.3. *The tangent space of the Schubert cell $X_w^\circ(F_\bullet)$ at the point $V_\bullet$ is given by*

$$T_{V_\bullet} X_w^\circ(F_\bullet) = \{\phi \in T_{V_\bullet}\mathrm{F}\ell(a,n) \mid \phi_i(F_j(V_i)) \subseteq F_{w_i(j)-j}(\mathbb{C}^n/V_i) \quad \forall\, i,j\}. \tag{3.4}$$

Proof. Similarly to Proposition 3.2.1, both sides of the equation (3.4) are vector subspaces of $T_{V_\bullet}\mathrm{F}\ell(a,n)$ of the same dimension and therefore suffices to show the $LHS \subseteq RHS$. Fix $i \in [r]$ and consider the map $f : \mathrm{F}\ell(a,n) \to \mathrm{Gr}(a_i, n)$ given by $f(V_\bullet) = V_i$. Clearly this map is surjective and induces a surjective map f_* on the tangent spaces at the point $V_\bullet$ given by $f_*(\phi) = \phi_i$. By Lemma 3.2.2, it suffices to show that $f_*(T_{V_\bullet} X_w^\circ(F_\bullet)) = T_{V_i} X_{w^i}^\circ(F_\bullet)$. However this is true since $f(X_w^\circ(F_\bullet)) = X_{w^i}^\circ(F_\bullet)$. □

Consider the point $\tilde{E} \in \mathrm{F}\ell(a,n)$ and choose $F_\bullet$ such that $\tilde{E} \in X_w^\circ(F_\bullet)$. We find that if $F_\bullet = lw^{-1}E_\bullet$ for some $l \in L$, then the space $T_{\tilde{E}} X_w^\circ(F_\bullet)$ decomposes "nicely" with respect to the decomposition (3.3).

3.2.1. Proof of Theorem 1.1.2

Consider the Levi subgroup $L \subseteq P$. It is easy to see that L decomposes into the product

$$L \simeq \{(g_1, \dots, g_{r+1}) \in GL(Q_1) \times \cdots \times GL(Q_{r+1}) \mid \prod_{i=1}^{r+1} \det(g_i) = 1\}. \tag{3.5}$$

Definition 3.2.4. *For any partial flag variety* $\mathrm{F}\ell(a, n)$ *and* $w \in S_n(a)$*, define*

$$\mathrm{F}\ell_{L,w}(n) := \{F_\bullet \in \mathrm{F}\ell(n) \mid F_\bullet = lw^{-1}E_\bullet \textit{ for some } l \in L\}.$$

Proposition 3.2.5. *With respect to equation* (3.3)*, if* $F_\bullet \in \mathrm{F}\ell_{L,w}(n)$*, then*

$$T_{\tilde{E}} X_w^\circ(F_\bullet) = \bigoplus_{i<j}^{r+1} \big(\mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}} X_w^\circ(F_\bullet)\big).$$

Proof. Let $t = \{t_1, t_2, \dots, t_{r+1}\}$ be a set of distinct positive integers. Let t act on $\mathbb{C}^n$ by scalar multiplication with respect to the splitting (3.1). Since $F_\bullet = lw^{-1}E_\bullet$ for some $l \in L$, each $F_j(E_{a_i})$ is fixed by the t action. By Lemma 3.1.1, we have

$$F_j(E_{a_i}) = \bigoplus_{m=1}^{i} (F_j(E_{a_i}) \cap Q_m)$$

and

$$F_{w_i(j)-j}(\mathbb{C}^n/E_{a_i}) = \bigoplus_{m=i+1}^{r+1} (F_{w_i(j)-j}(\mathbb{C}^n/E_{a_i}) \cap Q_m).$$

Therefore, for any $\phi = (\phi_1, \phi_2, \dots, \phi_r) \in T_{\tilde{E}} X_w^\circ(F_\bullet)$, the map ϕ_i can be written as the sum

$$\phi_i = \sum_{\substack{0<m_1\leq i \\ i<m_2\leq r+1}} \phi_{m_1,m_2} \quad \text{where } \phi_{m_1,m_2}(F_j \cap Q_{m_1}) \subseteq F_{w_i(j)-j} \cap Q_{m_2}. \tag{3.6}$$

Note that $\phi_{m_1,m_2} \in \mathrm{Hom}(Q_{m_1}, Q_{m_2})$. Let $t' = \{t_{i,j}\}_{i<j}$ be a set of distinct positive integers and let t' act on $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ by

$$t'\big(\sum_{i<j}^{r+1} \phi_{i,j}\big) = \sum_{i<j}^{r+1} t_{i,j}\phi_{i,j}$$

under the direct sum given in (3.3). By equation (3.6), we have that $t'\big(T_{\tilde{E}}X_w^\circ(F_\bullet)\big) = T_{\tilde{E}}X_w^\circ(F_\bullet)$. Thus by Lemma 3.1.1, the proposition is proved. □

Lemma 3.2.6. *Let $w \in S_n(a)$ and $\{i < j\} \subseteq [r+1]$. Choose m such that $i \leq m < j$ and let*

$$M^- = \{w(1), w(2), \ldots, w(a_m)\} \text{ and } M^+ = \{w(a_m+1), w(a_m+2), \ldots, w(n)\}.$$

Let

$$p_k^- := \#\{\alpha \in M^- \mid \alpha \leq w(a_{i-1}+k)\} \text{ and } p_k^+ := \#\{\alpha \in M^+ \mid \alpha \leq w(a_{j-1}+k)\}.$$

Then for any $k \in [b_i]$ and $l \in [b_j - 1]$, we have

$$w_{i,j}(k) = k + l \;\Leftrightarrow\; p_l^+ \leq w(a_{i-1}+k) - p_k^- < p_{l+1}^+$$

and

$$w_{i,j}(k) = k + b_j \;\Leftrightarrow\; p_{b_j}^+ \leq w(a_{i-1}+k) - p_k^-.$$

Proof. By the definition of $w_{i,j}$, $w_{i,j}(k) = k + l$ if and only if $w(a_{i-1}+k)$ is greater than exactly l elements in the set $\{w(a_{j-1}+1), w(a_{j-1}+2), \ldots, w(a_{j-1}+b_j)\}$. This is equivalent to

$$w(a_{j-1}+l) < w(a_{i-1}+k) < w(a_{j-1}+l+1)$$

which reduces to

$$p_l^+ + p_k^- \leq w(a_{i-1}+k) < p_{l+1}^+ + p_k^-.$$

In the case that $l = b_j$, we have that $w_{i,j}(k) = k + b_j$ if and only if $w(a_{i-1}+k)$ is greater than every element in the set $\{w(a_{j-1}+1), w(a_{j-1}+2), \ldots, w(a_{j-1}+b_j)\}$. This is equivalent to

$$p_{b_j}^+ + p_k^- \leq w(a_{i-1}+k).$$

□

Proposition 3.2.7. *For any* $\{i < j\} \subseteq [r+1]$*, identify* $T_{Q_i}\mathrm{Gr}(b_i, Q_i \oplus Q_j) \simeq \mathrm{Hom}(Q_i, Q_j)$*. If* $F_\bullet \in F\ell_{L,w}(n)$*, then*

$$\mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}} X_w^\circ(F_\bullet) = T_{Q_i} X_{w_{i,j}}^\circ(F_\bullet(Q_i \oplus Q_j))$$

where $F_\bullet(Q_i \oplus Q_j)$ *is the complete flag on* $Q_i \oplus Q_j$ *induced from* $F_\bullet$*.*

Proof. For any $(\phi_1, \phi_2, \cdots, \phi_r) \in T_{\tilde{E}}\mathrm{F}\ell(a, n)$ write

$$(\phi_1, \phi_2, \cdots, \phi_r) = \sum_{i<j}^{r+1} \phi_{i,j}$$

under the decomposition (3.3). Fix $\{i < j\} \subseteq [r+1]$ and choose m such that $i \leq m < j$. Then we have

$$\phi_m = \sum_{\substack{0<m_1\leq m \\ m<m_2\leq r+1}} \phi_{m_1,m_2}. \tag{3.7}$$

Observe that $\phi_{i,j}$ is included in the above summation (3.7). If $(\phi_1, \phi_2, \cdots, \phi_r) \in T_{\tilde{E}} X_w^\circ(F_\bullet)$, then

$$\phi_m(F_l(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m)) \subseteq F_{w_m(l)-l}(Q_{m+1} \oplus Q_{m+2} \oplus Q_{r+1}) \quad \forall\ l \in [a_m].$$

Define the set $M := \{w(1), w(2), \ldots, w(a_m)\}$ and for any $k \in [b_i]$, let

$$p_k := \#\{\alpha \in M \mid \alpha \leq w(a_{i-1} + k)\}.$$

Observe that $w_m(p_k) = w(a_i + k)$ and that $M = M^-$ and $p_k = p_k^-$ in Lemma 3.2.6. Since $F_\bullet \in F\ell_{L,w}(n)$, we have that p_k is the smallest number such that the flag $F_{p_k}(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m)$ induces the flag $F_k(Q_i)$ on Q_i. By Lemma 3.2.6, we have that the flag $F_{w_m(p_k)-p_k}(Q_{m+1} \oplus Q_{m+2} \oplus \cdots \oplus Q_{r+1})$ induces the flag $F_l(Q_j)$ on Q_j if and only if $w_{i,j}(k) = l + k$. Hence the map $\phi_{i,j}$ in the sum (3.7) of ϕ_m satisfies

$$\phi_{i,j}(F_k(Q_i)) \subseteq F_{w_{i,j}(k)-k}(Q_j) \tag{3.8}$$

Note that this result is independent of choice of $m \in \{i, i+1, \ldots, j-1\}$. This implies that

$$\mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}} X_w^\circ(F_\bullet) \subseteq T_{Q_i} X_{w_{i,j}}^\circ(F_\bullet(Q_i \oplus Q_j)).$$

To see the reverse containment, let $\phi_{i,j} \in T_{Q_i} X_{w_{i,j}}^\circ(F_\bullet(Q_i \oplus Q_j))$. We realize $\phi_{i,j}$ as an element of $T_{\tilde{E}}\mathrm{F}\ell(a,n)$ by setting $\phi_m = \phi_{i,j}$ under the sum (3.7) if $m \in \{i, i+1, \ldots, j-1\}$ and $\phi_m \equiv 0$ otherwise. It is easy to see that

$$\phi = (\phi_1, \phi_2, \cdots, \phi_r) \in \mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}} X_w^\circ(F_\bullet)$$

$\square$

Proof of *(ii)* $\Rightarrow$ *(iii)* **in Theorem 1.1.2:** If $(w^1, w^2, \ldots, w^s)$ is L-movable, then for a generic s-tuple $(l_1, l_2, \ldots l_s) \in L^s$ we have

$$\dim\Big(\bigcap_{i=k}^{s} T_{\tilde{E}} X_{w^k}^\circ(F_\bullet^k)\Big) = \dim \mathrm{F}\ell(a,n) - \sum_{k=1}^{s} \mathrm{codim}(X_{w^k}^\circ(F_\bullet^k)) = 0$$

where $F_\bullet^k = l_k(w^k)^{-1}E_\bullet$. By Proposition 3.2.5 and 3.2.7,

$$\bigcap_{k=1}^{s} \big(T_{\tilde{E}} X^\circ_{w^k}(F^k_\bullet)\big) = \bigoplus_{i<j}^{r+1} \Big(\bigcap_{k=1}^{s} T_{Q_i} X^\circ_{w^k_{i,j}}(F^k_\bullet(Q_i \oplus Q_j))\Big).$$

Hence by Lemma 3.1.2, for each $\{i < j\} \subseteq [r+1]$ we have

$$\dim\Big(\bigcap_{k=1}^{s} T_{Q_i} X^\circ_{w^k_{i,j}}(F^k_\bullet(Q_i \oplus Q_j))\Big) =$$

$$\dim(\mathrm{Hom}(Q_i, Q_j)) - \sum_{k=1}^{s} \mathrm{codim}\big(T_{Q_i} X^\circ_{w^k_{i,j}}(F^k_\bullet(Q_i \oplus Q_j))\big) = 0.$$

Thus Proposition 2.1.2(ii) proves $(ii) \Rightarrow (iii)$ in Theorem 1.1.2. □

Proof of $(iii) \Rightarrow (ii)$ in Theorem 1.1.2: Assume that $\prod_{k=1}^{s}[X_{w^k_{i,j}}] =$ a nonzero multiple of a class of a point in $H^*(\mathrm{Gr}(b_i, b_i+b_j))$ for all $\{i < j\} \subseteq [r+1]$. Since $\dim(Q_i \oplus Q_j) = b_i + b_j$, we can identify the homogeneous space $\mathrm{Gr}(b_i, b_i + b_j)$ with $SL(Q_i \oplus Q_j)/P_{i,j}$ where $P_{i,j}$ denotes the stabilizer of $Q_i \subseteq Q_i \oplus Q_j$. Since $P_{i,j}$ is maximal, by Proposition 2.1.9 we have that $(w^1_{i,j}, w^2_{i,j}, \dots, w^s_{i,j})$ is $L_{i,j}$-movable. Hence for generic $\mathcal{F}_{i,j} \in \prod_{k=1}^{s} \mathrm{F}\ell_{L_{i,j}, w^k_{i,j}}(Q_i \oplus Q_j)$, we have

$$\dim\Big(\bigcap_{k=1}^{s} T_{Q_i} X^\circ_{w^k_{i,j}}(F^k_{i,j\,\bullet})\Big) = \dim(\mathrm{Hom}(Q_i, Q_j)) - \sum_{k=1}^{s} \mathrm{codim}(X_{w^k_{i,j}}(F^k_{i,j\,\bullet})) = 0 \tag{3.9}$$

for each $i < j$. By Lemma 3.2.8 below, for every $i < j$, there exists a non-empty open set $U_{i,j} \subseteq \prod_{k=1}^{s} \mathrm{F}\ell_{L,w^k}(n)$ such that for every $\mathcal{F}_{i,j} \in (\psi_{w^1}, \psi_{w^2}, \dots, \psi_{w^s})(U_{i,j})$, equation (3.9) is satisfied. Choose $\mathcal{H} \in \bigcap_{i<j}^{r+1} U_{i,j}$. Then by Proposition 3.2.7,

$$\dim\Big(\bigcap_{k=1}^{s} T_{\tilde{E}_\bullet} X^\circ_{w^k}(H^k_\bullet)\Big) = \sum_{i<j}^{r+1} \dim\Big(\bigcap_{k=1}^{s} T_{Q_i} X^\circ_{w^k_{i,j}}(H^k_\bullet(Q_i \oplus Q_j))\Big) = 0.$$

By the codimension condition (1.2), the intersection of the Schubert cells $X^\circ_{w^k}(H^k_\bullet)$ is transverse at the point $\tilde{E}_\bullet$. Thus Proposition 2.1.2 proves $(iii) \Rightarrow (ii)$ in Theorem 1.1.2. □

Lemma 3.2.8. *Fix $\{i < j\} \subseteq [r+1]$. Let $L_{i,j}$ be the Levi subgroup of the parabolic $P_{i,j} \subseteq SL(Q_i \oplus Q_j)$ which stabilizes the space Q_i. Then for any $w \in S_n(a)$, the map*

$$\psi_w : \mathrm{F}\ell_{L,w}(n) \to \mathrm{F}\ell_{L_{i,j},w_{i,j}}(Q_i \oplus Q_j)$$

given by $\psi_w(F_\bullet) = F_\bullet(Q_i \oplus Q_j)$ is well defined and surjective.

Proof. Let $l = (g_1, g_2, \ldots, g_{r+1}) \in L$ with respect to equation (3.5). By the definition of $w_{i,j}$, we have $\psi_w(lw^{-1}E_\bullet) = (g_i, g_j)w_{i,j}^{-1}E_\bullet(Q_i \oplus Q_j)$. Thus ψ_w is well defined. Since ψ_w is L-equivariant and L acts transitively on $\mathrm{F}\ell_{L_{i,j},w_{i,j}}(Q_i \oplus Q_j)$, we also have that ψ_w is surjective. □

3.3. Tangent space of $\mathrm{IF}(a, 2n)$

We now describe the tangent space of $\mathrm{IF}(a, 2n)$ at some point $V_\bullet$. Since $\mathrm{IF}(a, 2n) \hookrightarrow \mathrm{F}\ell(a, 2n)$, we have a natural injection of tangent spaces

$$T_{V_\bullet}\mathrm{IF}(a, 2n) \hookrightarrow T_{V_\bullet}\mathrm{F}\ell(a, 2n) \hookrightarrow \bigoplus_{i=1}^{r} T_{V_i}\mathrm{Gr}(a_i, 2n).$$

There is also a natural embedding of $\mathrm{IF}(a, 2n) \hookrightarrow \prod_{i=1}^{r} \mathrm{IG}(a_i, 2n)$. This embedding induces another map on tangent spaces:

$$T_{V_\bullet}\mathrm{IF}(a, 2n) \hookrightarrow \bigoplus_{i=1}^{r} T_{V_i}\mathrm{IG}(a_i, 2n) \hookrightarrow \bigoplus_{i=1}^{r} T_{V_i}\mathrm{Gr}(a_i, 2n).$$

Let $\phi = (\phi_1, \phi_2, \ldots, \phi_r) \in \bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^{2n}/V_i)$. We have the following characterization of the tangent space of $T_{V_\bullet}\mathrm{IF}(a, 2n)$ with respect to the above two embeddings:

Proposition 3.3.1. *For any* $V_\bullet \in \mathrm{IF}(a, 2n)$*, the tangent space*

$$T_{V_\bullet}\mathrm{IF}(a, 2n) \simeq \{\phi \in T_{V_\bullet}\mathrm{F}\ell(a, 2n) \mid \phi_i \in T_{V_i}\mathrm{IG}(a_i, 2n)\}.$$

In other words

$$T_{V_\bullet}\mathrm{IF}(a, 2n) \simeq T_{V_\bullet}\mathrm{F}\ell(a, 2n) \cap \bigoplus_{i=1}^{r} T_{V_i}\mathrm{IG}(a_i, 2n) \subseteq \bigoplus_{i=1}^{r} \mathrm{Hom}(V_i, \mathbb{C}^{2n}/V_i).$$

Proof. We have that $\mathrm{IF}(a, 2n) = \mathrm{F}\ell(a, 2n) \cap \prod_{i=1}^{r} \mathrm{IG}(a_i, 2n)$ as subschemes of $\prod_{i=1}^{r} \mathrm{Gr}(a_i, 2n)$ and hence the proposition is true for generic $V_\bullet \in \mathrm{IF}(a, 2n)$. Since $\mathrm{IF}(a, 2n)$ is homogeneous, the proposition holds for all $V_\bullet \in \mathrm{IF}(a, 2n)$. □

We now take a closer look at $T_V\mathrm{IG}(d, 2n)$ for $d \leq n$. Consider the map $\psi : \mathrm{Hom}(V, \mathbb{C}^{2n}/V) \to \mathrm{Hom}(V, V^*)$ where

$$\psi(\phi)(v) = \langle \phi(v), * \rangle.$$

The proof of the following lemma can be found in [4].

Lemma 3.3.2. *For any* $V \in \mathrm{IG}(d, 2n)$*, we have*

$$T_V\mathrm{IG}(d, 2n) \simeq \psi^{-1}(\mathrm{sym}^2 V^*)$$

where $\mathrm{sym}^2 V^* \subseteq \mathrm{Hom}(V, V^*)$ *is the space of symmetric bilinear forms on* V*. Equivalently, we have*

$$T_V\mathrm{IG}(d, 2n) \simeq \{\phi \in \mathrm{Hom}(V, \mathbb{C}^{2n}/V) \mid \langle v, \phi(v') \rangle = \langle v', \phi(v) \rangle \ \forall\ v, v' \in V\}. \tag{3.10}$$

Recall the definitions of $I_i, \bar{I}_i$ and $\tilde{I}$ found in Section 1.2.1. For any $i \in [r]$, let

$$Q_i = \mathrm{Span}\{e_k \mid k \in I_i\} \quad \text{and} \quad \bar{Q}_i = \mathrm{Span}\{e_k \mid k \in \bar{I}_i\}$$

and

$$\tilde{Q} = \mathrm{Span}\{e_k \mid k \in \tilde{I}\}.$$

Clearly $\bar{Q}_i = V/Q_i^{\perp} = Q_i^*$ and we have the splitting:

$$\mathbb{C}^{2n} = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r \oplus \tilde{Q} \oplus \bar{Q}_r \oplus \cdots \oplus \bar{Q}_2 \oplus \bar{Q}_1. \tag{3.11}$$

Proposition 3.3.3. *With respect to the splitting* (3.11)*, we have the following description of the tangent space of* $\mathrm{IF}(a, 2n)$ *at the standard partial isotropic flag* $\tilde{E}_\bullet = E_{a_1} \subseteq E_{a_2} \subseteq \cdots \subseteq E_{a_r}$*:*

$$T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n) \simeq \bigoplus_{i<j}^{r} \big(\mathrm{Hom}(Q_i, Q_j) \oplus \mathrm{Hom}(Q_i, \bar{Q}_j)\big) \oplus \bigoplus_{i=1}^{r} \big(\mathrm{Hom}(Q_i, \tilde{Q}) \oplus \mathrm{sym}^2 Q_i^*\big). \tag{3.12}$$

Proof. Since $T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n) \subseteq T_{\tilde{E}_\bullet}\mathrm{F}\ell(a, 2n)$, we have that

$$T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n) \subseteq \bigoplus_{i<j}^{r} \mathrm{Hom}(Q_i, Q_j) \oplus \bigoplus_{i\neq j}^{r} \mathrm{Hom}(Q_i, \bar{Q}_j) \oplus \bigoplus_{i=1}^{r} \big(\mathrm{Hom}(Q_i, \tilde{Q}) \oplus \mathrm{Hom}(Q_i, \bar{Q}_i)\big).$$

If $\phi = (\phi_1, \phi_2, \ldots, \phi_r) \in T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n)$, the condition that $\phi_i \in T_{E_{a_i}}\mathrm{IG}(a_i, 2n)$ gives that

$$\langle v, \phi_i(\hat{v})\rangle = \langle \hat{v}, \phi_i(v)\rangle \quad \forall\ v, \hat{v} \in E_{a_i}.$$

With respect to (3.11), write

$$v = q_1 + q_2 + \cdots + q_i \text{ and } \hat{v} = \hat{q}_1 + \hat{q}_2 + \cdots + \hat{q}_i$$

and

$$\phi_i = \sum_{j=1}^{i} \left(\tilde{\phi}_j + \sum_{k=i+1}^{r} \phi_{j,k} + \sum_{k=1}^{r} \bar{\phi}_{j,k}\right)$$

where $\phi_{j,k} \in \mathrm{Hom}(Q_j, Q_k)$, $\bar{\phi}_{j,k} \in \mathrm{Hom}(Q_j, \bar{Q}_k)$ and $\tilde{\phi}_j \in \mathrm{Hom}(Q_j, \tilde{Q})$. By Lemma 3.3.2, we have that

$$\sum_{j=1}^{i}\sum_{k=1}^{r}\bar{\phi}_{j,k} \in \mathrm{sym}^2 E_{a_i}^*$$

and hence $\bar{\phi}_{j,k} = \bar{\phi}_{k,j}^*$. Thus ϕ_i is completely determined by the sum

$$\sum_{j=1}^{i}\Big(\tilde{\phi}_j + \bar{\phi}_{j,j} + \sum_{k=i+1}^{r}(\phi_{j,k} + \bar{\phi}_{j,k})\Big).$$

Since $\mathbb{C}^{2n}/Q_j^{\perp} = \bar{Q}_j$, we have that

$$\langle v, \phi_i(\hat{v})\rangle = \sum_{j=1}^{r}\langle q_j, \bar{\phi}_{j,j}(\hat{q}_j)\rangle = \sum_{j=1}^{r}\langle \hat{q}_j, \bar{\phi}_{j,j}(q_j)\rangle = \langle \hat{v}, \phi_i(v)\rangle.$$

The commuting conditions in (3.2) give that $\langle q_j, \bar{\phi}_{j,j}(\hat{q}_j)\rangle = \langle \hat{q}_j, \bar{\phi}_{j,j}(q_j)\rangle$ for each $j \in [r]$. Hence if we identify $\mathrm{Hom}(Q_i, \bar{Q}_j) \simeq \mathrm{Hom}(Q_j, \bar{Q}_i)$ by the dual map, we have

$$T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n) \subseteq \bigoplus_{i<j}^{r}\big(\mathrm{Hom}(Q_i, Q_j) \oplus \mathrm{Hom}(Q_i, \bar{Q}_j)\big) \oplus \bigoplus_{i=1}^{r}\big(\mathrm{Hom}(Q_i, \tilde{Q}) \oplus \mathrm{sym}^2 Q_i^*\big).$$

Since the dimensions of these vector spaces are equal, we have an isomorphism. □

We now look at tangent space of the Schubert cell $\Phi_w^\circ(F_\bullet) \subseteq \mathrm{IF}(a, 2n)$. For any $i \in [r]$ and $w \in S_{2n}^C(a)$, let w_i denote the image of w under the projection $S_{2n}^C(a) \twoheadrightarrow S_{2n}^C(a_i)$.

Proposition 3.3.4. *Let $w \in S_{2n}^C(a)$ and let $F_\bullet \in \mathrm{IF}(2n)$. Let $V_\bullet \in \Phi_w^\circ(F_\bullet) \subseteq \mathrm{IF}(a, 2n)$. The tangent space of the Schubert cell $\Phi_w^\circ(F_\bullet)$ at the point $V_\bullet$ is given by*

$$T_{V_\bullet}\Phi_w^\circ(F_\bullet) = \{\phi \in T_{V_\bullet}\mathrm{IF}(a, 2n) \mid \phi_i(F_j(V_i)) \subseteq F_{w_i(j)-j}(\mathbb{C}^{2n}/V_i) \quad \forall i, j\}. \tag{3.13}$$

Proof. As schemes, we have that $\Phi_w^\circ(F_\bullet) = X_w^\circ(F_\bullet) \cap \mathrm{IF}(a, 2n) \subseteq \mathrm{F}\ell(a, 2n)$. Therefore

$$T_{V_\bullet}\Phi_w^\circ(F_\bullet) = T_{V_\bullet}X_w^\circ(F_\bullet) \cap T_{V_\bullet}\mathrm{IF}(a, 2n).$$

Since the values $\{w_i(1), \ldots, w_i(a_i)\}$ for any $w \in S_{2n}^C(a)$ are fixed under the projection $S_{2n}^C(a_i) \to S_{2n}(a_i)$, the proposition is proved. $\square$

3.3.1. Proof of Theorem 1.2.1

For any $w \in S_{2n}^C(a)$ and $F_\bullet \in \mathrm{IF}(2n)$ such that $\tilde{E} \in \Phi_w^\circ(F_\bullet)$, let

$$\begin{aligned} H_{i,j}(w, F_\bullet) &:= \mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}_\bullet}\Phi_w^\circ(F_\bullet) \\ \bar{H}_{i,j}(w, F_\bullet) &:= \mathrm{Hom}(Q_i, \bar{Q}_j) \cap T_{\tilde{E}_\bullet}\Phi_w^\circ(F_\bullet) \\ \tilde{H}_i(w, F_\bullet) &:= \mathrm{Hom}(Q_i, \tilde{Q}) \cap T_{\tilde{E}_\bullet}\Phi_w^\circ(F_\bullet) \\ S_i(w, F_\bullet) &:= \mathrm{sym}^2 Q_i^* \cap T_{\tilde{E}_\bullet}\Phi_w^\circ(F_\bullet) \end{aligned}$$

with respect the decomposition (3.12). The Levi subgroup of $P^C = P$ is isomorphic to

$$L^C \simeq GL(Q_1) \times GL(Q_2) \times \cdots \times GL(Q_r) \times Sp(\tilde{Q}). \tag{3.14}$$

Note that the action of L^C on $T_{\tilde{E}_\bullet}\mathrm{IF}(a, 2n)$ fix the subspaces in Proposition 3.3.3. Similar to Definition 3.2.4, we have

Definition 3.3.5. *For any isotropic partial flag variety* $\mathrm{IF}(a, 2n)$ *and* $w \in S_{2n}^C(a)$*, define*

$$\mathrm{IF}_{L,w}(2n) := \{F_\bullet \in \mathrm{F}\ell(n) \mid F_\bullet = lw^{-1}E_\bullet \textit{ for some } l \in L^C\}$$

The following two propositions are the analogues of Propositions 3.2.5 and 3.2.7 for $\mathrm{IF}(a, 2n)$.

Proposition 3.3.6. *If* $F_\bullet \in \mathrm{IF}_{L,w}(2n)$*, then*

$$T_{\tilde{E}_\bullet}\Phi_w^\circ(F_\bullet) = \bigoplus_{i<j}^{r+1}\Big(H_{i,j}(w, F_\bullet) \oplus \bar{H}_{i,j}(w, F_\bullet)\Big) \oplus \bigoplus_{i=1}^{r}\Big(\tilde{H}_i(w, F_\bullet) \oplus S_i(w, F_\bullet)\Big). \tag{3.15}$$

Proof. The proof is exactly analogous to the proof of Proposition 3.2.5 where instead of using the splitting (3.1) of $\mathbb{C}^n$, we use the splitting (3.11) of $\mathbb{C}^{2n}$. $\square$

Proposition 3.3.7. *If $F_\bullet \in \mathrm{IF}_{L,w}(2n)$, then for any $i < j$, we have the following:*

(i) $H_{i,j}(w, F_\bullet) = T_{Q_i} X^\circ_{w_{i,j}}(F_\bullet(Q_i \oplus Q_j))$

(ii) $\bar{H}_{i,j}(w, F_\bullet) = T_{Q_i} X^\circ_{\bar{w}_{i,j}}(F_\bullet(Q_i \oplus \bar{Q}_j))$

(iii) $\tilde{H}_i(w, F_\bullet) = T_{Q_i} X^\circ_{\tilde{w}_i}(F_\bullet(Q_i \oplus \tilde{Q}))$

(iv) $S_i(w, F_\bullet) = T_{Q_i} \Phi^\circ_{\bar{w}_{i,i}}(F_\bullet(Q_i \oplus \bar{Q}_i))$

Proof. Recall that $\bar{a} := \{\bar{a}_1, \ldots, \bar{a}_r\}$ where $\bar{a}_i := 2n + 1 - a_i$ and consider the inclusion $\mathrm{IF}(a, 2n) \hookrightarrow \mathrm{F}\ell(a \cup \bar{a}, 2n)$ which maps

$$V_1 \subseteq \cdots \subseteq V_r \mapsto V_1 \subseteq \cdots \subseteq V_r \subseteq V_r^\perp \subseteq \cdots \subseteq V_1^\perp.$$

Using the natural inclusion of $S_n^C(a) \subseteq S_{2n}^A(a \cup \bar{a})$, it is easy to see that $\Phi^\circ_w(F_\bullet) \hookrightarrow X^\circ_w(F_\bullet)$. Since $L^C \subseteq L^A$, for any $F_\bullet \in \mathrm{IF}_{L,w}(2n)$ we can apply Proposition 3.2.7 to $X^\circ_w(F_\bullet)$. This gives

$$\begin{aligned} H_{i,j}(w, F_\bullet) &= \mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}_\bullet} \Phi^\circ_w(F_\bullet) \\ &\subseteq \mathrm{Hom}(Q_i, Q_j) \cap T_{\tilde{E}_\bullet} X^\circ_w(F_\bullet) \\ &= T_{Q_i} X^\circ_{w_{i,j}}(F_\bullet(Q_i \oplus Q_j)). \end{aligned}$$

Similarly, we have the inclusions

$$\bar{H}_{i,j}(w, F_\bullet) \subseteq T_{Q_i} X^\circ_{\bar{w}_{i,j}}(F_\bullet(Q_i \oplus \bar{Q}_j)) \text{ and } \tilde{H}_i(w, F_\bullet) \subseteq T_{Q_i} X^\circ_{\tilde{w}_i}(F_\bullet(Q_i \oplus \tilde{Q}))$$

and

$$S_i(w, F_\bullet) \subseteq T_{Q_i} X^\circ_{\bar{w}_{i,i}}(F_\bullet(Q_i \oplus \bar{Q}_i)).$$

Note that by definition, $S_i(w, F_\bullet) \subseteq \mathrm{sym}^2 Q_i^*$. Hence

$$\begin{aligned} S_i(w, F_\bullet) &\subseteq T_{Q_i} X^\circ_{\bar{w}_{i,i}}(F_\bullet(Q_i \oplus \bar{Q}_i)) \cap \mathrm{sym}^2 Q_i^* \\ &= T_{Q_i} X^\circ_{\bar{w}_{i,i}}(F_\bullet(Q_i \oplus \bar{Q}_i)) \cap T_{Q_i}\mathrm{LG}(b_i, Q_i \oplus \bar{Q}_i) \\ &= T_{Q_i} \Phi^\circ_{\bar{w}_{i,i}}(F_\bullet(Q_i \oplus \bar{Q}_i)). \end{aligned}$$

Calculating dimensions we have

$$\dim \Phi_w = \dim X_{w_{i,j}} + \dim X_{\bar{w}_{i,j}} + \dim X_{\tilde{w}_i} + \dim \Phi_{\bar{w}_{i,i}}.$$

By Proposition 3.3.6, the above inclusions must be isomorphisms. □

Proof of $(ii) \Rightarrow (iii)$ **in Theorem 1.2.1:** The proof is the same as the proof of Theorem 1.1.2 $(ii) \Rightarrow (iii)$ only we use Proposition 3.3.6 and Proposition 3.3.7. □

Before we prove $(iii) \Rightarrow (ii)$ in Theorem 1.2.1 we need the following two lemmas which serve as analogues of Lemma 3.2.8. The first is a technical lemma which is a consequence of recent work by Belkale-Kumar. For proof see [4, Theorem 34]. For any $i \in [r]$ and $w \in S^A_{b_i+2b_{r+1}}(b_i)$, consider the set

$$F(w, i) := \{g w^{-1} E_\bullet(Q_i \oplus \tilde{Q}) \mid g \in GL(Q_i) \times Sp(\tilde{Q})\}$$

Lemma 3.3.8. *Let* $(w^1, w^2, \ldots, w^s) \in (S^A_{b_i+2b_{r+1}}(b_i))^s$*, then the following are equivalent*

(i) $\prod_{k=1}^s [X_{w^k}] \neq 0$ *in* $H^*(\mathrm{Gr}(b_i, b_i + 2b_{r+1}))$

(ii) For generic $(F^1_\bullet, F^2_\bullet, \ldots, F^s_\bullet) \in \prod_{k=1}^s F(w^k, i)$*, the intersection* $\bigcap_{k=1}^s X^\circ_{w^k}(F^k_\bullet)$ *is trans-*

verse at $Q_i \in \mathrm{Gr}(b_i, Q_i \oplus \tilde{Q})$.

For any $\{i < j\} \subseteq [r]$, consider the following Levi factors with respect to the root system of $Sp(2n)$:

(i) Let $L_{i,j}$ be the Levi subgroup of the parabolic $P_{i,j} \subseteq SL(Q_i \oplus Q_j)$ which stabilizes the space Q_i.

(ii) Let $\bar{L}_{i,j}$ be the Levi subgroup of the parabolic $\bar{P}_{i,j} \subseteq SL(Q_i \oplus \bar{Q}_j)$ which stabilizes the space Q_i.

(iii) Let $\bar{L}_i$ be the Levi subgroup of the parabolic $\bar{P}_i \subseteq Sp(Q_i \oplus Q_i)$ which stabilizes the space Q_i.

Lemma 3.3.9. *For any* $w \in S_{2n}^C(a)$*, the following maps are surjective:*

(i) $\mathrm{IF}_{L,w}(2n) \to \mathrm{F}\ell_{L_{i,j},w_{i,j}}(Q_i \oplus Q_j)$ *given by* $F_\bullet \mapsto F_\bullet(Q_i \oplus Q_j)$.

(ii) $\mathrm{IF}_{L,w}(2n) \to \mathrm{F}\ell_{\bar{L}_{i,j},\bar{w}_{i,j}}(Q_i \oplus Q_j)$ *given by* $F_\bullet \mapsto F_\bullet(Q_i \oplus \bar{Q}_j)$.

(iii) $\mathrm{IF}_{L,w}(2n) \to \mathrm{IF}_{\bar{L}_i,\bar{w}_{i,i}}(Q_i \oplus Q_i)$ *given by* $F_\bullet \mapsto F_\bullet(Q_i \oplus Q_i)$.

(iv) $\mathrm{IF}_{L,w}(2n) \to F(\tilde{w}_i, i)$ *given by* $F_\bullet \mapsto F_\bullet(Q_i \oplus \tilde{Q})$.

Proof. Consider the Levi subgroup L^C as in (3.14). Clearly L^C acts transitively on all the range sets in items $(i) - (iv)$. Hence the maps are surjective. □

Proof of $(iii) \Rightarrow (ii)$ **in Theorem 1.2.1:** We follow a similar outline to the proof of $(iii) \Rightarrow (ii)$ in Theorem 1.1.2. Assume part (iii) in Theorem 1.2.1. Since $\mathrm{Gr}(b_i, b_i + b_j)$ and $\mathrm{LG}(b_i, 2b_i)$ are cominuscule flag varieties, by Proposition 2.1.9, the s-tuples $(w_{i,j}^1 w_{i,j}^2, \ldots, w_{i,j}^s)$, $(\bar{w}_{i,j}^1, \bar{w}_{i,j}^2, \ldots, \bar{w}_{i,j}^s)$ and $(\bar{w}_{i,i}^1, \bar{w}_{i,i}^2, \ldots, \bar{w}_{i,i}^s)$ are Levi-movable. By Lemmas 3.3.8 and 3.3.9

and Proposition 3.3.7, we can find a $\mathcal{F} \in \prod_{k=1}^{s} \mathrm{IF}_{L,w^k}(2n)$ such that

$$\begin{aligned}\dim\Big(\bigcap_{k=1}^{s} T_{\tilde{E}_\bullet}\Phi^\circ_{w^k}(F^k_\bullet)\Big) &= \sum_{i<j}^{r}\Big(\dim\bigcap_{k=1}^{s} H_{i,j}(w^k,F^k_\bullet)+\dim\bigcap_{k=1}^{s}\bar{H}_{i,j}(w^k,F^k_\bullet)\Big)\\ &+ \sum_{i=1}^{r}\Big(\dim\bigcap_{k=1}^{s}\tilde{H}_i(w^k,F^k_\bullet)+\dim\bigcap_{k=1}^{s} S_i(w^k,F^k_\bullet)\Big)=0.\end{aligned}$$

By the codimension condition (1.7), the Schubert cells $\Phi^\circ_{w^k}(F^k_\bullet)$ intersect transversally at $\tilde{E}$. Thus Proposition 2.1.2 proves $(iii) \Rightarrow (ii)$ in Theorem 1.2.1. □

3.4. Horn recursion for $(H^*(G/P), \odot_0)$

In [3], Belkale-Kumar give a list of necessary Horn-type inequalities which they call character inequalities satisfied by L-movable s-tuples. We state this result below in Theorem 3.4.2. They ask if these inequalities are sufficient to determine if a s-tuple is L-movable. Theorem 1.1.2 answers this question affirmatively for all type A flag varieties.

3.4.1. Refined inequalities for the new product

For any G/P, let c be any algebraic group homomorphism $Z(L) \to \mathbb{C}^*$, where $Z(L)$ denotes the center of the Levi subgroup $L \subseteq P$. We will call any such map a central character of L.

Definition 3.4.1. *For any $w \in W^P$ and central character c, define the character*

$$\chi^c_w = \sum_{\beta\in R(w,c)} \beta.$$

where

$$R(w,c) := \{\beta \in R^+ \backslash R^+_{\mathfrak{l}} \cap w^{-1}R^+ \mid e^{\beta}_{|Z(L)} = c\}$$

Note that

$$\chi_w = \sum_c \chi_w^c$$

where the sum runs over all central characters of L such that $\chi_1^c \neq 0$. Belkale-Kumar obtained the following result in [3, Thm 32].

Theorem 3.4.2. *Let $(w^1, w^2, \ldots, w^s) \in (W^P)^s$ be L-movable. Then the following two conditions are satisfied:*

(i) For any central character c of L such that $\chi_1^c \neq 0$,

$$\sum_{i=1}^{s} |R(w^i, c)| = |R(1, c)| \tag{3.16}$$

where $|\cdot|$ denotes the cardinality of the enclosed set.

(ii) For any maximal standard parabolic Q_L of L and any choice $(u^1, u^2, \ldots, u^s) \in (W^{Q_L})^s$ such that

$$\prod_{k=1}^{s} [\Lambda_{u^k}] \neq 0$$

in $H^*(L/Q_L)$ and any central character c of L such that $\chi_1^c \neq 0$, the following inequality is satisfied for any $\alpha_p \in \Delta(P) \backslash \Delta(Q_L)$:

$$\sum_{k=1}^{s} \chi_{w^k}^c (u^k x_p) \leq \chi_1^c (x_p) \tag{3.17}$$

3.4.2. Horn recursion for $(H^*(\mathrm{F}\ell(a, n)), \odot_0)$

We show that Theorem 1.1.2 implies that the conditions of Theorem 3.4.2 $(i), (ii)$ are sufficient to determine when $(w^1, w^2, \ldots, w^s)$ is L-movable for $SL(n)/P$. Recall that with respect to the splitting (3.1), we have the decomposition of L given in (3.5). Hence

$$Z(L) \simeq \{(z_1, z_2, \ldots, z_{r+1}) \in \mathbb{C}^{r+1} \mid \prod_{k=1}^{r+1} z_k^{b_k} = 1\}.$$

For any $\{i < j\} \subseteq [r+1]$, define the central character of L

$$c_{i,j}(z_1, z_2, \ldots, z_{r+1}) = z_i \cdot z_j^{-1}.$$

If $\beta = \varepsilon_a - \varepsilon_b \in R^+ \backslash R_{\mathfrak{l}}^+$, then $e^{\beta}_{|Z(L)} = c_{i,j}$ if and only if $a_{i-1} < a \leq a_i$ and $a_{j-1} < b \leq a_j$. Also, all central characters of L such that $\chi_1^c \neq 0$ are of the form $c_{i,j}$ for some $i < j$. By Lemma 2.1.6, we have that

$$|R(w, c_{i,j})| = \operatorname{codim}(X_{w_{i,j}})$$

and hence the equations in (3.16) are equivalent to the equations given in Theorem 1.1.2 (*iva*). For any $i \in [r+1]$ and any $d \in [b_i]$, let $P_{i,d}$ be the stabilizer of the subspace $\mathrm{Span}\{e_{a_{i-1}+1}, e_{a_{i-1}+2}, \ldots, e_{a_{i-1}+d}\} \subseteq Q_i$ in the group $GL(Q_i)$. Consider the parabolic subgroup of L:

$$Q_L := \{(g_1, \ldots, g_{r+1}) \in GL(Q_1) \times \cdots \times P_{i,d} \times \cdots \times GL(Q_{r+1}) \mid \prod_{i=1}^{r+1} \det(g_i) = 1\}.$$

The homogenous space $L/Q_L \simeq GL(Q_i)/P_{i,d} \simeq \mathrm{Gr}(d, b_i)$. Observe that the set $\Delta(P) \backslash \Delta(Q_L)$ consists of the single root $\alpha_{a_{i-1}+d}$. Calculating the action of $u \in S_d(b_i)$ on $x_{a_{i-1}+d}$ with respect to the positive roots in $R(1, c_{i,j})$, we have that

$$\beta(ux_{a_{i-1}+d}) = \begin{cases} 1 \text{ if } a - a_{i-1} \in \{u(1), u(2), \ldots, u(d)\} \\ 0 \text{ otherwise.} \end{cases}$$

where $\beta = \varepsilon_a - \varepsilon_b \in R(1, c_{i,j})$. By counting the number of positive roots that are sent to $R(1, c_{i,j})$ under the action of w^{-1}, we find that for any $w \in S_n(a)$ and $u \in S_{b_i}(d)$, the character

$$\chi_w^{c_{i,j}}(ux_{a_{i-1}+d}) = \sum_{l=1}^{d} \Big(b_j + u(l) - w_{i,j}(u(l))\Big).$$

In particular, we have that $\chi_1^{c_{i,j}}(ux_{a_{i-1}+d}) = db_j$. Hence the character inequalities (3.17) are the same as the inequalities (1.4).

Corollary 3.4.3. *Conditions* (i) *and* (ii) *in Theorem 3.4.2 are sufficient to determine when* $(w^1, w^2, \ldots, w^s) \in (W^P)^s$ *is* L*-movable for* $G = SL(n)$ *and* P *is any standard parabolic subgroup.*

Remark 3.4.4. *Theorem 1.2.1 does not prove the analogue of Corollary 3.4.3 for the group* $Sp(2n)$ *since the Horn recursion requires the Purbhoo-Sottile inequalities in [15] for the Lagrangian Grassmannians found in Theorem 1.2.1* $(iiib)$*.*

Question 3.4.5. *Are conditions* (i) *and* (ii) *in Theorem 3.4.2 sufficient to determine* L*-movable* s*-tuples in the case of the Lagrangian Grassmannian?*

If the answer is yes, then Theorem 1.2.1 would imply that (i) and (ii) in Theorem 3.4.2 are sufficient to determine L-movable s-tuples in all type C flag varieties.

3.5. A list of dimensional inequalities for $\mathrm{F}\ell(a, n)$

In this section we consider the entire cohomology $H^*(\mathrm{F}\ell(a, n))$ and prove Theorem 1.1.3. We start by considering the following lemma which gives an equivalent condition for a transversal intersection.

Lemma 3.5.1. *Let* $X_1, X_2, \ldots, X_s$ *be smooth subvarieties of a smooth variety* X *and let* $x \in \bigcap_{k=1}^s X_k$*. Then* $X_1, X_2, \ldots, X_s$ *intersect transversally at* x *if and only if the induced map of vector spaces* $\psi : T_xX \to \bigoplus_{k=1}^s T_xX/T_xX_k$ *is surjective.*

Proof. Note that $\ker(\psi) \simeq \bigcap_{k=1}^{s} T_x X_k \subseteq T_x X$. Then ψ surjective gives the short exact sequence

$$0 \longrightarrow \ker(\psi) \longrightarrow T_x X \xrightarrow{\psi} \bigoplus_{k=1}^{s} T_x X / T_x X_k \longrightarrow 0$$

and therefore $\dim(\bigcap_{k=1}^{s} T_x X_k) = \dim(T_x X) - \sum_{k=1}^{s} \operatorname{codim}(T_x X_k)$. $\square$

3.5.1. Proof of Theorem 1.1.3:

For any $V_\bullet \in \mathrm{F}\ell(a, n)$, consider the projection $f_i : T_{V_\bullet}(\mathrm{F}\ell(a, n)) \twoheadrightarrow T_{V_i}\mathrm{Gr}(a_i, n) \simeq \mathrm{Hom}(V_i, \mathbb{C}^n/V_i)$ given by $(\phi_1, \phi_2, \cdots, \phi_r) \mapsto \phi_i$. Using the proof of Proposition 3.2.3, for any $w \in S_n(a)$ and $F_\bullet \in \mathrm{F}\ell(n)$, the image of the subspace $T_{V_\bullet}(X_w^\circ(F_\bullet))$ is equal to

$$f_i(T_{V_\bullet} X_w^\circ(F_\bullet)) = T_{V_i} X_{w_i}^\circ(F_\bullet) \subseteq T_{V_i}\mathrm{Gr}(a_i, n).$$

For any $j \geq i$ define the map $f_{i,j} : T_{V_\bullet}(\mathrm{F}\ell(a, n)) \to \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)$ by composing the map f_i with the map $\rho_{i,j} : \mathbb{C}^n/V_i \twoheadrightarrow \mathbb{C}^n/V_j$. In other words, for any $v \in V_i$, we have

$$f_{i,j}((\phi_1, \phi_2, \cdots, \phi_r))(v) = \rho_{i,j} \circ \phi_i(v).$$

Note that the map $f_{i,j}$ is surjective since the map f_i is surjective. For any $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ and $\mathcal{F} \in \mathrm{F}\ell(n)^s$, we have the corresponding commuting diagram:

$$\begin{array}{ccc}
T_{V_\bullet}\mathrm{F}\ell(a, n) & \xrightarrow{\psi} & \bigoplus_{k=1}^{s} T_{V_\bullet}\mathrm{F}\ell(a, n)/T_{V_\bullet}X_{w^k}^\circ(F_\bullet^k) \\
\downarrow{\scriptstyle f_{i,j}} & & \downarrow{\scriptstyle \bar{f}_{i,j}^s} \\
\mathrm{Hom}(V_i, \mathbb{C}^n/V_j) & \xrightarrow{\psi_{i,j}} & \bigoplus_{k=1}^{s} \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)/f_{i,j}(T_{V_\bullet}X_{w^k}^\circ(F_\bullet^k))
\end{array}$$

where ψ and $\psi_{i,j}$ are diagonal embeddings and $\bar{f}_{i,j}^s$ coordinate wise projection of $f_{i,j}$. If

$(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ satisfies the condition $\prod_{k=1}^{s}[X_{w^k}] \neq 0$, then by Proposition 2.1.2 and Lemma 3.5.1, we can choose $\mathcal{F} \in \mathrm{F}\ell(n)^s$ such that the map ψ is surjective. Therefore, by the commutativity of the diagram, we have that $\psi_{i,j}$ is surjective and the images under $f_{i,j}$ of $T_{V_\bullet}X^\circ_{w^k}(F^k_\bullet) \subseteq \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)$ intersect transversally at the point $f_{i,j}(V_\bullet)$. Lemma 3.5.2 below completes the proof of Theorem 1.1.3 by computing the codimension of the space $f_{i,j}(T_{V_\bullet}X^\circ_w(F_\bullet)) \subseteq \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)$. □

Let $w \in S_n(a)$ and $\{i < j\} \subseteq [r]$. Let the set $A^j_w := \{w(a_j+1), w(a_j+2), \ldots, w(n)\}$. Define

$$p^{i,j}_w(l) := \#\{p \in A^j_w \mid p \leq w_i(a_i + w_i(l) - l)\}$$

where w_i is the image of w under the map $S_n(a) \to S_n(a_i)$.

Lemma 3.5.2. *Let $w \in S_n(a)$ and $\{i < j\} \subseteq [r]$. If we denote $p(l) := p^{i,j}_w(l)$, then*

$$f_{i,j}(T_{V_\bullet}X^\circ_w(F_\bullet)) = \{\phi \in \mathrm{Hom}(V_i, \mathbb{C}^n/V_j) \mid \phi(F_l(V_i)) \subseteq F_{p(l)}(\mathbb{C}^n/V_j)\}. \tag{3.18}$$

Proof. If $\phi = f_{i,j}((\phi_1, \phi_2, \cdots, \phi_r)) \in f_{i,j}(T_{V_\bullet}(X^\circ_w(F_\bullet)))$, by the definition of $p(l)$ we have

$$\phi(F_l(V_i)) \subseteq \rho_{i,j}(F_{w_i(l)-l}(\mathbb{C}^n/V_i)) \subseteq F_{p(l)}(\mathbb{C}^n/V_j) \quad \forall\, l \in [a_i].$$

This shows that the $LHS \subseteq RHS$ in equation (3.18). Let $\phi \in \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)$ be such that $\phi(F_l(V_i)) \subseteq F_{p(l)}(\mathbb{C}^n/V_j)\ \forall\, l \in [a_i]$. We construct $(\phi_1, \phi_2, \cdots, \phi_r) \in T_{V_\bullet}(X^\circ_w(F_\bullet))$ such that $f_{i,j}((\phi_1, \phi_2, \cdots, \phi_r)) = \phi$. Choose a splitting of

$$\mathbb{C}^n = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{r+1}$$

such that $V_k = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_k$ and identify $\mathbb{C}^n/V_k = Q_{k+1} \oplus Q_{k+2} \oplus \cdots \oplus Q_{r+1}\ \forall\, k \in [r]$. Then for any $k \in [r]$, the map ϕ induces a map $\phi_k : V_k \to \mathbb{C}^n/V_k$ by the following construction.

If $k \leq i$, then for any $\tilde{k} \in [k]$, define ϕ_k on $Q_{\tilde{k}}$ by $\phi_k(Q_{\tilde{k}}) := \phi(Q_{\tilde{k}})$.

If $k > i$, then for any $\tilde{k} \in [k]$, define ϕ_k on $Q_{\tilde{k}}$ by

$$\begin{aligned} \phi_k(Q_{\tilde{k}}) &:= \phi(Q_{\tilde{k}})\Big|_{Q_{k+1} \oplus Q_{k+2} \oplus \cdots \oplus Q_{r+1}} && \text{for } \tilde{k} < i \\ \phi_k(Q_{\tilde{k}}) &:= 0 && \text{for } \tilde{k} \geq i. \end{aligned}$$

It is easy to see that $(\phi_1, \phi_2, \ldots, \phi_r) \in T_{V_\bullet} X_w^\circ(F_\bullet)$ and that $f_{i,j}(\phi_1, \phi_2, \ldots, \phi_r) = \phi$. Therefore equation (3.18) is satisfied. □

By Lemma 3.5.2, the codimension of $f_{i,j}(T_{V_\bullet} X_w^\circ(F_\bullet)) \subseteq \mathrm{Hom}(V_i, \mathbb{C}^n/V_j)$ is given by

$$\operatorname{codim}(f_{i,j}(T_{V_\bullet} X_w^\circ(F_\bullet))) = a_i(n - a_j) - \sum_{l=1}^{a_i} p_w^{i,j}(l) = \sum_{l=1}^{a_i} (n - a_j - p_w^{i,j}(l))$$

which completes the proof of Theorem 1.1.3.

CHAPTER 4

Structure coefficients

Kleiman's transversality theorem [10] says that for generic $(g_1, \dots g_s) \in G^s$, the intersection $\bigcap_{k=1}^s g_k \Lambda_{w^k}$ is transverse and dense in the intersection $\bigcap_{k=1}^s g_k \bar{\Lambda}_{w^k}$. If $(w^1, w^2, \dots, w^s) \in (W^P)^s$ are such that $\prod_{k=1}^s [\Lambda_{w^k}] = c[\Lambda_e]$ for some positive integer c, then for generic $(g_1, \dots g_s) \in G^s$, we have

$$\left| \bigcap_{k=1}^s g_k \Lambda_{w^k} \right| = \left| \bigcap_{k=1}^s g_k \bar{\Lambda}_{w^k} \right| = c.$$

Let $(w^1, w^2, \dots, w^s) \in (W^P)^s$ be L-movable. The goal of this chapter is prove Theorems 1.1.4 and 1.2.2 which give a formula for the structure coefficient c as a product of structure coefficients coming from induced maximal flag varieties in the type A and C cases. Since the proof in each of these cases is similar, we will go over the type A case in detail and refer back to this case when needed in the type C case.

4.1. Induced Schubert varieties

Let P_i be the standard maximal parabolic of G containing P associated to the root $\alpha_{a_i} \in \Delta \backslash \Delta(P)$ and let L_i denote the Levi subgroup of P_i. Let U_{P_i} be the unipotent radical of P_i and observe that $P_i = L_i \cdot U_{P_i}$. Since $P \subseteq P_i$, there exists a standard parabolic subgroup $Q_i \subseteq L_i$ (Q_i contains $B_{L_i} := B \cap L_i$) such that $Q_i = P \cap L_i$ and $P = Q_i \cdot U_{P_i}$.

Consider the projection $f : G/P \twoheadrightarrow G/P_i$. This gives rise to the fibration diagram

$$\begin{array}{ccc} L_i/Q_i = P_i/P & \longrightarrow & G/P \\ & & \downarrow f \\ & & G/P_i \end{array}$$

In particular, for any $g \in G$, we have $f^{-1}(gP_i) = \{gpP \mid p \in P_i\} \simeq gL_i/Q_i$. For any $w \in W^P$, there exist $w_i \in W^{P_i}$ and $w_\gamma \in W_{L_i}^{Q_i}$ such that $w = w_i \cdot w_\gamma$. Since w, w_i, w_γ are minimal length, this product is unique [15]. Clearly the image $f(\Lambda_w) = \Lambda_{w_i}$.

Lemma 4.1.1. *For any $gP_i \in \Lambda_{w_i}$, we have that $f^{-1}(gP_i) \cap \Lambda_w \simeq g\Lambda_{w_\gamma}$.*

Proof. Write $g = bw_i$ for some $b \in B$. We have that

$$f^{-1}(gP_i) \cap \Lambda_w = (bw_iP_i \cap Bw_\gamma w_i)P = bw_i(P_i \cap w_i^{-1}Bw_iw_\gamma)P.$$

Since $w_i \in W^{P_i}$, the Borel $B_{L_i} = L_i \cap w_i^{-1}Bw_i$. Hence, under the identification $f^{-1}(gP_i) \simeq gL_i/Q_i$, we have the above expression equal to

$$bw_i(L_i \cap w_i^{-1}Bw_iw_\gamma)Q_i = bw_i(L_i \cap w_i^{-1}Bw_i)w_\gamma Q_i = bw_iB_{L_i}w_\gamma Q_i = g\Lambda_{w_\gamma}.$$

□

In the next two sections, we show that if $(w^1, \ldots, w^s)$ is L-movable, then the associated structure constant is a product of the structure constant coming from $(w_i^1, \ldots, w_i^s)$ and $(w_\gamma^1, \ldots, w_\gamma^s)$ for certain $i \in [r]$.

4.2. A formula for type A structure coefficients

In this section we focus on the case where $G/P = \mathrm{F}\ell(a,n)$ and $i = 1$. Hence the projection we consider is $f : \mathrm{F}\ell(a,n) \twoheadrightarrow \mathrm{Gr}(a_1,n)$. The techniques used in this section are inspired by techniques used by Belkale in his proof of Horn's conjecture in [2]. Recall Definition 1.1.1 of induced permutations. Define $\gamma := \{a_1+1 < a_1+2 < \cdots < n\} \subseteq [n]$ and for any $w \in S_n(a)$, consider the induced permutation $w_\gamma \in S_{n-a_1}$. For any point $V \in \mathrm{Gr}(a_1,n)$, the fiber $f^{-1}(V)$ is isomorphic to $\mathrm{F}\ell(a_\gamma, n-a_1)$ where $a_\gamma = \{a_2 - a_1 < a_3 - a_1 < \cdots < a_r - a_1\}$. Applying Lemma 4.1.1, for any $w \in S_n(a)$ and $F_\bullet \in \mathrm{F}\ell(n)$ such that $V \in f(X_w^\circ(F_\bullet))$, we have that

$$X_w^\circ(F_\bullet) \cap f^{-1}(V) \simeq X_{w_\gamma}^\circ(F_\bullet(\mathbb{C}^n/V)).$$

Let $(w^1, w^2, \ldots, w^s) \in S_n(a)^s$ and let $\mathcal{F} \in \mathrm{F}\ell(n)^s$ be such that $\bigcap_{k=1}^s X_{w_1^k}^\circ(F_\bullet^k)$ is not empty. For any $V \in \bigcap_{k=1}^s X_{w_1^k}^\circ(F_\bullet^k)$, we have

$$\bigcap_{k=1}^{s} X_{w^k}^\circ(F_\bullet^k) \cap f^{-1}(V) \simeq \bigcap_{k=1}^{s} X_{w_\gamma^k}^\circ(F_\bullet^k(\mathbb{C}^n/V)). \tag{4.1}$$

Note that set (4.1) could possibly be empty. We will show later in Proposition 4.2.6, that if $(w^1, w^2, \ldots, w^s)$ is L-movable, then we can choose $\mathcal{F} \in \mathrm{F}\ell(n)^s$ "generic" enough so that (4.1) is nonempty for all $V \in \bigcap_{k=1}^s X_{w_1^k}^\circ(F_\bullet^k)$. We first show that an L-movable s-tuple induces a Levi-movable s-tuple in the projection and fiber of f. We have the following relationships between the lengths of w, w_γ, and w_1.

Remark 4.2.1. *For any $w \in S_n(a)$, we have*

$$\ell^A(w_\gamma) = \ell^A(w) - \ell^A(w_1) \tag{4.2}$$

and for any $i \in [r-1]$,

$$\ell^A((w_\gamma)_i) = \ell^A(w_{i+1}) - \ell^A(w_1) + \sum_{k=1}^{i} \ell^A(w_{1,k}). \tag{4.3}$$

For any $V \in \mathrm{Gr}(a_1, n)$ and $w \in S_n(a_1)$, define

$$Y_V^w := \{F_\bullet \in \mathrm{F}\ell(n) \mid V \in X_w^\circ(F_\bullet)\}.$$

Lemma 4.2.2. *For any* $V \in \mathrm{Gr}(a_1, n)$ *and* $w \in S_n(a_1)$*, the map* $Y_V^w \to \mathrm{F}\ell(\mathbb{C}^n/V)$ *given by* $F_\bullet \mapsto F_\bullet(\mathbb{C}^n/V)$ *is surjective.*

Proof. Let $G_V := \{g \in SL(n) \mid gV = V\}$. It is easy to see that the map $Y_V^w \to \mathrm{F}\ell(\mathbb{C}^n/V)$ is G_V-equivariant. Since G_V acts transitively on $\mathrm{F}\ell(\mathbb{C}^n/V)$, the map is surjective. □

Using notation at the beginning of this chapter, we can identify $\mathrm{F}\ell(a_\gamma, n - a_1) = L_1/Q_1 \simeq P_1/P$. Let L_Q denote the Levi-subgroup of Q_1 in L_1.

Proposition 4.2.3. *If* $(w^1, w^2, \ldots, w^s)$ *is* L*-movable, then the following are true:*

(i) The s*-tuple* $(w_1^1, w_1^2, \ldots, w_1^s)$ *is* L_1*-movable.*

(ii) The s*-tuple* $(w_\gamma^1, w_\gamma^2, \ldots, w_\gamma^s)$ *is* L_Q*-movable.*

Proof. Since $(w^1, w^2, \ldots, w^s)$ is L-movable, for generic $\mathcal{F} \in \mathrm{F}\ell(n)^s$ the intersection $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$ is nonempty. By the numerical conditions (1.3), the expected dimension of the set $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$ is zero. Hence the set $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$ is finite and transverse. Since this is an intersection of Schubert cells in a Grassmannian, by Proposition 2.1.9, it is also L_1-movable. This proves part (i).

For part (ii), fix $V \in \mathrm{Gr}(a_1, n)$ and consider $\prod_{k=1}^s Y_V^{w_1^k} \subseteq \mathrm{F}\ell(n)^s$. By part (i) and the assumption, for generic $\mathcal{F} \in \prod_{k=1}^s Y_V^{w_1^k}$, the intersections $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$ and $\bigcap_{k=1}^s X^\circ_{w^k}(F_\bullet^k)$ are nonempty and transversal. Since $f(\bigcap_{k=1}^s X^\circ_{w^k}(F_\bullet^k))$ is contained in $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$, we can further assume that there exists a $V_\bullet \in \bigcap_{k=1}^s X^\circ_{w^k}(F^k_\bullet)$ such that $f(V_\bullet) = V$. By equation (4.1), $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(\mathbb{C}^n/V))$ is nonempty and finite and by Lemma 4.2.2, the induced flags $\mathcal{F}(\mathbb{C}^n/V)$ are generic in $\mathrm{F}\ell(\mathbb{C}^n/V)^s$. Hence the intersection $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(\mathbb{C}^n/V))$ is transverse. By Proposition 2.1.2, we have that $\prod_{k=1}^s [X_{w^k_\gamma}]$ is a nonzero multiple of a class of a point in $H^*(\mathrm{F}\ell(a_\gamma, n-a_1))$. By Theorem 1.1.2, it suffices to check that the s=tuple $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$ satisfies the numerical conditions for L_Q-movability. Since $(w^1, w^2, \ldots, w^s)$ is L-movable, we have the following numerical conditions (Note that (4.5) requires Theorem 1.1.2 (iii)):

$$\sum_{k=1}^s \left(a_i(n-a_i) - \ell(w^k_i)\right) = a_i(n-a_i) \tag{4.4}$$

$$\sum_{k=1}^s \left(a_1(a_i - a_{i-1}) - \ell(w^k_{1,i})\right) = a_1(a_i - a_{i-1}). \tag{4.5}$$

for any $i \in [r]$. For any $i \in \{2, 3, \ldots, r\}$ rewrite the dimension of $\mathrm{Gr}(a_i - a_1, n - a_1)$ as

$$\dim(\mathrm{Gr}(a_i - a_1, n-a_1)) = a_i(n-a_i) - a_1(n-a_1) + \sum_{k=1}^i a_1(a_k - a_{k-1}) \tag{4.6}$$

Combining (4.3),(4.4),(4.5) and (4.6) shows that $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$ satisfies the numerical conditions for L_Q-movability in Theorem 1.1.2 (i). □

Alternate proof of Proposition 4.2.3 (ii)**:** Consider the list of products given in Theorem 1.1.2 (iii). Applying Theorem 1.1.2 again to a certain subset of these products implies that $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$ is L_Q-movable. □

We now fix $(w^1, w^2, \ldots, w^s)$ to be L-movable and show that for generic $\mathcal{F} \in \mathrm{F}\ell(n)^s$, the intersection $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(\mathbb{C}^n/V))$ is nonempty for every $V \in \bigcap_{k=1}^s X^\circ_{w^k_1}(F^k_\bullet)$. Define the variety $Y \subseteq \mathrm{Gr}(a_1, n) \times \mathrm{F}\ell(n)^s$ by the following:

$$Y := \{(V, \mathcal{F}) \mid V \in \bigcap_{k=1}^{s} X^\circ_{w_1^k}(F^k_\bullet)\}.$$

Note that the variety Y is irreducible and smooth. See [2, Section 8] for a similar example of this result.

Definition 4.2.4. *For any $(V, \mathcal{F}) \in Y$, we say that $(V, \mathcal{F})$ has property $P1$ if the intersection $\bigcap_{k=1}^s X^\circ_{w_\gamma^k}(F^k_\bullet(\mathbb{C}^n/V))$ is transverse and $\bigcap_{k=1}^s X^\circ_{w_\gamma^k}(F^k_\bullet(\mathbb{C}^n/V)) = \bigcap_{k=1}^s X_{w_\gamma^k}(F^k_\bullet(\mathbb{C}^n/V))$.*

Note that if $(V, \mathcal{F})$ has property $P1$, then $\bigcap_{k=1}^s X^\circ_{w^k}(F^k_\bullet) \cap f^{-1}(V)$ is not empty.

Proposition 4.2.5. *Property $P1$ is an open condition on Y.*

Proof. Consider Y as a fiber bundle on $\mathrm{Gr}(a_1, n)$ with fiber $\prod_{k=1}^s Y_V^{w_1^k}$ over the point $V \in \mathrm{Gr}(a_1, n)$. Let Z be the quotient flag bundle on $\mathrm{Gr}(a_1, n)$ with fiber $\mathrm{F}\ell(\mathbb{C}^n/V)^s$ over the point $V \in \mathrm{Gr}(a_1, n)$. By Lemma 3.2.8, the fiber bundle map $\eta : Y \twoheadrightarrow Z$ given by $\mathcal{F} \mapsto \mathcal{F}(\mathbb{C}^n/V)$ is surjective. Choose an open set $U \subseteq \mathrm{Gr}(a_1, n)$ such that fiber bundle Z is trivial. Over the set U, choose a local trivialization

$$Z|_U \simeq U \times \mathrm{F}\ell(\mathbb{C}^{n-a_1})^s.$$

Since $(w_\gamma^1, w_\gamma^2, \ldots, w_\gamma^s)$ is L_Q-movable, there exists an open subset $O \subset \mathrm{F}\ell(\mathbb{C}^{n-a_1})^s$ such that for every $\mathcal{H} \in O$, $\bigcap_{k=1}^s X^\circ_{w_\gamma^k}(H^k_\bullet)$ is transverse and $\bigcap_{k=1}^s X^\circ_{w_\gamma^k}(H^k_\bullet) = \bigcap_{k=1}^s X_{w_\gamma^k}(H^k_\bullet)$. Moreover, we can choose O to be $SL(n-a_1)$-invariant under the diagonal action on $\mathrm{F}\ell(\mathbb{C}^{n-a_1})^s$. Consider the fiber bundle $\eta^{-1}(O)$ over U. Since O is $SL(n-a_1)$-invariant, $\eta^{-1}(O)$ is independent of choice of local trivialization. It is easy to see that $\eta^{-1}(O)$ is an open set of Y and every $(V, \mathcal{F}) \in \eta^{-1}(O)$ satisfies property $P1$. □

Proposition 4.2.6. *Let $\tilde{O} \subseteq Y$ be an open subset of Y such that every point in $\tilde{O}$ has* *property $P1$. Let $g : Y \twoheadrightarrow \mathrm{F}\ell(n)^s$ be the projection of Y onto its second factor. For generic $\mathcal{F} \in \mathrm{F}\ell(n)^s$, the set $g^{-1}(\mathcal{F}) \subseteq \tilde{O}$.*

Proof. The fiber of g over any point $\mathcal{F}$ is isomorphic to $\bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$. Choose an open subset of $\dot{U} \subseteq \mathrm{F}\ell(n)^s$ such that for every $\mathcal{F} \in \dot{U}$, the set $g^{-1}(\mathcal{F})$ is finite. Let $\tilde{Y}$ be the closure of $g(Y \backslash \tilde{O})$ in $\mathrm{F}\ell(n)^s$. Since Y is irreducible, we have $\dim(Y \backslash \tilde{O}) \geq \dim(\tilde{Y})$. Since g is generically finite to one, we have that $\dim(\mathrm{F}\ell(n)^s) > \dim(\tilde{Y})$ and hence there exists an open subset $\ddot{U} \subseteq \mathrm{F}\ell(n)^s \backslash \tilde{Y}$. For any $\mathcal{F} \in \dot{U} \cap \ddot{U}$, we have $g^{-1}(\mathcal{F}) \subseteq \tilde{O}$. □

4.2.1. Proof of Theorem 1.1.4

Proof. Chose $\mathcal{F} \in \mathrm{F}\ell(n)^s$ generically so that,

$$\left| \bigcap_{k=1}^{s} X^\circ_{w_1^k}(F_\bullet^k) \right| = c_1 \quad \text{and} \quad \left| \bigcap_{k=1}^{s} X^\circ_{w^k}(F_\bullet^k) \right| = c.$$

By Proposition 4.2.6, the flags $\mathcal{F}$ can also be generically chosen so that for any $V \in \bigcap_{k=1}^s X^\circ_{w_1^k}(F_\bullet^k)$ the point $(V, \mathcal{F})$ satisfies property $P1$. Therefore the map

$$f : \bigcap_{k=1}^{s} X^\circ_{w^k}(F_\bullet^k) \twoheadrightarrow \bigcap_{k=1}^{s} X^\circ_{w_1^k}(F_\bullet^k)$$

is surjective. Since the number of points in each fiber $f^{-1}(V)$ is exactly c_γ, we have that $c = c_1 \cdot c_\gamma$. □

Let w_0 be the longest element in W and w_P be the longest element in W_P. For any $w \in S_n(a)$, define $w^\vee := w_0 w w_P$. Note that $w^\vee \in S_n(a)$.

Corollary 4.2.7. *Let $(w, u, v^\vee)$ be L-movable. Then $c^v_{w,u} = c^{v_1}_{w_1,u_1} \cdot c^{v_\gamma}_{w_\gamma,u_\gamma}$.*

Proof. Since the Poincaré pair $(w, w^\vee)$ is L-movable, by Proposition 4.2.3, we have that $(w_1, (w^\vee)_1)$ and $(w_\gamma, (w^\vee)_\gamma)$ are Levi-movable. Hence $(w_1)^\vee = (w^\vee)_1$ and $(w_\gamma)^\vee = (w^\vee)_\gamma$.

Apply Theorem 1.1.4 to the triple $(w, u, v^\vee)$. □

Recall that Proposition 4.2.3 says that if $(w, u, v^\vee)$ is L-movable then $(w_\gamma, u_\gamma, v_\gamma^\vee)$ is L_Q-movable. Hence we can apply Theorem 1.1.4 to $(w_\gamma, u_\gamma, v_\gamma^\vee)$. This process gives an inductive way to write $c_{w,u}^v$ as a product of Littlewood-Richardson coefficients coming from the Grassmannians $\mathrm{Gr}(b_i, n - a_{i-1})$ where $i \in [r]$.

Corollary 4.2.8. *The non-zero structure coefficients of $(H^*(\mathrm{F}\ell(n), \mathbb{Z}), \odot_0)$ are all equal to 1.*

Proof. Since $\mathrm{F}\ell(n)$ is the complete flag variety, we have $b_i = a_i - a_{i-1} = 1$ for all $i \in [n]$. Hence $\mathrm{Gr}(b_i, n - a_{i-1})$ is projective space where all structure coefficients are equal to 1. □

Remark 4.2.9. *Analogues of Theorem 1.1.4 exist for any projection $f_i : \mathrm{F}\ell(a, n) \twoheadrightarrow \mathrm{Gr}(a_i, n)$ and fiber*

$$f_i^{-1}(V) \simeq \mathrm{F}\ell((a_1, \ldots, a_{i-1}), a_i) \times \mathrm{F}\ell((a_{i+1} - a_i, \ldots, a_r - a_i), n - a_i).$$

with corresponding induced coefficients. The proof is similar to that of Theorem 1.1.4. Comparing these formulas gives many interesting relations between type A structure coefficients.

4.3. A formula for type C structure coefficients

The arguments in this section are very similar to those in the previous section. We focus on the case where $G/P = \mathrm{IF}(a, 2n)$ and $i = r$. Consider the projection $f : \mathrm{IF}(a, 2n) \twoheadrightarrow \mathrm{IG}(a_r, 2n)$. Clearly, the image $f(\Phi_w^\circ(F_\bullet)) = \Phi_{w_r}^\circ(F_\bullet)$. For any $V \in \mathrm{IG}(a_r, 2n)$, the fiber $f^{-1}(V) = \mathrm{F}\ell(a_\gamma^C, V) \simeq \mathrm{F}\ell(a_\gamma^C, a_r)$ where $a_\gamma^C := \{a_1 < a_2 < \cdots < a_{r-1}\}$. If $V \in f(\Phi_w^\circ(F_\bullet))$, then

$$\Phi_w^\circ(F_\bullet) \cap f^{-1}(V) \simeq X_{w_\gamma}^\circ(F_\bullet(V))$$

where $w_\gamma := w_{I^r}$ (Recall the definition of w_{I^r} in Section 1.2.2). Analogous to Proposition 4.2.3, we show that Levi-movable s-tuples induce Levi-movable s-tuples in the projection and fiber of f. Let L_Q denote the Levi subgroup of Q_r in L_r under the identification $\mathrm{F}\ell(a^C_\gamma, a_r) \simeq L_r/Q_r$. Note that for any $w \in S^C_n(a)$, the length $\ell^C(w) = \ell^A(w_r) + \ell^C(w_\gamma)$.

Proposition 4.3.1. *If $(w^1, w^2, \ldots, w^s) \in (S^C_n(a))^s$ is L-movable, then the following are true:*

(i) The s-tuple $(w^1_r, w^2_r, \ldots, w^s_r)$ is L_r-movable.

(ii) The s-tuple $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$ is L_Q-movable.

Proof. Note that part (ii) of the proposition is an immediate consequence of applying the results in Theorem 1.2.1$(iiia)$ to Theorem 1.1.2. To prove part (i), note that the s-tuple $(w^1_r, w^2_r, \ldots, w^s_r)$ satisfies the numerical condition in Theorem 1.2.1(i) since $(w^1, w^2, \ldots, w^s)$ is L-movable. It suffices to show that $(w^1_r, w^2_r, \ldots, w^s_r)$ satisfies the codimension condition (1.7). To see this, we combine the codimension conditions of $(w^1_\gamma, w^2_\gamma, \ldots, w^s_\gamma)$ and $(w^1, w^2, \ldots, w^s)$. □

Fix $(w^1, w^2, \ldots, w^s)$ to be L-movable. We show that for generic $\mathcal{F} \in \mathrm{IF}(2n)^s$, the intersection $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(V))$ is nonempty for every $V \in \bigcap_{k=1}^s \Phi^\circ_{w^k_r}(F^k_\bullet)$. Define the variety $Y^C \subseteq \mathrm{IG}(a_r, 2n) \times \mathrm{IF}(2n)^s$ by the following:

$$Y^C := \{(V, \mathcal{F}) \mid V \in \bigcap_{k=1}^s \Phi^\circ_{w^k_r}(F^k_\bullet)\}.$$

Definition 4.3.2. *For any $(V, \mathcal{F}) \in Y^C$, we say that $(V, \mathcal{F})$ has property $P2$ if the intersection $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(V))$ is transverse and $\bigcap_{k=1}^s X^\circ_{w^k_\gamma}(F^k_\bullet(V)) = \bigcap_{k=1}^s X_{w^k_\gamma}(F^k_\bullet(V))$.*

Note that if $(V, \mathcal{F})$ has property $P2$, then $\bigcap_{k=1}^s \Phi^\circ_{w^k}(F^k_\bullet) \cap f^{-1}(V)$ is not empty.

Proposition 4.3.3. *Property $P2$ is an open condition on Y^C.*

Proof. The proof is exactly analogous to the proof of Proposition 4.2.5. Here we use the fact that $(w_\gamma^1, w_\gamma^2, \ldots, w_\gamma^s)$ is L_Q-movable and that the map $Y^C|_V \to \mathrm{F}\ell(V)^s$ given by $(V, \mathcal{F}) \mapsto \mathcal{F}(V)$ is surjective. □

Proposition 4.3.4. *Let $\tilde{O} \subseteq Y^C$ be an open subset of Y^C such that every point in $\tilde{O}$ has property $P2$. Let $g : Y \twoheadrightarrow \mathrm{IF}(2n)^s$ be the projection of Y^C onto its second factor. For generic $\mathcal{F} \in \mathrm{IF}(2n)^s$, the set $g^{-1}(\mathcal{F}) \subseteq \tilde{O}$.*

Proof. This follows from the proof of Proposition 4.2.6 and the fact that the map g is generically finite to one. □

4.3.1. Proof of Theorem 1.2.2

Proof. Once again, this follows from the previous section using the proof of Theorem 1.1.4 replacing Proposition 4.2.6 with Proposition 4.3.4. □

CHAPTER 5

Branching Schubert calculus and Horn recursion

In this chapter we address a more general version of the original question found at the beginning of Chapter 1. Let $\tilde{G}$ be a semisimple algebraic subgroup of G and choose parabolic subgroups $\tilde{P}$ and P such that $\phi : \tilde{G}/\tilde{P} \hookrightarrow G/P$. If $[\Lambda_w]$ is a Schubert class in $H^*(G/P)$, we can ask: Under what conditions is $\phi^*([\Lambda_w]) \neq 0$? In [3, Theorem 29], Belkale-Kumar construct a list of necessary criteria in the form of inequalities in the case of the diagonal embedding $\tilde{G}/\tilde{P} \hookrightarrow G/P = (\tilde{G}/\tilde{P})^s$. We find that their work generalizes to the setting where G is any semisimple algebraic group containing $\tilde{G}$. In this thesis, the choice of $\tilde{P}$ and P are dependant on a fixed one parameter subgroup of $\tilde{G}$. In the first three sections we give background information for this problem. In the last three sections we state and prove the main results.

5.1. Dominant weights and parabolic subgroups

We reestablish many of the objects associated to G initially defined in Chapter 2. Fix a torus $H \subseteq G$ and let $R \subseteq \mathfrak{h}^*$ be the root system of G and choose a set of positive roots R^+. Choose a set of simple roots $\Delta \subseteq R^+$. For any $\alpha \in R^+$, let $t_\alpha \in \mathfrak{h}$ denote the unique element such that $\alpha(\mathbf{t}) = (t_\alpha, \mathbf{t})$ for all $\mathbf{t} \in \mathfrak{h}$ where $(,)$ denotes the Killing form on $\mathfrak{h}$. Define $\mathbb{E} := \mathbb{R}\{t_\alpha \mid \alpha \in \Delta\} \subseteq \mathfrak{h}$ be the real span of the t_α. Let $\mathfrak{h}_+ := \{x \in \mathbb{E} \mid \alpha(x) \geq 0 \ \forall \alpha \in \Delta\}$ denote the corresponding positive Weyl chamber in $\mathbb{E}$. Let B denote the Borel subgroup with respect to the choice of positive roots R^+ and let W denote the Weyl group of G. Let $\tau \in OPS(G)$ (i.e. an algebraic group homomorphism $\tau : \mathbb{C}^* \to G$) and define the associated tangent vector in $\mathfrak{g}$ to be

$$\dot{\tau} := \frac{d\tau}{dt}(1) \in \mathfrak{g}.$$

Observe that if $\tau \in OPS(H)$, then $\dot{\tau} \in \mathbb{E} \subseteq \mathfrak{h}$. We say τ is ***dominant*** with respect to G if $\dot{\tau} \in \mathfrak{h}_+$. Let $P^G(\tau)$ be the associated parabolic subgroup defined by:

$$P^G(\tau) := \{g \in G \mid \lim_{t \to 0} \tau(t) g \tau(t)^{-1} \text{ exists in } G\}.$$

We say that $P^G(\tau)$ is ***standard*** if $B \subseteq P^G(\tau)$.

Proposition 5.1.1. *Let $\tau \in OPS(H)$. The parabolic subgroup $P^G(\tau)$ is standard if and only if τ is dominant with respect to G.*

Proof. Consider the map $\exp : \mathfrak{b} \to B$. Since B is generated by $\exp(\mathfrak{b})$, it suffices to show τ is dominant if and only if $\lim_{t \to 0} \tau(t) \exp(X) \tau(t)^{-1}$ exists for all $X \in \mathfrak{b}$. To do this, we show the limit exists for a basis of $\mathfrak{b}$. Consider the Cartan decomposition

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha.$$

If $X \in \mathfrak{h}$ we have that $\tau(t) \exp(X) \tau(t)^{-1} = \exp(X)$ since $\tau \in OPS(H)$ and hence the limit above exists. If $X \in \mathfrak{g}_\alpha$, then

$$\tau(t) \exp(X) \tau(t)^{-1} = \exp(\mathrm{Ad}(\tau(t))(X)).$$

Observe that $\tau(t) = \exp(\dot{\tau} \ln t)$. By a simple calculation, we have

$$\mathrm{Ad}(\tau(t))(X) = \mathrm{Ad}(\exp(\dot{\tau} \ln t))(X) = \exp(\mathrm{ad}(\dot{\tau} \ln t)(X)) = t^{\alpha(\dot{\tau})X}.$$

Thus $\lim_{t \to 0} \tau(t) \exp(X) \tau(t)^{-1}$ exists if and only if $\alpha(\dot{\tau}) \geq 0$. This proves the proposition.

□

Note that if $P^G(\tau)$ is standard, then its Lie algebra $\mathfrak{p}$ corresponds to the set of simple roots $\Delta(P^G(\tau)) := \{\alpha \in \Delta \mid \alpha(\dot{\tau}) = 0\}$.

5.2. Subgroups and the Weyl group

In this section we give a brief survey of [5, Section 2.2], in which Berenstein-Sjamaar determine the relative Weyl set W_{rel}. The only difference is that we do not require $1 \in W_{\text{rel}}$. We provide basic proofs of the statements needed in this thesis. Let $\tilde{G}$ be semisimple algebraic subgroup of G and let $f : \tilde{G} \hookrightarrow G$ denote the embedding of $\tilde{G}$ into G. Fix a torus $\tilde{H}$ such that $\tilde{H} = H \cap \tilde{G}$ and let $\tilde{R}$ be the root system of $\tilde{G}$ with respect to $\tilde{H}$. Consider the induced map $f_* : \tilde{\mathfrak{h}} \hookrightarrow \mathfrak{h}$. Note that $f_*(\tilde{\mathbb{E}}) \subseteq \mathbb{E}$ since $\tilde{R} \subseteq f^*(R)$. Choose a set of positive roots $\tilde{R}^+$ and simple roots $\tilde{\Delta} \subseteq \tilde{R}^+$. Since W acts on the space $\mathbb{E}$, we can ask: How does this action affect $\tilde{\mathfrak{h}}_+ \subset \tilde{\mathbb{E}} \subseteq \mathbb{E}$?

Definition 5.2.1. *Define*

$$W_{\text{com}} := \{v \in W \mid \dim \tilde{\mathfrak{h}}_+ = \dim(\tilde{\mathfrak{h}}_+ \cap v\mathfrak{h}_+)\}$$

to be the compatible subset of W.

Let $\tilde{B}$ be the Borel subgroup of $\tilde{G}$ with respect to $\tilde{R}^+$.

Proposition 5.2.2. *If $v \in W_{\text{com}}$, then $\tilde{B} = vBv^{-1} \cap \tilde{G}$.*

Proof. By the assumption, choose λ such that $\dot{\lambda} \in \tilde{\mathfrak{h}}_+^\circ \cap v\mathfrak{h}_+$. This implies that $\tilde{B} = P^{\tilde{G}}(\lambda)$ and $B \subseteq P^G(v^{-1}\lambda v) = v^{-1}P^G(\lambda)v$. Therefore, we have $\tilde{B} \subseteq vBv^{-1} \cap \tilde{G}$ and thus $vBv^{-1} \cap \tilde{G}$ is a standard parabolic of $\tilde{G}$. To show that $\tilde{B} = vBv^{-1} \cap \tilde{G}$, it suffices show that the Lie algebra of $vBv^{-1} \cap \tilde{G}$ contains no negative root spaces of the Lie algebra of $\tilde{G}$. Let $\beta \in \tilde{R}$ be any root corresponding to a root space of the Lie algebra of $vBv^{-1} \cap \tilde{G}$. Since $\tilde{R} \subseteq f^*(R)$, there exists an $\alpha \in R^+$ such that $\beta = vf^*(\alpha)$. For any $x \in \tilde{\mathfrak{h}}_+^\circ \cap v\mathfrak{h}_+$, we have

$$\beta(x) = vf^*(\alpha)(x) = v\alpha(x) = \alpha(v^{-1}x) \geq 0$$

since $v^{-1}x \in \mathfrak{h}_+$. Thus β cannot be a negative root. □

Define $\mathfrak{h}_v := \tilde{\mathfrak{h}}_+ \cap v\mathfrak{h}_+$ and note that $\tilde{\mathfrak{h}}_+ = \bigcup_{v \in W_{\mathrm{com}}} \mathfrak{h}_v$. The set W_{com} may be over determined in the sense that there may exist $u, v \in W_{\mathrm{com}}$ such that $\mathfrak{h}_u = \mathfrak{h}_v$. Define

$$\overline{W} := N_{Z_G(\tilde{H})}(H)/H \subseteq N_G(H)/H = W$$

where $Z_G(\tilde{H})$ is the centralizer of $\tilde{H}$ in G. Note that $\overline{W}$ is well defined since $H \subseteq Z_G(\tilde{H})$.

Proposition 5.2.3. *The right action of $\overline{W}$ on W fixes the subset W_{com}.*

Proof. Let $w \in W_{\mathrm{com}}$ and let $\bar{w} \in \overline{W}$. Then $\tilde{\mathfrak{h}}_+ \cap \bar{w}w\mathfrak{h}_+ = \bar{w}^{-1}\tilde{\mathfrak{h}}_+ \cap w\mathfrak{h}_+$. Hence, it suffices to show that $\overline{W}$ fixes the space $\tilde{\mathfrak{h}}_+$. In fact, we will show that $\overline{W}$ acts trivially on $\tilde{\mathfrak{h}}_+$. Let $h \in \tilde{\mathfrak{h}}_+$ and choose $\lambda \in OPS(\tilde{H})$ such that $\dot{\lambda} = h$. Let $\bar{w}_0 \in N_{Z_G(\tilde{H})}(H)$ denote any representative of $\bar{w} \in \overline{W}$. We have that

$$\bar{w}h = \frac{d}{dt}(\bar{w}_0\lambda(t)\bar{w}_0^{-1})\big|_{t=1} = \frac{d}{dt}\lambda(t)\big|_{t=1} = h.$$

This proves the proposition. □

We consider the orbit space of W_{com} with respect to the action of $\overline{W}$. There exists a unique element in each orbit of minimal length which leads to the following definition.

Definition 5.2.4. *Define W_{rel} be the set of minimal length representatives in W_{com} of the orbit space $\overline{W}\backslash W_{\mathrm{com}}$.*

The following are some basic properties of the set W_{rel}. For the proof see [5].

Proposition 5.2.5. *The following are true:*

(i) $\tilde{\mathfrak{h}}_+ = \bigcup_{v \in W_{\mathrm{rel}}} \mathfrak{h}_v$.

(ii) If $\mathfrak{h}_v^\circ \cap \mathfrak{h}_u \neq \emptyset$ *for some* $v, u \in W_{\mathrm{rel}}$*, then* $v = u$.

(iii) $|W_{\mathrm{rel}}| = 1$ *if and only if* $\tilde{\mathfrak{h}}_+ \subseteq \mathfrak{h}_+$.

5.3. Flag varieties and the statement of results

Fix $\tau \in OPS(\tilde{H})$ to be dominant with respect to $\tilde{G}$. Clearly, we have that $\tau \in OPS(H)$, although it may not be dominant with respect to G. Choose $v \in W_{\mathrm{rel}}$ such that $\tau_v := v^{-1}\tau v$ is dominant with respect to G. By Proposition 5.1.1, $P^G(\tau_v)$ is a standard parabolic subgroup of G. We simplify notation by denoting $P^{\tilde{G}}(\tau)$ and $P^G(\tau_v)$ by $\tilde{P}$ and P respectively. Note that $\tilde{P} = vPv^{-1} \cap \tilde{G}$. Let

$$\phi_{\tau,v} : \tilde{G}/\tilde{P} \hookrightarrow G/P$$

be the $\tilde{G}$-equivariant map defined by mapping $\phi_{\tau,v}(g\tilde{P}) = gvP$. We will denote $\phi_{\tau,v}$ by ϕ when the choice of τ and v are clear.

Lemma 5.3.1. *The map ϕ is well defined and injective.*

Proof. Let $g_1, g_2 \in \tilde{G}$ be such that $g_1\tilde{P} = g_2\tilde{P}$. Abusing notation, let $v \in N_G(H)$ be a representative of $v \in W_{\mathrm{rel}}$. Then there exists a $p' = vpv^{-1}$ such that $p' \in \tilde{P}$, $p \in P$, and $g_1 = g_2p'$. Hence

$$g_1v = g_2p'v = g_2vpv^{-1}v = g_2vp.$$

Since P contains H, the map ϕ not depend on the choice of $v \in N_G(H)$. Thus ϕ is well defined. Now suppose $g_1, g_2 \in \tilde{G}$ are such that $\phi(g_1\tilde{P}) = \phi(g_1\tilde{P})$. Then there exists a $p \in P$

such that $g_1 v = g_2 v p$. Hence $g_2^{-1} g_1 = v p v^{-1}$. Therefore $g_2^{-1} g_1 \in (vPv^{-1} \cap \tilde{G}) = \tilde{P}$. This proves that ϕ is injective. □

Consider the induced map on cohomology

$$\phi^* : H^*(G/P) \to H^*(\tilde{G}/\tilde{P}).$$

For any $w \in W^P$, we are interested in constructing a list of necessary conditions given that

$$\phi^*([\Lambda_w]) \neq 0. \tag{5.1}$$

It is sufficient to consider only the case where

$$\phi^*([\Lambda_w]) = c[\Lambda_e] \text{ for some } c \neq 0. \tag{5.2}$$

To see this we consider the diagonal embedding

$$\bar{\phi} : \tilde{G}/\tilde{P} \hookrightarrow G/P \times \tilde{G}/\tilde{P}$$

given by $g\tilde{P} \mapsto (\phi(gP), g\tilde{P})$. If (5.1) is satisfied, then

$$\phi^*([\Lambda_w]) = \sum_{u \in \tilde{W}^{\tilde{P}}} c_w^u [\Lambda_u]$$

where $c_w^u \in \mathbb{Z}_{\geq 0}$. If $c_w^u \neq 0$, then we have that

$$\bar{\phi}^*([\Lambda_{(w,u^\vee)}]) = \phi^*([\Lambda_w]) \cdot [\Lambda_{u^\vee}] = c_w^u [\bar{\Lambda}_e]$$

where $u^\vee \in \tilde{W}^{\tilde{P}}$ is the Poincaré dual of $u \in \tilde{W}^{\tilde{P}}$. Hence determining (5.1) is equivalent to determining when $\bar{\phi}^*([\Lambda_{(w,u^\vee)}])$ is a nonzero multiple of a class a point. By replacing G

with $G \times \tilde{G}$, and ϕ with $\bar{\phi}$, we see that determining (5.1) is equivalent to determining (5.2).

5.3.1. Applications to Representation Theory

The motivation for studying the pullbacks (5.1) comes from representation theory. Let $\chi \in \mathfrak{h}_+^*$ be an integral dominant weight of G and let V_χ denote the corresponding irreducible representation of G of highest weight χ. We ask the question: For which χ does V_χ contain a nonzero $\tilde{G}$-invariant vector? In [5], Berenstein-Sjamaar give an answer to the asymptotic version of this question.

Theorem 5.3.2. *Let $\chi \in \mathfrak{h}_+$ be an integral dominant weight of G. Then there exists an integer $N \in \mathbb{Z}_{\geq 0}$ such that $V_{N\chi}$ contains a nonzero $\tilde{G}$-invariant vector if and only if for every $(\tau, v, w) \in OPS(\tilde{G}) \times W_{\text{rel}} \times W^P$ such that $\phi^*([\Lambda_w]) \neq 0$, the following inequality is satisfied:*

$$f^*(vw^{-1}\chi)(\dot{\tau}) \leq 0. \tag{5.3}$$

What the above theorem says is that the set of such dominant weights generate a convex cone in $\mathfrak{h}_+^*$, in which the walls are indexed by the triples (τ, v, w) which satisfy (5.1).

5.3.2. Levi subgroups

For any parabolic $P = P^G(\tau)$, we define the Levi subgroup of P by

$$L = L^G(\tau) := \{g \in G \mid \lim_{t \to 0} \tau(t) g \tau(t)^{-1} = g\}.$$

We remark that L is a maximal reductive subgroup of P. Define the L-dominant chamber of $\mathfrak{h}$ by

$$\mathfrak{h}_+^L := \{x \in \mathbb{E} \mid \alpha(x) \geq 0 \ \forall \alpha \in \Delta(P)\}.$$

Observe that $\mathfrak{h}_+ \subseteq \mathfrak{h}_+^L$. We say $\tau \in OPS(G)$ is ***L*-dominant** if $\dot{\tau} \in \mathfrak{h}_+^L$. Let $B_L := B \cap L$ denote the Borel subgroup of L. We say a subgroup $Q \subseteq L$ is a ***standard*** parabolic subgroup if $B_L \subseteq Q$. Let $R_{\mathfrak{l}}^+ \subseteq R^+$ denote the set of positive roots generated by $\Delta(P)$.

Proposition 5.3.3. *Let L be the Levi subgroup of a standard parabolic $P \subseteq G$. A subgroup $Q \subseteq L$ is a standard parabolic if and only if $Q = P^L(\lambda)$ for some L-dominant $\lambda \in OPS(H)$.*

Proof. Since P is standard, we have that the Lie algebra of L is equal to

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathfrak{l}}^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in R_{\mathfrak{l}}^+} \mathfrak{g}_{-\alpha}$$

and that the Lie algebra of B_L is equal to

$$\mathfrak{b}_L = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathfrak{l}}^+} \mathfrak{g}_\alpha.$$

Following the proof of Proposition 5.1.1, the parabolic subgroup Q contains B_L if and only if $\alpha(\lambda) \geq 0$ for all $\alpha \in \Delta(P)$. □

For the rest of this chapter, we fix $L := L^G(\tau_v) \subseteq P$ and $\tilde{L} := L^{\tilde{G}}(\tau) \subseteq \tilde{P}$.

Lemma 5.3.4. $\tilde{L} = vLv^{-1} \cap \tilde{G}$.

Proof. Let $g \in vLv^{-1} \cap \tilde{G}$. We have that

$$\lim_{t\to 0} v^{-1}\tau(t)vv^{-1}gvv^{-1}\tau(t)^{-1}v = \lim_{t\to 0} v^{-1}\tau(t)g\tau(t)^{-1}v = v^{-1}gv.$$

Thus $\lim_{t\to 0} \tau(t)g\tau(t)^{-1} = g$ and $g \in \tilde{L}$. The reverse argument is the same. □

5.3.3. Admissibility of one parameter subgroups

The proofs of the results in this chapter require techniques from Geometric Invariant Theory (GIT). Since the groups we consider are not necessarily reductive and varieties are not necessarily projective, we will need the notion of an ***admissible*** one parameter subgroup. Consider the variety P/B_L.

Definition 5.3.5. *Let $\lambda \in OPS(\tilde{P})$. We say λ is P-admissible or admissible if the limit*

$$\lim_{t\to 0} v^{-1}\lambda(t)vpB_L$$

exists in P/B_L for all $p \in P$.

This definition is a generalization of the definition of admissibility given by Belkale-Kumar in [3] where $P/B_L = (\tilde{P}/\tilde{B}_{\tilde{L}})^s$ and v is taken to be the identity. We now give a characterization of P-admissible $\lambda \in OPS(\tilde{H})$. Consider the cone

$$\mathcal{C}_P := \{\dot{\lambda} \in \tilde{\mathbb{E}} \mid v\beta(\dot{\lambda}) \geq 0 \ \forall \beta \in R^+ \backslash R^+_{\mathfrak{l}}\} \subseteq \tilde{\mathbb{E}}.$$

Lemma 5.3.6. *Let $\lambda \in OPS(\tilde{H})$. Then λ is P-admissible if and only if $\dot{\lambda} \in \mathcal{C}_P$.*

Proof. Let $P = U \cdot L$ be the Levi decomposition of P and let $\lambda_v := v^{-1}\lambda(t)v$. If λ is P-admissible, then the limit

$$\lim_{t\to 0} \lambda_v(t)pB_L = \lim_{t\to 0} \lambda_v(t)u\lambda_v(t)^{-1}\lambda_v(t)lB_L$$

exists in P/B_L. Since L/B_L is compact, λ is P-admissible if and only if the limit

$$\lim_{t\to 0} \lambda_v(t)u\lambda_v(t)^{-1}$$

exists in G . This is equivalent to $\beta(\dot{\lambda}_v) \geq 0$ for all $\beta \in R^+ \backslash R_{\mathfrak{l}}^+$. □

Lemma 5.3.7. *If $\lambda \in OPS(\tilde{P})$ is P-admissible, then $\lambda_0 := \tilde{p}\lambda\tilde{p}^{-1}$ is P-admissible for any $\tilde{p} \in \tilde{P}$.*

Proof. For any $p \in P$, we have the limit

$$\lim_{t\to 0} v^{-1}\lambda_0(t)vpB_L = \lim_{t\to 0} v^{-1}\tilde{p}\lambda(t)\tilde{p}vpB_L = \lim_{t\to 0} v^{-1}\tilde{p}vv^{-1}\lambda(t)vv^{-1}\tilde{p}vpB_L.$$

Since $v^{-1}\tilde{p}v \in P$ and λ is P-admissible, the above limit exists. □

5.3.4. The main result on necessary Horn conditions

By an abuse of notation, we fix $v \in N_G(H)$ to be a representative of $v \in W_{\text{rel}}$. There is no loss of generality in the final results by making such a choice. Let $\mathfrak{h}_+^L$ and $\tilde{\mathfrak{h}}_+^{\tilde{L}}$ denote the corresponding dominant chambers of $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ with respect to L and $\tilde{L}$ and let $W_L = W_P$ be the Weyl group of L. We now construct an $\tilde{L}$-equivariant embedding of Levi flag varieties given a $\tilde{L}$-dominant OPS λ. By Lemma 5.3.4, we can define $(W_L)_{\text{rel}}$ as in Section 5.2. The only difference is that we must consider the roots $v^{-1}\tilde{R}_{\mathfrak{l}}$. Let $\lambda \in OPS(\tilde{H})$ be $\tilde{L}$-dominant and choose $\hat{v} \in (W_L)_{\text{rel}}$ such that $\dot{\lambda} \in \tilde{\mathfrak{h}}_+^{\tilde{L}} \cap v\hat{v}\mathfrak{h}_+^L$. Consider the standard parabolic subgroups $\tilde{Q}(\lambda) := P^{\tilde{L}}(\lambda)$ and $Q(\lambda_{\hat{v}}) := P^L(\lambda_{\hat{v}})$ where $\lambda_{\hat{v}} := (v\hat{v})^{-1}\lambda v\hat{v}$. Define

$$\phi_{\lambda,\hat{v}}^L : \tilde{L}/\tilde{Q}(\lambda) \hookrightarrow L/Q(\lambda_{\hat{v}})$$

to be the $\tilde{L}$-equivariant map which takes $\phi_{\lambda,\hat{v}}^L(l\tilde{Q}(\lambda)) = v^{-1}lv\hat{v}Q(\lambda_{\hat{v}})$.

Let $\tilde{R}_{\mathfrak{l}}^+ \subseteq \tilde{R}^+$ denote the set of roots generated by the simple roots $\tilde{\Delta}(\tilde{P})$. We define characters similar to the one in Definition 2.1.4. For any $w \in W^P$, define the character

$$\chi_w := \sum_{\beta \in R^+ \setminus R_{\mathfrak{l}}^+ \cap w^{-1} R^+} \beta.$$

Also define the character

$$\tilde{\chi} := \sum_{\beta \in \tilde{R}^+ \setminus \tilde{R}_{\mathfrak{l}}^+} \beta.$$

We now state the first main result of this chapter.

Theorem 5.3.8. *Let $w \in W^P$ be such that $\phi^*([\Lambda_w]) =$ a nonzero multiple of a class of a point in $H^*(\tilde{G}/\tilde{P})$. Then for any admissible, $\tilde{L}$-dominant $\lambda \in OPS(\tilde{H})$ and $(\hat{w}, \hat{v}) \in W_L^{Q(\lambda_{\hat{v}})} \times (W_L)_{\text{rel}}$ such that $\dot{\lambda} \in \tilde{\mathfrak{h}}_+^{\tilde{L}} \cap v\hat{v}\mathfrak{h}_+^L$ and $(\phi_{\lambda,\hat{v}}^L)^*([\Lambda_{\hat{w}}]) \neq 0$ in $H^*(\tilde{L}/\tilde{Q}(\lambda))$, the following inequality holds:*

$$(f^*(v\hat{v}\hat{w}^{-1}\chi_w) - \tilde{\chi})(\dot{\lambda}) \leq 0. \tag{5.4}$$

In particular, we can choose $\lambda = \tau \in OPS(\tilde{H})$. By definition, τ is P-admissible and $\tilde{L}$-dominant, and τ_v is L-dominant. Hence $\tau \in \tilde{\mathfrak{h}}_+^{\tilde{L}} \cap v\mathfrak{h}_+^L$.

Corollary 5.3.9. *For any $w \in W^P$ such that $\phi^*([\bar{\Lambda}_w]) =$ a nonzero multiple of a class of a point in $H^*(\tilde{G}/\tilde{P})$, we have that*

$$(f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) \leq 0.$$

Proof. Observe that $\tilde{Q}(\tau) = \tilde{L}$ and $Q(\tau_v) = L$. Thus $\hat{w}$ is the identity in Theorem 5.3.8. Since τ_v is L-dominant, $\hat{v}$ is also the identity. □

The inequality in the above corollary is important since we will later see in Section 5.6 that the condition that w is (L, ϕ)-movable is characterized by when this inequality is an equality.

Remark 5.3.10. *Theorem 5.3.8 together with Corollary 5.3.9 generalizes [3, Theorem 29] where Belkale-Kumar prove this result in the case of the diagonal embedding.*

Question 5.3.11. *In what cases are the inequalities* (5.4) *sufficient to determine when* $\phi^*([\Lambda_w]) \neq 0$ *in Theorem 5.3.8?*

If the inequalities (5.4) are sufficient, then by Theorem 5.3.2, determining when $\phi^*([\Lambda_w]) \neq 0$ would be equivalent to solving a certain asymptotic representation theory restriction problem with respect to the embedding $v^{-1}\tilde{L}v \subseteq L$.

Considering all admissible, $\tilde{L}$-dominant λ in Theorem 5.3.8 produces a highly redundant list of inequalities. The conclusion can be replaced by an equivalent statement involving only finitely many $\lambda \in OPS(\tilde{H})$. By Proposition 5.2.5, there exists a cubicle division of the $\tilde{L}$-dominant chamber of $\tilde{\mathfrak{h}}$

$$\tilde{\mathfrak{h}}_+^{\tilde{L}} = \bigcup_{\hat{v} \in (W_L)_{\mathrm{rel}}} \mathfrak{h}_{\hat{v}}^{\tilde{L}}$$

where $\mathfrak{h}_{\hat{v}}^{\tilde{L}} := \tilde{\mathfrak{h}}_+^{\tilde{L}} \cap v\hat{v}\mathfrak{h}_+^{L}$. Consider the intersection $\tilde{\mathfrak{h}}_+^{\tilde{L}} \cap \mathcal{C}_P$. Since $\mathcal{C}_P$ is closed, we can choose a finite collection of admissible, $\tilde{L}$-dominant $\lambda_1, \ldots, \lambda_s \in OPS(\tilde{H})$ such that the appropriate sub-collection span the cubicles $\mathfrak{h}_{\hat{v}}^{\tilde{L}} \cap \mathcal{C}_P$. Fix $\hat{v}_1, \ldots, \hat{v}_s \in (W_L)_{\mathrm{rel}}$ such that each $(v\hat{v}_k)^{-1}\lambda_k v\hat{v}_k$ is L-dominant. For every $k \in [s]$, let $\tilde{Q}_k$ denote the standard parabolic subgroups of $\tilde{L}$ associated to λ_k and let Q_k denote the standard parabolic subgroups of L associated to $(v\hat{v}_k)^{-1}\lambda_k v\hat{v}_k$. For each $k \in [s]$, we also have the induced $\tilde{L}$-equivariant map on flag varieties

$$\phi_k : \tilde{L}/\tilde{Q}_k \hookrightarrow L/Q_k$$

which sends $\phi_k(l\tilde{Q}_k) = v^{-1}lv\hat{v}_kQ_k$ and the induced map ϕ_k^* on cohomology.

Theorem 5.3.12. *Let* $w \in W^P$ *be such that* $\phi^*([\Lambda_w]) =$ *a nonzero multiple of a class of a point in* $H^*(\tilde{G}/\tilde{P})$*. Then for all* $k \in [s]$ *and* $\hat{w} \in W_L^{Q_k}$ *such that* $\phi_k^*([\Lambda_{\hat{w}}]) \neq 0$ *the following* *inequality is satisfied:*

$$(f^*(v\hat{v}_k\hat{w}^{-1}\chi_w) - \tilde{\chi})(\dot{\lambda}_k) \leq 0.$$

5.3.5. Proof of Theorem 5.3.8 ⇔ Theorem 5.3.12

Clearly Theorem 5.3.8 implies Theorem 5.3.12, so we focus on the reverse implication. Let λ be admissible and $\tilde{L}$-dominant. Then $\dot{\lambda} \in \tilde{\mathfrak{h}}_+^{\tilde{L}} \cap \mathcal{C}_P$ and therefore lies in one of the cubicles $\mathfrak{h}_{\hat{v}}^{\tilde{L}} \cap \mathcal{C}_P$. Choose $\hat{v} \in (W_L)_{\text{rel}}$ such that $\dot{\lambda} \in \mathfrak{h}_{\hat{v}}^{\tilde{L}} \cap \mathcal{C}_P$ and write

$$\dot{\lambda} = \sum a_k \dot{\lambda}_k$$

where the sum runs over the $\dot{\lambda}_k$ which span $\mathfrak{h}_{\hat{v}}^{\tilde{L}} \cap \mathcal{C}_P$ and $a_k \geq 0$. For fixed $\hat{w}$, the functional $f^*(v\hat{v}_k\hat{w}^{-1}\chi_w) - \tilde{\chi}$ is linear on $\mathfrak{h}_{\hat{v}}^{\tilde{L}} \cap \mathcal{C}_P$ since χ_w and $\tilde{\chi}$ are linear. Hence Theorem 5.3.12 suffices to prove Theorem 5.3.8. □

5.4. Tangent space analysis

The proof of Theorem 5.3.8 relies on the Hilbert-Mumford numerical criterion for semistability in which we consider certain $\tilde{P}$-equivariant line bundles on the space P/B_L. These line bundles are derived by analyzing the tangent spaces of G/P and $\tilde{G}/\tilde{P}$ at the points vP and $e\tilde{P}$ respectively. By an abuse of notation we let $\Lambda_w := w^{-1}BwP$ denote the Schubert cell shifted by w^{-1}. Consider a generic translate $g\Lambda_w \subseteq G/P$. Since G/P is homogeneous, without loss of generality, we can assume that this translate contains the point $vP \in G/P$. Note that we still fix $v \in N_G(H)$ to be a representative of $v \in W_{\text{rel}}$.

Lemma 5.4.1. *Suppose $vP \in g\Lambda_w$ for some $g \in G$, then there exists a $p \in P$ such that $g\Lambda_w = vp\Lambda_w$.*

Proof. By the assumption, there exist $p \in P$ and $b \in B$ such that $v = gw^{-1}bwp^{-1}$. Hence $g = vpw^{-1}b^{-1}w$ and $g\Lambda_w = vpw^{-1}b^{-1}w\Lambda_w = vp\Lambda_w$. □

By the above lemma, the assumption in Theorem 5.3.8 is equivalent to the condition that

$$\dim(T_{e\tilde{P}}(\phi^{-1}(vp\Lambda_w))) = 0$$

for generic $p \in P$. Let

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to T_{vP}(G/P)$$

be the induced map between tangent spaces at the points $e\tilde{P}$ and vP. For any $p \in P$ and $w \in W^P$, consider the subspace $T_{vP}(vp\Lambda_w) \subseteq T_{vP}(G/P)$. Analogous to Proposition 2.1.2, the following proposition in a basic fact that relates nonvanishing cohomology to tangent spaces.

Proposition 5.4.2. *Let $w \in W^P$ be such that*

$$\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}. \tag{5.5}$$

Then the following are equivalent.

(i) $\phi^*([\Lambda_w]) =$ *a nonzero multiple of a class of a point in* $H^*(\tilde{G}/\tilde{P})$.

(ii) *For generic $p \in P$, the induced map*

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to T_{vP}(G/P)/T_{vP}(vp\Lambda_w)$$

is an isomorphism.

Observe that part *(ii)* of Proposition 5.4.2 is equivalent to saying the point $e\tilde{P}$ is scheme theoretically isolated in $\phi^{-1}(vp\Lambda_w)$ for generic $p \in P$.

5.4.1. $\tilde{P}$-equivariant bundles

In this section, we define several vector bundles on the variety P/B_L. Recall that $v^{-1}\tilde{P}v \subseteq P$ and define the $\tilde{P}$-equivariant product bundle

$$\tilde{\mathcal{T}} := P/B_L \times T_{e\tilde{P}}(\tilde{G}/\tilde{P})$$

on P/B_L where the $\tilde{P}$ action is given by the diagonal action $\tilde{p}(pB_L, l) = (v^{-1}\tilde{p}vpB_L, \tilde{p}l)$. Note that while the vector bundle structure of $\tilde{\mathcal{T}}$ is trivial, the action of $\tilde{P}$ is nontrivial. Consider the conjugated action of P on G/P given by

$$p(gP) = vpv^{-1}gP. \tag{5.6}$$

Clearly this action of P fixes the point vP and thus the vector space $T_{vP}(G/P)$ is a P-module. Define

$$\mathcal{T}' := P/B_L \times T_{vP}(G/P)$$

be the P-equivariant vector bundle on P/B_L where the action of P acts diagonally (Note that the action on the first factor is not conjugated). The map ϕ_* induces a $\tilde{P}$-equivariant map $\Theta : \tilde{\mathcal{T}} \to \mathcal{T}'$ given by $(pB_L, l) \mapsto (pB_L, \phi_*(l))$. Since $T_{vP}(G/P)$ is a P-module, is it also a B_L-module. Hence we can define the P-equivariant bundle

$$\mathcal{T} := P \times_{B_L} T_{vP}(G/P)$$

where $(pb, l) \sim (p, bl)$. We now show that $\mathcal{T}'$ and $\mathcal{T}$ are P-equivariantly isomorphic. Define $\xi : \mathcal{T}' \to \mathcal{T}$ by mapping $(pB_L, l) \mapsto (p, p^{-1}l)$.

Lemma 5.4.3. *The map ξ is a well defined P-equivariant isomorphism of vector bundles* *on* P/B_L.

Proof. We first show ξ is well defined. For any $b \in B_L$, we have that

$$\xi((pbB_L, l)) = (pb, (pb)^{-1}l) = (p, bb^{-1}p^{-1}l) = (p, p^{-1}l)$$

and hence ξ is well defined. Since $\mathcal{T}'$ and $\mathcal{T}$ are vector bundles of the same rank, it suffices to show that ξ is injective and P-equivariant. To show that ξ is injective, suppose $p, p' \in P$ are such that $\xi((pB_L, l)) = \xi((p'B_L, l))$. Then there exists a $b \in B_L$ such that $p = p'b$. This implies that ξ is injective. The following calculation shows that ξ is P-equivariant:

$$\xi(p'(pB_L, l)) = (p'p, p^{-1}p'^{-1}p'l) = (p'p, p^{-1}l) = p'(p, p^{-1}l) = p'\xi((pB_L, l)).$$

□

Observe that for any $w \in W^P$, the action of B_L on G/P given in (5.6) fixes the space $v\Lambda_w$ and hence, $T_{vP}(v\Lambda_w)$ is a B_L-module. Define

$$\mathcal{T}_w := P \times_{B_L} T_{vP}(v\Lambda_w)$$

be the corresponding P-equivariant vector bundle on P/B_L. Note that $\mathcal{T}_w$ is a sub-bundle of $\mathcal{T}$ and that $\xi^{-1}(\mathcal{T}_w)\big|_{pB_L} = pB_L \times T_{vP}(vp\Lambda_w)$. Consider the $\tilde{P}$-equivariant map

$$\xi \circ \Theta : \tilde{\mathcal{T}} \to \mathcal{T}/\mathcal{T}_w.$$

If equation (5.5) is satisfied, then the rank of the vector bundles $\tilde{\mathcal{T}}$ and $\mathcal{T}/\mathcal{T}_w$ is the same. Consider the determinant map of line bundles

$$\theta : \det(\tilde{\mathcal{T}}) \to \det(\mathcal{T}/\mathcal{T}_w)$$

induced from the map $\xi \circ \Theta$. The map θ can be viewed as a $\tilde{P}$-invariant section of the space

$$H^0(P/B_L,\ \det(\tilde{\mathcal{T}})^* \otimes \det(\mathcal{T}/\mathcal{T}_w)). \tag{5.7}$$

We now have the following addition to Proposition 5.4.2:

Proposition 5.4.4. *Let $w \in W^P$ be such that*

$$\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}.$$

Then the following are equivalent.

(i) $\phi^*([\Lambda_w]) =$ *a nonzero multiple of a class of a point in* $H^*(\tilde{G}/\tilde{P})$.

(ii) For generic $p \in P$, the induced map

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to T_{vP}(G/P)/T_{vP}(vp\Lambda_w)$$

is an isomorphism.

(iii) For generic $p \in P$, the section $\theta \in H^0(P/B_L,\ \det(\tilde{\mathcal{T}})^* \otimes \det(\mathcal{T}/\mathcal{T}_w))^{\tilde{P}}$ does not vanish at pB_L.

Note that part *(iii)* of Proposition 5.4.4 is equivalent to $\theta(pB_L) \neq 0$ for some $p \in P$.

5.5. Geometric Invariant Theory

We review some basic properties of Geometric Invariant Theory. Let S be a complex algebraic group acting on a variety $\mathbb{X}$ and let $\mathbb{L}$ be a S-equivariant line bundle on $\mathbb{X}$. The following definition is for Mumford's numerical measure of instability:

Definition 5.5.1. *Let $\lambda \in OPS(S)$ be such that for any $x \in \mathbb{X}$, the limit $\lim_{t\to 0}\lambda(t)x$ exists. Let $x_0 \in \mathbb{X}$ denote this limit. Then the fiber over x_0 in $\mathbb{L}$ is fixed under the action of $\lambda(t)$. In particular this action λ is given by some character $z \mapsto z^r$. Define*

$$\mu^{\mathbb{L}}(x, \lambda) := r$$

The following are some basic properties of $\mu^{\mathbb{L}}(x,\lambda)$ (see [14] for details):

Proposition 5.5.2. *Suppose $x \in \mathbb{X}$ and $\lambda \in OPS(S)$ are such that the limit $\lim_{t\to 0}\lambda(t)x$ exists in $\mathbb{X}$. Let $\mathbb{L}, \mathbb{L}'$ be S-equivariant line bundles on $\mathbb{X}$. Then:*

(i) $\mu^{\mathbb{L}}(gx, g\lambda g^{-1}) = \mu^{\mathbb{L}}(x,\lambda)$ *for all* $g \in S$.

(ii) $\mu^{\mathbb{L}\otimes\mathbb{L}'}(x,\lambda) = \mu^{\mathbb{L}}(x,\lambda) + \mu^{\mathbb{L}'}(x,\lambda)$.

(iii) If there exists a $\sigma \in H^0(\mathbb{X},\mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^{\mathbb{L}}(x,\lambda) \geq 0$.

(iv) If $\mu^{\mathbb{L}}(x,\lambda) = 0$, then any element of $H^0(\mathbb{X},\mathbb{L})^S$ which does not vanish at x, does not vanish at $\lim_{t\to 0}\lambda(t)x$ as well.

We apply the above proposition to the situation in Proposition 5.4.4 (*iii*). Let $\mathbb{L} := \det(\tilde{\mathcal{T}})^* \otimes \det(\mathcal{T}/\mathcal{T}_w)$ denote the $\tilde{P}$-equivariant line bundle on P/B_L. If $\phi^*([\Lambda_w])$ = a nonzero multiple of a class of a point, then Propositions 5.4.4 and 5.5.2 imply that $\mu^{\mathbb{L}}(pB_L,\lambda) \geq 0$ for any P-admissible $\lambda \in OPS(\tilde{P})$ and generic $p \in P$. We prove Theorem 5.3.8 by determining $\mu^{\mathbb{L}}(pB_L,\lambda)$ explicitly for certain cases. By Proposition 5.5.2 (*ii*), it suffices to consider $\det(\tilde{\mathcal{T}})^*$ and $\det(\mathcal{T}/\mathcal{T}_w)$ separately.

5.5.1. Computing Mumford's number

Since $\det(\tilde{\mathcal{T}})^*$ is a trivial line bundle on P/B_L, the action of $\tilde{P}$ on the fiber is independent of the base point. Hence we only need to consider how $\tilde{P}$ acts on $\det(T_{e\tilde{P}}(\tilde{G}/\tilde{P}))^*$.

Let $\chi : \tilde{P} \to \mathbb{C}^*$ be the character such that the action of $\tilde{P}$ is given by $pl = \chi(p)l$ for any vector $l \in \det(T_{e\tilde{P}}(\tilde{G}/\tilde{P}))^*$. If λ is P-admissible and $\tilde{L}$-dominant, then

$$\mu^{\det(\tilde{\mathcal{T}})^*}(pB_L, \lambda) = d\chi(\dot{\lambda}).$$

Since the tangent vector $\dot{\lambda} \in \tilde{\mathfrak{h}}$, it suffices to consider $d\chi\big|_{\tilde{\mathfrak{h}}}$. Recall in Section 5.3, we defined

$$\tilde{\chi} := \sum_{\beta \in \tilde{R}^+ \setminus \tilde{R}^+_{\mathfrak{l}}} \beta.$$

Lemma 5.5.3. *The character $d\chi\big|_{\tilde{\mathfrak{h}}} = \tilde{\chi}$.*

Proof. Consider the Cartan decomposition of $T_{e\tilde{P}}(\tilde{G}/\tilde{P}) = \bigoplus_{\beta \in \tilde{R}^+ \setminus \tilde{R}^+_{\mathfrak{l}}} \mathfrak{g}_{-\beta}$. Hence $\tilde{\mathfrak{h}}$ acts on $\det(T_{e\tilde{P}}(\tilde{G}/\tilde{P}))$ by $-\tilde{\chi}$. Thus $\tilde{\mathfrak{h}}$ acts on the dual space by $\tilde{\chi}$. □

We now consider the line bundle $\det(\mathcal{T}/\mathcal{T}_w)$ on P/B_L. Recall that the $\tilde{P}$-equivariant structure of $\det(\mathcal{T}/\mathcal{T}_w)$ is the restriction of its P-equivariant structure under the action $\tilde{P} \subseteq vPv^{-1}$. Hence we can compute $\mu^{\det(\mathcal{T}/\mathcal{T}_w)}(pB_L, \lambda)$ with respect to the P action and then consider the restriction to $\tilde{P}$. For any character $\nu : H \to \mathbb{C}^*$ (note that any character on H uniquely extends to a character on B_L), we have the corresponding P-equivariant line bundle $\mathbb{L}(\nu) := P \times_{B_L} \mathbb{C}_\nu$ where

$$(p, c) \sim (pb, \nu(b)c).$$

Since any P-equivariant line bundle on P/B_L can be realized by this construction, there exists a character $\chi_w : H \to \mathbb{C}^*$ such that

$$\det(\mathcal{T}/\mathcal{T}_w) \simeq \mathbb{L}(\chi_w)$$

as a P-equivariant line bundles.

Lemma 5.5.4. *The character* $d\chi_w = \sum_{\beta \in R^+ \setminus R_{\mathfrak{l}}^+ \cap w^{-1}R^+} \beta$.

Proof. Consider the untwisted line bundle $\det(v^{-1}\mathcal{T}/v^{-1}\mathcal{T}_w)$ on P/B_L where

$$v^{-1}\mathcal{T} := P \times_{B_L} T_{eP}(G/P) \quad \text{and} \quad v^{-1}\mathcal{T}_w := P \times_{B_L} T_{eP}(\Lambda_w).$$

By [3, Lemma 6], we have that $\det(v^{-1}\mathcal{T}/v^{-1}\mathcal{T}_w) \simeq \mathbb{L}(\hat{\chi}_w)$. Where

$$d\hat{\chi}_w = \sum_{\beta \in R^+ \setminus R_{\mathfrak{l}}^+ \cap w^{-1}R^+} \beta.$$

Since P acts on $\det(\mathcal{T}/\mathcal{T}_w)$ through conjugated group vPv^{-1}, we have that the character $\hat{\chi}_w = \chi_w$. □

For any P-admissible, $\tilde{L}$-dominant $\lambda \in OPS(\tilde{H})$, choose $\hat{v} \in (W_L)_{\text{rel}} \subseteq W_L$ such that $\lambda_{\hat{v}} := (v\hat{v})^{-1}\lambda v\hat{v}$ is L-dominant. Let $Q = P^L(\lambda_{\hat{v}}) \subseteq L$ denote the standard parabolic associated to $\lambda_{\hat{v}}$ and let $P = U \cdot L$ be the Levi-decomposition of P (U denotes the unipotent radical of P).

Proposition 5.5.5. *For any P-admissible, $\tilde{L}$-dominant $\lambda \in OPS(\tilde{H})$ and $p = ul \in P$, we have*

$$\mu^{\det(\mathcal{T}/\mathcal{T}_w)}(pB_L, \lambda) = -f^*(v\hat{v}\hat{w}^{-1}\chi_w)(\dot{\lambda})$$

where $\hat{w} \in W_L^Q$ is determined by $\hat{v} \in lB_L\hat{w}Q \subseteq L$.

Proof. Let $\hat{v} \in N_L(H)$ be a representative of $\hat{v} \in W_L$. We remark that the conclusions of this lemma are independent of this choice. We analyze the action of λ on a generic fiber (p, c) of the line bundle $\mathbb{L}(\chi_w)$. We have that

$$\lambda(t)(p, c) = (\lambda_v(t)ul, c) = (\lambda_v(t)u\lambda_v(t)^{-1}\lambda_v(t)l, c).$$

Write $l = \hat{v}q\hat{w}^{-1}b$, where $q \in Q$ and $b \in B_L$. Since $Q = P^L(\lambda_{\hat{v}}) = \hat{v}^{-1}P^L(\lambda_v)\hat{v}$, there exists a $q' \in P^L(\lambda_v)$ such that $q = \hat{v}^{-1}q'\hat{v}$. Thus

$$\lambda_v(t)l = \lambda_v(t)q'\hat{v}\hat{w}^{-1}b = \lambda_v(t)q'\lambda_v^{-1}(t)\lambda_v(t)\hat{v}\hat{w}^{-1}b.$$

Define $b(t) \in OPS(B_L)$ by $b(t) := b^{-1}\hat{w}\lambda_{\hat{v}}^{-1}(t)\hat{w}^{-1}b$. Combining the above expressions, we get

$$\lambda(t)(p, c) = (\lambda_v(t)u\lambda_v(t)^{-1}\lambda_v(t)q'\lambda_v^{-1}(t)\lambda_v(t)\hat{v}\hat{w}^{-1}bb(t), \chi_w(b(t))c).$$

Since λ is P-admissible and $q' \in P^L(\lambda_v)$, we get the limit of the expression in the first factor exists in P. Finally, we write out

$$\chi_w(b(t)) = \chi_w(\hat{w}\lambda_{\hat{v}}^{-1}(t)\hat{w}^{-1}) = v\hat{v}\hat{w}^{-1}\chi_w(\lambda^{-1}(t)).$$

Thus

$$\mu^{\det(\mathcal{T}/\mathcal{T}_w)}(pB_L, \lambda) = -f^*(v\hat{v}\hat{w}^{-1}\chi_w)(\dot{\lambda}).$$

$\square$

5.5.2. Proof of Theorem 5.3.8

Proof. Let $w \in W^P$ be such that $\phi^*([\Lambda_w])$ = a nonzero multiple of a class of a point in $H^*(\tilde{G}/\tilde{P})$. This implies that

$$\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}$$

and by Proposition 5.4.4, the section $\theta \in H^0(P/B_L, \mathbb{L})^{\tilde{P}}$ is not identically zero. Hence, there exists an open subset $Z_\theta \subseteq P$ such that $\theta(pB_L) \neq 0$ for all $p \in Z_\theta$.

Let $\lambda \in OPS(\tilde{H})$ be admissible and $\tilde{L}$-dominant and assume $(\hat{w}, \hat{v}) \in W_L^{Q(\lambda_{\hat{v}})} \times (W_L)_{\text{rel}}$ satisfy the conditions that $\lambda \in \tilde{\mathfrak{h}}_+^L \cap v\hat{v}\mathfrak{h}_+^L$ and $(\phi_{\lambda,\hat{v}}^L)^*([\Lambda_{\hat{w}}]) \neq 0$. We abbreviate $\phi_{\lambda,\hat{v}}^L$ by simply ϕ_λ. For generic $l \in L$, we have that

$$\phi_\lambda^{-1}(lB_L\hat{w}Q) \neq \emptyset.$$

Let $\pi : P \to L$ denote the projection of P onto L induced by the Levi decomposition $U \cdot L$. Since π is surjective, there exists an open subset $Z_\lambda \subseteq P$ such that for every $p \in Z_\lambda$, we have $\phi_\lambda^{-1}(\pi(p)B_L\hat{w}Q) \neq \emptyset$. Define

$$Z := Z_\theta \cap Z_\lambda$$

Clearly Z is an open subset of P. Choose $p_0 \in Z$ and let $p_0 = u_0 l_0$ under the Levi decomposition. Now choose $l \in \tilde{L}$ such that $\phi_\lambda(l\tilde{Q}) \in l_0 BwQ$ and consider $\lambda_0 := l\lambda l^{-1} \in OPS(\tilde{P})$. By Lemma 5.3.7, λ_0 is P-admissible. By Proposition 5.5.2 (i), we have that

$$\mu^{\det(\tilde{\mathcal{T}})^*}(p_0 B_L, \lambda_0) = \mu^{\det(\tilde{\mathcal{T}})^*}(v^{-1}l^{-1}vp_0 B_L, \lambda) = \tilde{\chi}(\dot{\lambda}).$$

By Lemma 5.3.4, we have that $v^{-1}l^{-1}v \in L$. Since L normalizes U, we have that $u' := v^{-1}l^{-1}vu_0v^{-1}lv \in U$ and

$$v^{-1}l^{-1}vp_0 = v^{-1}l^{-1}vu_0l_0 = u'v^{-1}l^{-1}vl_0.$$

If we let $l' = v^{-1}l^{-1}vl_0 \in L$, then the Levi decomposition of $v^{-1}l^{-1}vp_0 = u'l'$. Note that $\hat{v} \in l'B_L\hat{w}Q$, since $v^{-1}lv\hat{v} \in l_0B_L\hat{w}Q$. By Proposition 5.5.5, we have that

$$\mu^{\det(\mathcal{T}/\mathcal{T}_w)}(p_0 B_L, \lambda_0) = \mu^{\det(\mathcal{T}/\mathcal{T}_w)}(v^{-1}l^{-1}vp_0 B_L, \lambda) = -f^*(v\hat{v}\hat{w}^{-1}\chi_w)(\dot{\lambda}).$$

Finally, since $p_0 \in Z$ and by Proposition 5.5.2 (ii) and (iii), we have

$$u^{\mathbb{L}}(p_0 B_L, \lambda_0) = (\tilde{\chi} - f^*(v\hat{v}\hat{w}^{-1}\chi_w))(\dot{\lambda}) \geq 0.$$

□

5.6. Levi-movability and Branching Schubert Calculus

In this section we generalize ideas of Levi-movability to branching Schubert calculus. The main results are generalizations of Proposition 2.1.5 and Theorem 3.4.2 both initially established by Belkale-Kumar in [3]. Once again, we fix $v \in N_G(H)$ to be a representative of $v \in W_{\text{rel}}$ in this analysis.

Definition 5.6.1. *We say $w \in W^P$ is (L, ϕ)-movable if $\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}$ and for generic $l \in L$, the point $e\tilde{P}$ is scheme theoretically isolated in $\phi^{-1}(vl\Lambda_w)$.*

Clearly, if w is (L, ϕ)-movable, then $\phi^*([\Lambda_w]) \neq 0$, however the converse is not true in general. Consider the restriction of the line bundle $\mathbb{L} = \det(\tilde{\mathcal{T}})^* \otimes \det(\mathcal{T}/\mathcal{T}_w)$ on P/B_L to L/B_L. Since $v^{-1}\tilde{L}v \subset L$, we can view $\det(\tilde{\mathcal{T}})^*$ and $\det(\mathcal{T}/\mathcal{T}_w)$ as $\tilde{L}$-equivariant line bundles on L/B_L. Let $\hat{\theta}$ denote the restriction of the section $\theta \in H^0(P/B_L, \mathbb{L})^{\tilde{P}}$ defined in Section 5.4.1 to L/B_L. By Proposition 5.4.4, the following proposition is immediate

Proposition 5.6.2. *We have that $w \in W^P$ is (L, ϕ)-movable if and only if the section*

$$\hat{\theta} \in H^0(L/B_L,\ \mathbb{L})^{\tilde{L}}$$

does not vanish for some $l \in L$.

5.6.1. Generalized Proposition 2.1.5: A numerical condition for (L, ϕ)-movability

The following theorem is a numerical criterion for (L, ϕ)-movability which generalizes the condition stated in Proposition 2.1.5.

Theorem 5.6.3. *Let $w \in W^P$ be such that $\dim \Lambda_w = \dim G/P - \dim \tilde{G}/\tilde{P}$. Then w is (L, ϕ)-movable if and only if $\phi^*([\Lambda_w]) \neq 0$ in $H^*(\tilde{G}/\tilde{P})$ and $(f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) = 0$.*

By Corollary 5.3.9, for any $w \in W^P$ such that $\phi^*([\Lambda_w]) \neq 0$, we have $(f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) \leq 0$. Hence (L, ϕ)-movability is still a boundary condition to a linear inequality as in the case of the diagonal embedding. Recall equations (1.1) and (1.6) in Chapter 1 for examples. Before we can prove Theorem 5.6.3, we need the following lemma.

Lemma 5.6.4. *The image of τ is contained in $Z(\tilde{L})$, the center of $\tilde{L}$.*

Proof. By definition, $l \in \tilde{L}$ if and only if $\lim_{t \to 0} \tau(t) l \tau^{-1}(t) = l$. Hence for any $s \in \mathbb{C}^*$, we have that

$$\tau(s) l \tau^{-1}(s) = \tau(s)(\lim_{t \to 0} \tau(t) l \tau^{-1}(t))\tau^{-1}(s) = \lim_{t \to 0} \tau(ts) l \tau^{-1}(ts) = l.$$

Hence $\tau(s) l = l \tau(s)$ for all $l \in \tilde{L}$. □

Note that by the same argument, we have that τ_v is contained in $Z(L)$, the center of L.

Proof of Theorem 5.6.3: If w is (L, ϕ)-movable, then $\hat{\theta}$ is non-vanishing in $H^0(L/B_L, \mathbb{L})^{\tilde{L}}$. Hence the center of $\tilde{L}$ acts trivially on $\mathbb{L}$. By Lemma 5.6.4, the OPS τ is in the center of $\tilde{L}$ and thus $u^{\mathbb{L}}(lB_L, \tau) = 0$ for generic $l \in L$. By Corollary 5.3.9, we have that

$$u^{\mathbb{L}}(lB_L, \tau) = (f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) = 0.$$

Now assume that $\phi^*([\Lambda_w]) \neq 0$ and $(f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) = 0$. The first assumption implies that for some $p \in P$, the section $\theta(pB_L) \neq 0$. If we write $p = ul$ with the respect to the Levi decomposition $P = U \cdot L$, then

$$\lim_{t \to 0} \tau_v(t) p B_L = \lim_{t \to 0} \tau_v(t) u \tau_v^{-1}(t) \tau_v(t) l p B_L.$$

Since τ_v is L-dominant, we get

$$\lim_{t \to 0} \tau_v(t) u \tau_v^{-1}(t) = 1.$$

Hence $\lim_{t \to 0} \tau_v(t) p B_L = l B_L \in L/B_L$. Since τ_v is central in L and by the second assumption, for generic $p \in P$ we have that

$$u^{\mathbb{L}}(pB_L, \tau) = (f^*(v\chi_w) - \tilde{\chi})(\dot{\tau}) = 0.$$

By Proposition 5.5.2 (iv), we have that $\theta(lB_L) = \hat{\theta}(lB_L) \neq 0$ since $\theta(pB_L) \neq 0$. By Proposition 5.6.2, w is (L, ϕ)-movable. □

5.6.2. Generalized Theorem 3.4.2: Horn recursion for (L, ϕ)-movability

In Theorem 3.4.2, Belkale-Kumar construct a list of necessary Horn inequalities which are partly indexed by central characters $c : Z(\tilde{L}) \to \mathbb{C}^*$. In the branching Schubert calculus setting we consider a different subgroup of $\tilde{L}$. Recall that $\tilde{L} \subseteq vLv^{-1}$ and consider the subgroup

$$Z := Z(vLv^{-1}) \cap \tilde{L} \subseteq vLv^{-1}$$

where $Z(vLv^{-1})$ denotes the center of vLv^{-1}. Clearly we have that $Z \subseteq Z(\tilde{L})$ and note that in the case of the diagonal embedding, we have that $Z = Z(\tilde{L})$.

Definition 5.6.5. *For any character $c : Z \to \mathbb{C}^*$ and $w \in W^P$, define*

$$\tilde{R}(c) := \{\beta \in \tilde{R}^+ \backslash \tilde{R}^+_{\mathfrak{l}} \ : \ e^{\beta}\big|_Z = c\}.$$

and

$$R(c,w) := \{\beta \in R^+ \backslash R^+_{\mathfrak{l}} \cap w^{-1}R^+ \ : \ e^{(v\beta)}\big|_Z = c\}.$$

Definition 5.6.6. *For any character $c : Z \to \mathbb{C}^*$ and $w \in W^P$, define the characters*

$$\chi^c_w := \sum_{\beta \in R(c,w)} \beta \quad \text{and} \quad \tilde{\chi}^c := \sum_{\beta \in \tilde{R}(c)} \beta.$$

Theorem 5.6.7. *Let $w \in W^P$ be (L, ϕ)-movable. Then the following are true:*

(i) For any character c of Z such that $\tilde{\chi}^c \neq 0$, we have

$$|\tilde{R}(c)| = |R(c,w)|$$

where $|\cdot|$ denotes cardinality of the enclosed set.

(ii) For any $\tilde{L}$-dominant $\lambda \in OPS(\tilde{H})$ and $(\hat{w}, \hat{v}) \in W_L^{Q(\lambda_{\hat{v}})} \times (W_L)_{\mathrm{rel}}$ such that $\lambda \in \tilde{\mathfrak{h}}^L_+ \cap v\hat{v}\mathfrak{h}^L_+$ and $(\phi^L_{\lambda,\hat{v}})^*([\Lambda_{\hat{w}}]) \neq 0$ in $H^*(\tilde{L}/\tilde{Q}(\lambda))$ and for any character c of Z such that $\tilde{\chi}^c \neq 0$, the following inequality holds:

$$(f^*(v\hat{v}\hat{w}^{-1}\chi^c_w) - \tilde{\chi}^c)(\dot{\lambda}) \leq 0.$$

Observe that there is no need for P-admissibility in the above theorem since L/B_L is a projective variety. To prove Theorem 5.6.7, we follow the same setup and proof of Theorem

5.3.8. For any character c of Z define

$$\tilde{T}^c := \{m \in T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \mid t \cdot m = c(t)m, \ \forall t \in Z\}.$$

Since $Z \subseteq Z(\tilde{L})$, for any character c of Z, the action of $\tilde{L}$ fixes $\tilde{T}^c \subseteq T_{e\tilde{P}}(\tilde{G}/\tilde{P})$. As in Section 5.4.1, we can define the $\tilde{L}$-equivariant bundle $\tilde{\mathcal{T}}^c := L/B_L \times \tilde{T}^c$ on L/B_L. Similarly, define

$$T^c := \{m \in T_{vP}(G/P) \mid t \cdot m = c(t)m, \ \forall t \in Z\}$$

and note that the vLv^{-1} action fixes T^c since $Z \subseteq Z(vLv^{-1})$. Let $\mathcal{T}^c := L/B_L \times T^c$ denote the corresponding L-equivariant vector bundle on L/B_L. Observe that T^c is also a B_L-module and hence we can define $(\mathcal{T}^c)' := L \times_{B_L} T^c$. Note that the vector bundle $\mathcal{T}^c$ is L-equivariantly isomorphic to $(\mathcal{T}^c)'$. Finally, for any $w \in W^P$, define

$$T^c_w := \{m \in T_{vP}(v\Lambda_w) \mid t \cdot m = c(t)m, \ \forall t \in Z\}$$

and the corresponding sub-bundle $\mathcal{T}^c_w := L \times_{B_L} T^c_w \subseteq \mathcal{T}^c$. Consider the tangent space map

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to T_{vP}(G/P).$$

Since ϕ_* is $\tilde{P}$-equivariant, it is also $\tilde{L}$-equivariant and hence $\phi_*(\tilde{T}^c) \subseteq T^c$.

Proof of Theorem 5.6.7 (i): If w is (L, ϕ)-movable, then the induced map

$$\phi_* : T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to T_{vP}(G/P)/T_{vP}(vl\Lambda_w)$$

is an isomorphism for generic $l \in L$. Consider the decompositions

$$T_{e\tilde{P}}(\tilde{G}/\tilde{P}) = \bigoplus_c \tilde{T}^c, \quad T_{vP}(G/P) = \bigoplus_c T^c \quad \text{and} \quad T_{vP}(vl\Lambda_w) = \bigoplus_c lT^c_w$$

as c ranges over all characters of Z such that $\tilde{R}(c) \neq \emptyset$. Note that there are only a finite number of such c and that this condition is equivalent to $\tilde{\chi}_c \neq 0$. Since $\phi_*(\tilde{T}^c) \subseteq T^c$, we have that

$$\phi_*|_{\tilde{T}^c} : \tilde{T}^c \to T^c/lT^c_w$$

is also an isomorphism. By Lemma 2.1.6, the rank of $\tilde{T}^c$ is equal to $|\tilde{R}(c)|$ and the rank of T^c/T^c_w is equal to $|R(c,w)|$. This proves Theorem 5.6.7 (i). □

Consider the induced map on vector bundles over L/B_L

$$\Theta^c : \tilde{\mathcal{T}}^c \to \mathcal{T}^c/\mathcal{T}^c_w.$$

As in Section 5.4.1, let $\theta^c : \det(\tilde{\mathcal{T}}^c) \to \det(\mathcal{T}^c/\mathcal{T}^c_w)$ denote the determinant map of Θ^c. Observe that we can view θ^c as a $\tilde{L}$-invariant section of the line bundle

$$\det(\tilde{\mathcal{T}}^c)^* \otimes \det(\mathcal{T}^c/\mathcal{T}^c_w).$$

Proof of Theorem 5.6.7 (ii): If w is (L,ϕ)-movable, the map Θ^c is an isomorphism over an open set of L/B_L. This implies that $\theta^c(lB_L) \neq 0$ for generic $l \in L$. Following the proof of Theorem 5.3.8 we have that for any $\tilde{L}$-dominant $\lambda \in OPS(\tilde{H})$ and $(\hat{w},\hat{v}) \in W_L^{Q(\lambda_{\hat{v}})} \times (W_L)_{\mathrm{rel}}$ satisfying the conditions in Theorem 5.6.7, the following inequality is valid:

$$(f^*(v\hat{v}\hat{w}^{-1}\chi^c_w) - \tilde{\chi}^c)(\dot{\lambda}) \leq 0.$$

This completes the proof. □

CHAPTER 6

Examples of determining L-movability and structure coefficients

In this chapter we give a summary of the induced Weyl groups elements used throughout this thesis and basic examples of applying Theorems 1.1.2 and 1.1.4 on type A flag varieties and Theorems 1.2.1 an 1.2.2 on type C flag varieties. There are two major types of induced Weyl group elements. The first type is by simply taking the image under the projection $W^P \to W^{P_i}$ for any maximal parabolic subgroups P_i which contains P. We always denote the image of w as w_i. The second uses Definition 1.1.1 which gives an induced permutation associated to a subset of $[n]$ or $[2n]$.

6.1. Type A example

6.1.1. Type A permutations

For $\mathrm{F}\ell(a, n)$, we identify W^P with

$$S_n(a) := \{(w(1), w(2), \ldots, w(n)) \in S_n \mid w(i) < w(i+1) \ \forall \ i \notin a\}.$$

Consider the example when $n = 8$ and $a = \{2, 5\}$. We have the following splitting found in equation (3.1):

$$\mathbb{C}^8 = Q_1 \oplus Q_2 \oplus Q_3.$$

For the following induced permutations, we use short lines “$\shortmid$” to indicate the breaks corresponding to the set $a = \{2,5\}$ and framed boxes to indicate the relevant parts of the permutation. Let

$$w = (3,7 \shortmid 1,4,5 \shortmid 2,6,8) \in S_8(\{2,5\}).$$

Theorem 1.1.2 (*iii*) uses the following induced permutations:

$$w = (\boxed{3,7} \shortmid \boxed{1,4,5} \shortmid 2,6,8) \quad \rightarrow \quad w_{1,2} = (2,5 \shortmid 1,3,4) \in S_5(2)$$

$$w = (\boxed{3,7} \shortmid 1,4,5 \shortmid \boxed{2,6,8}) \quad \rightarrow \quad w_{1,3} = (2,4 \shortmid 1,3,5) \in S_5(2)$$

$$w = (3,7 \shortmid \boxed{1,4,5} \shortmid \boxed{2,6,8}) \quad \rightarrow \quad w_{2,3} = (1,3,4 \shortmid 2,5,6) \in S_6(3)$$

Theorem 1.1.4 uses the following induced permutations:

$$w = (\boxed{3,7} \shortmid \boxed{1,4,5 \shortmid 2,6,8}) \quad \rightarrow \quad w_1 = (3,7 \shortmid 1,2,4,5,6,8) \in S_8(2)$$

$$w = (3,7 \shortmid \boxed{1,4,5} \shortmid \boxed{2,6,8}) \quad \rightarrow \quad w_\gamma = (1,3,4 \shortmid 2,5,6) \in S_3(6).$$

The proof of Theorem 1.1.2 (*i*) $\Leftrightarrow$ (*ii*) uses the following induced permutations:

$$w = (\boxed{3,7} \shortmid \boxed{1,4,5 \shortmid 2,6,8}) \quad \rightarrow \quad w_1 = (3,7 \shortmid 1,2,4,5,6,8) \in S_8(2)$$

$$w = (\boxed{3,7 \shortmid 1,4,5} \shortmid \boxed{2,6,8}) \quad \rightarrow \quad w_2 = (1,3,4,5,7 \shortmid 2,6,8) \in S_8(5).$$

6.1.2. Example

Let $n = 7$ and $a = \{1,4\}$ and consider the flag variety $\mathrm{F}\ell(\{1,4\},7)$. Let

$$w^1 = w^2 = (7,2,4,6,1,3,5), \; w^3 = (1,3,5,7,2,4,6).$$

We apply Theorem 1.1.2 (iii) to show that the 3-tuple (w^1, w^2, w^3) is L-movable. The induced Grassmannians are $\mathrm{Gr}(1,3)$, $\mathrm{Gr}(1,3)$ and $\mathrm{Gr}(2,4)$. We have that

$$w^1_{1,2} = w^2_{1,2} = (4,1,2,3) \quad w^3_{1,2} = (1,2,3,4)$$
$$w^1_{1,3} = w^2_{1,3} = (4,1,2,3) \quad w^3_{1,3} = (1,2,3,4)$$
$$w^1_{2,3} = w^2_{2,3} = w^3_{2,3} = (2,4,6,1,3,5)$$

Since the structure constants associated to these induced 3-tuples are nonzero, we have that (w^1, w^2, w^3) is L-movable. Note that if an induced 3-tuple's associated structure constant is unknown, we can apply Theorem 1.1.2 (iv).

We now apply Theorem 1.1.4 to compute the associated structure constant. Consider the projection $f : \mathrm{F}\ell(\{1,4\},7) \to \mathrm{Gr}(1,7)$. The fiber $f^{-1}(V)$ is isomorphic to the Grassmannian $\mathrm{Gr}(3,6)$. We have that

$$w^1_1 = w^2_1 = (7,1,2,3,4,5,6) \quad w^3_1 = (1,2,3,4,5,6,7)$$
$$w^1_\gamma = w^2_\gamma = w^3_\gamma = (2,4,6,1,3,5)$$

Hence $c = c_1 \cdot c_\gamma = 1 \cdot 2 = 2$

6.2. Type C example

6.2.1. Type C permutations

For $\mathrm{IF}(a, 2n)$, we identify W^P with

$$S^C_{2n}(a) := \{w \in S_{2n} \mid w(2n+1-i) = 2n+1-w(i) \ \forall i \in [n] \text{ and } w(i) < w(i+1) \ \forall\ i \notin a\}.$$

Consider the example when $n = 6$ and $a = \{2, 5\}$. We have the following splitting found in equation (3.11):

$$\mathbb{C}^{12} = Q_1 \oplus Q_2 \oplus \tilde{Q} \oplus \bar{Q}_2 \oplus \bar{Q}_1.$$

Let

$$w = (3,6 \mid 1,8,11 \mid 4,9 \mid 2,5,12 \mid 7,10) \in S_{12}^{C}(\{2,5\}).$$

Theorem 1.2.1 (*iii*) uses the following induced permutations:

$$w = (\, \boxed{3,6} \mid \boxed{1,8,11} \mid 4,9 \mid 2,5,12 \mid 7,10 \,) \quad \rightarrow \quad w_{1,2} = (2,3 \mid 1,4,5) \in S_5(2)$$

$$w = (\, \boxed{3,6} \mid 1,8,11 \mid \boxed{4,9} \mid 2,5,12 \mid 7,10 \,) \quad \rightarrow \quad \tilde{w}_1 = (1,3 \mid 2,4) \in S_4(2)$$

$$w = (\, \boxed{3,6} \mid 1,8,11 \mid 4,9 \mid \boxed{2,5,12} \mid 7,10 \,) \quad \rightarrow \quad \bar{w}_{1,2} = (2,4 \mid 1,3,5) \in S_5(2)$$

$$w = (\, \boxed{3,6} \mid 1,8,11 \mid 4,9 \mid 2,5,12 \mid \boxed{7,10} \,) \quad \rightarrow \quad \bar{w}_{1,1} = (1,2 \mid 3,4) \in S_4^C(2)$$

$$w = (\, 3,6 \mid \boxed{1,8,11} \mid \boxed{4,9} \mid 2,5,12 \mid 7,10 \,) \quad \rightarrow \quad \tilde{w}_2 = (1,3,5 \mid 2,4) \in S_5(3)$$

$$w = (\, 3,6 \mid \boxed{1,8,11} \mid 4,9 \mid \boxed{2,5,12} \mid 7,10 \,) \quad \rightarrow \quad \bar{w}_{2,2} = (1,4,5 \mid 2,3,6) \in S_6^C(3).$$

Theorem 1.2.2 uses the following induced permutations:

$$w \quad = \quad (\, \boxed{3,6 \mid 1,8,11} \mid \boxed{4,9} \mid \boxed{2,5,12 \mid 7,10} \,) \rightarrow$$

$$w_2 = (1,3,6,8,11 \mid 4,9 \mid 2,5,7,10,12) \in S_{12}^C(5)$$

$$w \quad = \quad (\, \boxed{3,6} \mid \boxed{1,8,11} \mid 4,9 \mid 2,5,12 \mid 7,10 \,) \rightarrow w_\gamma = (2,3 \mid 1,4,5) \in S_5(2).$$

The proof of Theorem 1.2.1 $(i) \Leftrightarrow (ii)$ uses the following induced permutations:

$$w = (\boxed{3,6} \mid \boxed{1,8,11 \mid 4,9 \mid 2,5,12} \mid \boxed{7,10}) \rightarrow$$

$$w_1 = (3,6 \mid 1,2,4,5,8,9,11,12 \mid 7,10) \in S_{12}^C(2)$$

$$w = (\boxed{3,6 \mid 1,8,11} \mid \boxed{4,9} \mid \boxed{2,5,12 \mid 7,10}) \rightarrow$$

$$w_2 = (1,3,6,8,11 \mid 4,9 \mid 2,5,7,10,12) \in S_{12}^C(5)$$

$$w = (\boxed{3,6} \mid 1,8,11 \mid 4,9 \mid 2,5,12 \mid \boxed{7,10}) \rightarrow \bar{w}_1 = (1,2 \mid 3,4) \in S_4^C(2)$$

$$w = (\boxed{3,6 \mid 1,8,11} \mid 4,9 \mid \boxed{2,5,12 \mid 7,10}) \rightarrow$$

$$\bar{w}_2 = (1,3,5,7,9 \mid 2,4,6,8,10) \in S_{10}^C(5).$$

6.2.2. Example

Let $n = 4$ and $a = \{2,3\}$ and consider the flag variety $\mathrm{IF}(\{2,3\},8)$. Let

$$w^1 = (1,7,6,4,5,3,2,8),\ \ w^2 = (3,8,4,2,7,5,1,6),\ \ w^3 = (6,8,5,2,7,4,1,3).$$

We apply Theorem 1.2.1 (iii) to show that the 3-tuple (w^1, w^2, w^3) is L-movable. The induced Grassmannians are $\mathrm{Gr}(2,3)$, $\mathrm{Gr}(2,3)$, $\mathrm{Gr}(2,4)$, $\mathrm{Gr}(1,3)$, $\mathrm{LG}(2,4)$ and $\mathrm{LG}(1,2)$. We have that

$$\begin{array}{ccc}
w^1_{1,2} = (1,3,2) & w^2_{1,2} = (1,3,2) & w^3_{1,2} = (2,3,1) \\
\bar{w}^1_{1,2} = (1,3,2) & \bar{w}^2_{1,2} = (1,3,2) & \bar{w}^3_{1,2} = (2,3,1) \\
\tilde{w}^1_1 = (1,4,2,3) & \tilde{w}^2_1 = (2,4,1,3) & \tilde{w}^3_1 = (2,4,1,3) \\
\tilde{w}^1_2 = (3,1,2) & \tilde{w}^2_2 = (2,1,3) & \tilde{w}^3_2 = (2,1,3) \\
\bar{w}^1_{1,1} = (1,3,2,4) & \bar{w}^2_{1,1} = (2,4,1,3) & \bar{w}^3_{1,1} = (3,4,1,2) \\
\bar{w}^1_{2,2} = (2,1) & \bar{w}^2_{2,2} = (2,1) & \bar{w}^3_{2,2} = (1,2)
\end{array}$$

It is easy to see that the structure constants associated to these induced 3-tuples are nonzero and hence (w^1, w^2, w^3) is L-movable.

We now apply Theorem 1.2.2 to compute the associated structure constant. Consider the projection $f : \mathrm{IF}(\{2,3\}, 8) \to \mathrm{IG}(3, 8)$. The fiber $f^{-1}(V)$ is isomorphic to the Grassmannian $\mathrm{Gr}(2, 3)$. We have that

$$w_2^1 = (1,6,7,4,5,2,3,8) \quad w_2^2 = (3,4,8,2,7,1,5,6) \quad w_2^3 = (5,6,8,2,7,1,3,4)$$
$$w_\gamma^1 = (1,3,2) \qquad\qquad w_\gamma^2 = (1,3,2) \qquad\qquad w_\gamma^3 = (2,3,1)$$

It is easy to see that $c_\gamma = 1$. To compute the structure constant associated to (w_2^1, w_2^2, w_2^3) we fix generic complete isotropic flags $F_\bullet, G_\bullet, H_\bullet \in \mathrm{IF}(8)$ and consider the intersection

$$\Phi^\circ_{w_2^1}(F_\bullet) \cap \Phi^\circ_{w_2^2}(G_\bullet) \cap \Phi^\circ_{w_2^3}(H_\bullet).$$

If we consider the corresponding type A Schubert cells in $\mathrm{Gr}(3, 8)$ with respect to generic type A complete flags, we have that the only point in the intersection is the 3 dimensional vector space $V := \mathrm{Span}\{F_1, G_6 \cap H_4\}$. It suffices to check that if $F_\bullet, G_\bullet, H_\bullet$ are isotropic, then V is isotropic. Let $v_1 = f_1 + h_1$ and $v_2 = f_2 + h_2$ where $v_1, v_2 \in V$ and $f_1, f_2 \in F_1$ and $h_1, h_2 \in H_4 \cap V$. Then

$$\langle v_1, v_2\rangle = \langle f_1, f_2\rangle + \langle f_1, h_2\rangle + \langle h_1, f_2\rangle + \langle h_1, h_2\rangle = \langle f_1, h_2\rangle + \langle h_1, f_2\rangle$$

since F_1 and H_4 are isotropic. Since $V \in \Phi^\circ_{w_2^1}(F_\bullet)$, we have that $V \subset F_7 = F_1^\perp$. Hence

$$\langle f_1, h_2\rangle + \langle h_1, f_2\rangle = 0$$

since $h_1, h_2 \in V$ and thus V is isotropic. This implies that the structure constant $c_2 = 1$. By Theorem 1.2.2, the structure constant associated to (w^1, w^2, w^3) is $c = c_2 \cdot c_\gamma = 1$.

CHAPTER 7

Examples in Branching Schubert calculus

In this chapter, we work out some examples regarding Theorem 5.3.8 on branching Schubert calculus. The embeddings we consider are $Sp(2n) \subseteq SL(2n)$, $SO(2n) \subseteq SL(2n)$ and $SL(n) \subseteq SL(V)$ where V is an irreducible representation of $SL(n)$.

7.1. The Symplectic group embedding

Let $\mathbb{C}^{2n}$ be a $2n$ dimensional complex vector space with a skew-symmetric bilinear form. Consider the groups $\tilde{G} = Sp(2n)$ and $G = SL(2n) \times Sp(2n)$ and consider the diagonal embedding

$$f : Sp(2n) \hookrightarrow SL(2n) \times Sp(2n)$$

induced from the natural inclusion $Sp(2n) \subseteq SL(2n)$. Please refer to Chapter 2 for details on these groups. We fix the following objects associated to $\tilde{G} = Sp(2n)$ and $G = SL(2n) \times Sp(2n)$.

$$\begin{aligned}
\tilde{H} &= \{\mathrm{diag}(t_1, t_2, \ldots, t_n, t_n^{-1}, t_{n-1}^{-1}, \ldots, t_1^{-1})\} \\
\tilde{\mathfrak{h}} &= \{\mathrm{diag}(t_1, t_2, \ldots, t_n, -t_n, -t_{n-1}, \ldots, -t_1)\} \\
\tilde{\mathfrak{h}}_+ &= \{\mathbf{t} \in \tilde{\mathfrak{h}} \mid t_i \in \mathbb{R} \text{ and } t_1 \geq t_2 \geq \cdots \geq t_n \geq 0\} \\
H &= \{\mathbf{t} = \mathrm{diag}(t'_1, t'_2, \ldots, t'_{2n}) \mid \prod_{i=1}^{2n} t'_i = 1\} \times \tilde{H} \\
\mathfrak{h} &= \{\mathbf{t'} = \mathrm{diag}(t'_1, t'_2, \ldots, t'_{2n}) \mid \sum_{i=1}^{2n} t'_i = 0\} \times \tilde{\mathfrak{h}} \\
\mathfrak{h}_+ &= \{\mathbf{t'} \in \mathfrak{h} \mid t'_i \in \mathbb{R} \text{ and } t'_1 \geq t'_2 \geq \cdots \geq t'_{2n}\} \times \tilde{\mathfrak{h}}_+
\end{aligned}$$

Immediately, we see that $f_*(\tilde{\mathfrak{h}}_+) \subseteq \mathfrak{h}_+$ and hence the relative Weyl set $W_{\text{rel}} = \{1\}$. We now work out some examples with respect to certain choices of $\tau \in OPS(\tilde{H})$.

7.1.1. The Isotropic Grassmannian embedding

Fix $r \leq n$ and choose the dominant $\tau \in OPS(\tilde{H})$ defined as

$$\tau(t) := \text{diag}(t, \dots, t, 1, \dots, 1, t^{-1}, \dots, t^{-1})$$

where the value of the first r entries is t and the value of the next $n-r$ entries is 1. Clearly, τ viewed as an OPS of $SL(2n) \times Sp(2n)$ gives

$$f(\tau(t)) = \text{diag}(t, \dots, t, 1, \dots, 1, t^{-1}, \dots, t^{-1}) \times \text{diag}(t, \dots, t, 1, \dots, 1, t^{-1}, \dots, t^{-1}).$$

Since $W_{\text{rel}} = \{1\}$, the OPS τ is also dominant for $SL(2n) \times Sp(2n)$. The parabolic subgroup of $Sp(2n)$ with respect to τ is the maximal parabolic $P^{Sp(2n)}(\tau) = P_r^C$ and the parabolic subgroup of $SL(2n) \times Sp(2n)$ is the product of parabolics $P^{SL(2n) \times Sp(2n)}(\tau) = P^A \times P_r^C$ where $\Delta \backslash \Delta(P^A) = \{\alpha_r, \alpha_{2n-r}\}$. Hence the map of flag varieties is

$$\phi : \text{IG}(r, 2n) \hookrightarrow \text{F}\ell(\{r, 2n-r\}, 2n) \times \text{IG}(r, 2n)$$

given by $V \mapsto (V \subseteq V^\perp, V)$. The Levi subgroup of P_r^C is

$$\tilde{L} = GL(r) \times Sp(2n - 2r)$$

and the Levi subgroup of $P^A \times P_r^C$ is

$$L = \{(g_1, \ldots, g_5) \in GL(r)^3 \times GL(2n-2r) \times Sp(2n-2r) \mid \prod_{i=1,2,4} \det g_i = 1\}.$$

Note that we identify the second $GL(r)$ factor with its anti-diagonal transpose. For the Levi dominant chambers, we have

$$\begin{aligned} \tilde{\mathfrak{h}}_+^{\tilde{L}} &= \{\mathbf{t} \in \tilde{\mathfrak{h}} \mid t_i \in \mathbb{R} \text{ and } t_1 \geq \cdots \geq t_r \text{ and } t_{r+1} \geq \cdots \geq t_n \geq 0\} \\ \mathfrak{h}_+^L &= \{\mathbf{t'} \in \mathfrak{h} \mid t'_i \in \mathbb{R} \text{ and } t'_i \geq t'_{i+1} \ \forall \ i \neq r, 2n-r\} \times \tilde{\mathfrak{h}}_+^{\tilde{L}}. \end{aligned}$$

Since $f_*(\tilde{\mathfrak{h}}_+^{\tilde{L}}) \subseteq \mathfrak{h}_+^L$, the relative Weyl set $(W_L)_{\text{rel}} = \{1\}$. The map $\tilde{L} \hookrightarrow L$ is given by $(g_1, g_2) \mapsto (g_1, g_1^{-1}, g_1, g_2, g_2)$. We consider a particular set of admissible, $\tilde{L}$-dominant one parameter subgroups. For any $d_1 \in [r-1]$ and $d_2 \in [n-r]$, let $\lambda_{d_1,d_2} \in OPS(\tilde{H})$ be defined by

$$\lambda_{d_1,d_2}(t) := \operatorname{diag}(t^{m_1}, t^{m_2}, \ldots, t^{m_n}, t^{-m_n}, \ldots, t^{-m_2}, t^{-m_1})$$

where

$$m_i := \begin{cases} 3 & \text{if} \quad 1 \leq i \leq d_1 \\ 2 & \text{if} \quad d_1 < i \leq r \\ 1 & \text{if} \quad r < i \leq r + d_2 \\ 0 & \text{if} \quad r + d_2 < i \leq n \end{cases}$$

It is easy to see that λ_{d_1,d_2} is both $\tilde{L}$ dominant and L-dominant. Since $P^{Sp(2n)}(\tau) \subseteq P^{Sp(2n)}(\lambda_{d_1,d_2})$, we have that λ_{d_1,d_2} is also admissible. The flag varieties

$$\tilde{L}/\tilde{Q}(\lambda_{d_1,d_2}) \simeq \operatorname{Gr}(d_1, r) \times \operatorname{IG}(d_2, 2n-2r)$$

and

$$L/Q(\lambda_{d_1,d_2}) \simeq \operatorname{Gr}(d_1, r)^3 \times \operatorname{Gr}(d_2, 2n-2r) \times \operatorname{IG}(d_2, 2n-2r).$$

We analyze the map

$$\tilde{L}/\tilde{Q}(\lambda_{d_1,d_2}) \hookrightarrow L/Q(\lambda_{d_1,d_2})$$

on each factor of $\tilde{L}/\tilde{Q}(\lambda_{d_1,d_2})$. This map breaks up into the diagonal embeddings

$$\phi_1 : \mathrm{Gr}(d_1, r) \hookrightarrow \mathrm{Gr}(d_1, r)^3$$

and

$$\phi_2 : \mathrm{IG}(d_2, 2n-2r) \hookrightarrow \mathrm{Gr}(d_2, 2n-2r) \times \mathrm{IG}(d_2, 2n-2r).$$

Let $(w, \tilde{w}) \in W^P \simeq S^A_{2n}(\{r, 2n-r\}) \times S^C_{2n}(r)$ be such that

$$\dim \Lambda_{(w,\tilde{w})} = \dim G/P - \dim \tilde{G}/\tilde{P} = \dim \mathrm{F}\ell(\{r, 2n-r\}, 2n) = 4nr - 3r^2.$$

This condition reduces to $\ell^A(w) + \ell^C(\tilde{w}) = 4nr - 3r^2$. Assume that $\phi^*([\Lambda_{(w,\tilde{w})}]) \neq 0$, then by Theorem 5.3.8, for any

$$u = (u^1, \dots, u^5) \in S^A_r(d_r)^3 \times S^A_{2n-2r}(d_2) \times S^C_{2n-2r}(d_2)$$

such that

$$\phi_1^*(\Lambda_{(u^1,u^2,u^3)}) = [X_{u^1}] \cdot [X_{u^2}] \cdot [X_{u^3}] \neq 0 \in H^*(\mathrm{Gr}(d_1, r))$$

and

$$\phi_2^*(\Lambda_{(u^4,u^5)}) \neq 0 \in H^*(\mathrm{IG}(d_2, 2n-2r))$$

have that

$$(f^*(u^{-1}\chi_{(w,\tilde{w})}) - \tilde{\chi})(\dot{\lambda}_{d_1,d_2}) \leq 0.$$

7.1.2. The Lagrangian embedding

We consider the special case where $r = n$ in the previous section in more detail. Consider the one parameter subgroup

$$\tau(t) := \mathrm{diag}(t, \ldots, t, t^{-1}, \ldots, t^{-1})$$

where the value of the first n entries is t. The parabolic subgroup of $Sp(2n)$ with respect to τ is the maximal parabolic $P^{Sp(2n)}(\tau) = P_n^C$ and the parabolic subgroup of $SL(2n) \times Sp(2n)$ is the product of maximal parabolics $P^{SL(2n)\times Sp(2n)}(\tau) = P_n^A \times P_n^C$. Hence the map of flag varieties is

$$\phi : \mathrm{LG}(n, 2n) \hookrightarrow \mathrm{Gr}(n, 2n) \times \mathrm{LG}(n, 2n).$$

The Levi subgroup of P_n^C is $\tilde{L} = GL(n)$ and the Levi subgroup of $P_n^A \times P_n^C$ is

$$L = \{(g_1, g_2, g_3) \in GL(n)^3 \mid \det g_1 \cdot \det g_2 = 1\}.$$

The map $\tilde{L} \hookrightarrow L$ is given by $g \mapsto (g, g^{-1}, g)$. Since $n = r$, we consider a different set of $\tilde{L}$-dominant one parameter subgroups which serve as an analogue of the OPS λ_{d_1,d_2} found in the previous section. For any $d \in [n-1]$, let $\lambda_d \in OPS(\tilde{H})$ be defined by

$$\lambda_d(t) = \mathrm{diag}(t^2, \ldots, t^2, t, \ldots, t, t^{-1}, \ldots, t^{-1}, t^{-2}, \ldots, t^{-2})$$

where the value of the first d entries is t^2 and the next $n - d$ entries is t. It is easy to see that

$$\dot{\lambda}_d := \mathrm{diag}(2, \ldots, 2, 1, \ldots, 1, -1, \ldots, -1, -2, \ldots, -2)$$

and that λ_d is both $\tilde{L}$ dominant and L-dominant. Since $P^{Sp(2n)}(\tau) \subseteq P^{Sp(2n)}(\lambda_d)$, we have that λ_d is also admissible. The flag varieties $\tilde{L}/\tilde{Q}(\lambda_d) \simeq \mathrm{Gr}(d,n)$ and $L/Q(\lambda_d) \simeq \mathrm{Gr}(d,n)^3$ and the map

$$\mathrm{Gr}(d,n) \hookrightarrow \mathrm{Gr}(d,n)^3$$

is the diagonal embedding.

Let $(w,\tilde{w}) \in W^P \simeq S^A_{2n}(n) \times S^C_{2n}(n)$ be such that

$$\dim \Lambda_{(w,\tilde{w})} = \dim G/P - \dim \tilde{G}/\tilde{P} = \dim \mathrm{Gr}(n,2n) = n^2.$$

Assume that $\phi^*([\Lambda_{(w,\tilde{w})}]) \neq 0$, then by Theorem 5.3.8, for any $u = (u^1,u^2,u^3) \in S_n(d)^3$ such that

$$[X_{u^1}] \cdot [X_{u^2}] \cdot [X_{u^3}] \neq 0$$

we have that

$$(f^*(u^{-1}\chi_{(w,\tilde{w})}) - \tilde{\chi})(\dot{\lambda}_d) \leq 0.$$

We explicitly calculate the action of u on $\dot{\lambda}_d$. To do this, we consider the action of u on any $\mathbf{t'} \times \mathbf{t} \in \mathfrak{h}$. Denote $\mathbf{t'}$ with the following twisted index

$$\mathbf{t'} = \mathrm{diag}\{t'_1, t'_2 \ldots, t'_n, t'_{n+n}, \ldots, t'_{n+2}, t'_{n+1}\}$$

and denote $\mathbf{t}$ by its first n terms $\mathbf{t} = \mathrm{diag}\{t_1,t_2,\ldots,t_n\}$. Then $u(\mathbf{t'} \times \mathbf{t}) =$

$$\mathrm{diag}\{t'_{u^1(1)}, t'_{u^1(2)} \ldots, t'_{u^1(n)}, t'_{n+u^2(n)}, \ldots, t'_{n+u^2(2)}, t'_{n+u^2(1)}\} \times \mathrm{diag}\{t_{u^3(1)}, t_{u^3(2)}, \ldots, t_{u^3(n)}\}.$$

Hence if we write $\dot{\lambda}_d = \dot{\lambda}'_d \times \dot{\lambda}_d$ viewed as a vector in $\mathfrak{h}$, then

$$(f^*(u^{-1}\chi_{(w,\tilde{w})}) - \tilde{\chi})(\dot{\lambda}_d) = \chi^A_w((u^1,u^2)\dot{\lambda}'_d) + \chi^C_{\tilde{w}}(u^3\dot{\lambda}_d) - \tilde{\chi}(\dot{\lambda}_d).$$

The character

$$\chi^A_w := \sum_{\beta \in R^+ \setminus R^+_{\mathfrak{l}} \cap w^{-1}R^+} \beta$$

where R denotes the roots of $SL(2n)$ and

$$\chi^C_{\tilde{w}} := \sum_{\beta \in R^+ \setminus R^+_{\mathfrak{l}} \cap w^{-1}R^+} \beta$$

where R denotes the roots of $Sp(2n)$.

Example 7.1.1. *Let $n = 2$ and consider*

$$\phi : \mathrm{LG}(2, 4) \hookrightarrow \mathrm{Gr}(2, 4) \times \mathrm{LG}(2, 4).$$

There are five pairs $(w, \tilde{w}) \in S^A_4(2) \times S^C_4(2)$ which satisfy $\ell^A(w) + \ell^C(\tilde{w}) = 4$:

$$(3,4,1,2) \times (1,2,3,4) \quad (2,4,1,3) \times (1,3,2,4) \quad (2,3,1,4) \times (2,4,1,3)$$
$$(1,4,2,3) \times (2,4,1,3) \quad (1,3,2,4) \times (3,4,1,2)$$

The map on Levi factors is $SL(2) \hookrightarrow SL(2)^3$. The OPS λ_1 induces the embedding of Levi factors $\mathrm{Gr}(1, 2) \hookrightarrow \mathrm{Gr}(1, 2)^3$. Hence there are 4 inequalities coming from the following 3-tuples in $S_2(1)^3$:

$$((2,1),(2,1),(1,2)),\ ((2,1),(1,2),(2,1)),\ ((1,2),(2,1),(2,1)),\ ((2,1),(2,1),(2,1)).$$

It turns out that all five pairs satisfy these inequalities.

Example 7.1.2. *Let $n = 3$ and consider*

$$\phi : \mathrm{LG}(3, 6) \hookrightarrow \mathrm{Gr}(3, 6) \times \mathrm{LG}(3, 6).$$

There are 19 pairs $(w, \tilde{w}) \in S^A_6(3) \times S^C_6(3)$ which satisfy $\ell^A(w) + \ell^C(\tilde{w}) = 9$. The map *on Levi factors is* $SL(3) \hookrightarrow SL(3)^3$. *The OPS* λ_1 *induces the embedding of Levi factors* $\mathrm{Gr}(1,3) \hookrightarrow \mathrm{Gr}(1,3)^3$ *and the OPS* λ_2 *induces the embedding* $\mathrm{Gr}(2,3) \hookrightarrow \mathrm{Gr}(2,3)^3$. *Each OPS gives a list of 10 inequalities for a total of 20 between* λ_1 *and* λ_2. *Of the 19 pairs* $(w, \tilde{w})$, *we find that 11 of them violate some inequality with respect to* λ_1 *or* λ_2 *and hence* $\phi^*([\Lambda_{(w,\tilde{w})}]) = 0$ *for these pairs. These are listed below:*

$$(3,5,6,1,2,4) \times (1,2,4,3,5,6) \quad (3,4,6,1,2,5) \times (1,3,5,2,4,6)$$
$$(2,5,6,1,3,4) \times (1,3,5,2,4,6) \quad (2,4,6,1,3,5) \times (1,4,5,2,3,6)$$
$$(1,5,6,2,3,4) \times (1,4,5,2,3,6) \quad (2,3,5,1,4,6) \times (3,5,6,1,2,4)$$
$$(1,4,5,2,3,6) \times (3,5,6,1,2,4) \quad (1,3,6,2,4,5) \times (3,5,6,1,2,4)$$
$$(2,3,4,1,5,6) \times (4,5,6,1,2,3) \quad (1,3,5,2,4,6) \times (4,5,6,1,2,3)$$
$$(1,2,6,3,4,5) \times (4,5,6,1,2,3)$$

Corollary 7.1.3. *Consider the pair* $(w, \tilde{w}) = (2,3,4,1,5,6) \times (4,5,6,1,2,3) \in S_6^A(3) \times S_6^C(3)$. *Since* $\phi^*([\Lambda_{(w,\tilde{w})}]) = 0$, *we have that a generic 4 dimensional subspace of* $\mathbb{C}^6$ *does not contain a 3 dimensional isotropic subspace.*

7.2. The Even orthogonal group embedding

Let $\mathbb{C}^{2n}$ be a $2n$ dimensional complex vector space with a symmetric bilinear form with basis $\{e_1, e_2, \ldots, e_{2n}\}$ such that

$$\langle e_i, e_{2n+1-i} \rangle = 1 \quad \text{and} \quad \langle e_i, e_j \rangle = 0 \text{ if } j \neq 2n+1-i.$$

Let $\tilde{G} = SO(2n)$ be the special orthogonal group with respect to this form and let $G = SL(2n) \times SO(2n)$. Consider the diagonal embedding

$$f : SO(2n) \hookrightarrow SL(2n) \times SO(2n)$$

induced from the natural inclusion $SO(2n) \subseteq SL(2n)$. Note that the objects $\tilde{H}, \tilde{\mathfrak{h}}$ and $H, \mathfrak{h}$ have the exact same description as for the embedding $Sp(2n) \hookrightarrow SL(2n) \times Sp(2n)$. However, the positive Weyl chambers $\tilde{\mathfrak{h}}_+$ and $\mathfrak{h}_+$ are different. We have that

$$\begin{aligned} \tilde{\mathfrak{h}}_+ &= \{\mathbf{t} \in \tilde{\mathfrak{h}} \mid t_i \in \mathbb{R} \text{ and } t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq |t_n|\} \\ \mathfrak{h}_+ &= \{\mathbf{t'} \in \mathfrak{h} \mid t'_i \in \mathbb{R} \text{ and } t'_1 \geq t'_2 \geq \cdots \geq t'_{2n}\} \times \tilde{\mathfrak{h}}_+. \end{aligned}$$

The set of roots of $SO(2n)$ is

$$R^D = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n\} \subseteq \tilde{\mathfrak{h}}^*$$

with positive roots $(R^+)^D = \{(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq n\}$ where $\varepsilon_i(\mathbf{t}) = t_i$. Let $\Delta^D := \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ denote the set of simple roots where $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n := \varepsilon_{n-1} + \varepsilon_n$. Denote the Weyl group of $SO(2n)$ by

$$W^D := \{w \in S_{2n} \mid w(2n+1-i) = 2n+1-w(i) \text{ and } \#\{w(i) > n \mid i \in [n]\} \text{ is even}\}.$$

Under the inclusion $W^D \subseteq S_{2n}$, the reflections corresponding to Δ^D are $s_i := (i, i+1)(2n-i, 2n-i+1)$ for $i < n$ and $s_n := (n-1, n+1)(n, n+2)$.

Observe that $f_*(\tilde{\mathfrak{h}}_+) \nsubseteq \mathfrak{h}_+$ and hence the relative Weyl set is nontrivial. We have that $W_{\text{rel}} = \{1, v\} \subseteq W = W^A \times W^D$ where v is the simple transposition $(n, n+1)$ in the first factor W^A. We now work out some examples with respect to certain choices of $\tau \in OPS(\tilde{H})$.

7.2.1. The orthogonal Grassmannian embedding

Choose the dominant $\tau \in OPS(\tilde{H})$ defined as

$$\tau(t) := \mathrm{diag}(t, \ldots, t, t^{-1}, \ldots, t^{-1})$$

where the value of the first n entries is t. The parabolic subgroup $P^{SO(2n)}(\tau) = P_{n-1}^D$ where $\Delta^D \backslash \Delta^D(P_n^D) = \{\alpha_{n-1}\}$. The flag variety

$$SO(2n)/P_{n-1}^D \simeq \mathrm{OG}^+(n, 2n)$$

where $\mathrm{OG}^+(n, 2n)$ is the connected set of n-dimensional isotropic subspaces which contains the standard isotropic flag $E^+ := \mathrm{Span}\{e_1, \ldots, e_n\}$. The OPS τ is also dominant with respect to $SL(2n) \times SO(2n)$. The parabolic

$$P^{SL(2n)\times SO(2n)}(\tau) \simeq P_n^A \times P_{n-1}^D$$

and the flag variety

$$SL(2n) \times SO(2n)/P_n^A \times P_{n-1}^D \simeq \mathrm{Gr}(n, 2n) \times \mathrm{OG}^+(n, 2n).$$

The map

$$\phi : \mathrm{OG}^+(n, 2n) \hookrightarrow \mathrm{Gr}(n, 2n) \times \mathrm{OG}^+(n, 2n)$$

is simply the diagonal embedding. Similarly to the Lagrangian Grassmannian, the Levi subgroup of P^D_{n-1} is $\tilde{L} = GL(n)$ and the Levi subgroup of $P^A_n \times P^D_{n-1}$ is

$$L = \{(g_1, g_2, g_3) \in GL(n)^3 \mid \det g_1 \cdot \det g_2 = 1\}.$$

The map $\tilde{L} \hookrightarrow L$ is given by $g \mapsto (g, g^{-1}, g)$. For the Levi dominant chambers, we have

$$\begin{aligned} \tilde{\mathfrak{h}}^{\tilde{L}}_+ &= \{\mathbf{t} \in \tilde{\mathfrak{h}} \mid t_i \in \mathbb{R} \text{ and } t_1 \geq \cdots \geq t_n\} \\ \mathfrak{h}^L_+ &= \{\mathbf{t'} \in \mathfrak{h} \mid t'_i \in \mathbb{R} \text{ and } t'_i \geq t'_{i+1} \ \forall\ i \neq n\} \times \tilde{\mathfrak{h}}^{\tilde{L}}_+. \end{aligned}$$

Since the Levi factors and Levi dominant chambers are the same as they are for the Lagrangian embedding, we have that $(W_L)_{\text{rel}}$ is trivial and for any $d \in [n-1]$, we can define the admissible, $\tilde{L}$-dominant $\lambda_d \in OPS(\tilde{H})$ as in the previous section. The flag varieties $\tilde{L}/\tilde{Q}(\lambda_d) \simeq \text{Gr}(d, n)$ and $L/Q(\lambda_d) \simeq \text{Gr}(d, n)^3$ and the map

$$\text{Gr}(d, n) \hookrightarrow \text{Gr}(d, n)^3$$

is the diagonal embedding. Let $(w, \tilde{w}) \in W^P \simeq S^A_{2n}(n) \times (W^D)^{P^D_{n-1}}$ be such that

$$\dim \Lambda_{(w,\tilde{w})} = \dim G/P - \dim \tilde{G}/\tilde{P} = \dim \text{Gr}(n, 2n) = n^2.$$

Assume that $\phi^*([\Lambda_{(w,\tilde{w})}]) \neq 0$, then by Theorem 5.3.8, for any $u = (u^1, u^2, u^3) \in S_n(d)^3$ such that

$$[X_{u^1}] \cdot [X_{u^2}] \cdot [X_{u^3}] \neq 0$$

we have that

$$(f^*(u^{-1}\chi_{(w,\tilde{w})}) - \tilde{\chi})(\dot{\lambda}_d) \leq 0.$$

7.2.2. The twisted orthogonal Grassmannian embedding

In this section, we choose a $\tau \in OPS(\tilde{H})$ which is dominant with respect to $\tilde{G} = SO(2n)$ but not dominant with respect to $G = SL(2n) \times SO(2n)$. Choose the dominant $\tau \in OPS(\tilde{H})$ defined as

$$\tau(t) := \mathrm{diag}(t, \ldots, t, t^{-1}, t, t^{-1}, \ldots, t^{-1})$$

where the value of the first $n-1$ entries is t and the value of the nth entry is t^{-1}. The OPS τ is not dominant with respect to $SL(2n) \times SO(2n)$, however $\tau_v := v^{-1}\tau v$ is dominant. The parabolic subgroup $P^{SO(2n)}(\tau) = P_n^D$ where $\Delta^D \backslash \Delta^D(P_n^D) = \{\alpha_n\}$ and the flag variety

$$SO(2n)/P_n^D \simeq \mathrm{OG}^-(n, 2n)$$

where $\mathrm{OG}^-(n, 2n)$ is the connected set of n-dimensional isotropic subspaces which contains the isotropic flag $E^- := \mathrm{Span}\{e_1, \ldots, e_{n-1}, e_{n+1}\}$. The parabolic

$$P^{SL(2n)\times SO(2n)}(\tau_v) \simeq P_n^A \times P_n^D$$

and the flag variety

$$SL(2n) \times SO(2n)/P_n^A \times P_n^D \simeq \mathrm{Gr}(n, 2n) \times \mathrm{OG}^-(n, 2n).$$

The map

$$\phi : \mathrm{OG}^-(n, 2n) \hookrightarrow \mathrm{Gr}(n, 2n) \times \mathrm{OG}^-(n, 2n)$$

is twisted embedding $gE^- \mapsto (gvE^-, gE^-) = (gE^+, gE^-)$ where $g \in SO(2n)$. Note that

$$\mathrm{Im}\phi = \mathrm{OG}^+(n, 2n) \times \mathrm{OG}^-(n, 2n).$$

As with the standard embedding, the Levi factors are $\tilde{L} = GL(n)$ and

$$L = \{(g_1, g_2, g_3) \in GL(n)^3 \mid \det g_1 \cdot \det g_2 = 1\}.$$

Note that $v^{-1}\tilde{L}v$ is the Levi factor of P^D_{n-1} and that map $\tilde{L} \hookrightarrow L$ is given by $g \mapsto (v^{-1}gv, v^{-1}g^{-1}v, g)$. For the Levi dominant chambers, we have

$$\begin{aligned}
\tilde{\mathfrak{h}}^{\tilde{L}}_+ &= \{\mathbf{t} \in \tilde{\mathfrak{h}} \mid t_i \in \mathbb{R} \text{ and } t_1 \geq \cdots \geq t_{n-1} \geq -t_n\} \\
\mathfrak{h}^L_+ &= \{\mathbf{t'} \in \mathfrak{h} \mid t'_i \in \mathbb{R} \text{ and } t'_i \geq t'_{i+1} \ \forall\ i \neq n\} \times \tilde{\mathfrak{h}}^{\tilde{L}}_+.
\end{aligned}$$

Note that $f_*(\tilde{\mathfrak{h}}^{\tilde{L}}_+) \subseteq v\mathfrak{h}^L_+$ and hence $(W_L)_{\mathrm{rel}}$ is again trivial. The one parameter subgroups $\lambda_d \in OPS(\tilde{H})$ are still $\tilde{L}$-dominant, however the parabolic subgroup $\tilde{Q}(\lambda_d)$ is not maximal in $\tilde{L}$. The flag varieties

$$\tilde{L}/\tilde{Q}(\lambda_d) \simeq \mathrm{F}\ell(\{d, n-1\}, n)$$

and

$$L/Q((\lambda_d)_v) \simeq \mathrm{F}\ell(\{d, n-1\}, n)^3$$

and the map

$$\mathrm{F}\ell(\{d, n-1\}, n) \hookrightarrow \mathrm{F}\ell(\{d, n-1\}, n)^3$$

is the diagonal embedding. To generate the same Grassmannians as in the untwisted embedding in the previous section consider $\lambda'_d \in OPS(\tilde{H})$ defined by

$$\lambda'_d(t) = \mathrm{diag}(t^2, \ldots, t^2, t, \ldots, t, t^{-1}, t, t^{-1}, \ldots, t^{-1}, t^{-2}, \ldots, t^{-2})$$

where the value of the first d entries is t^2 and the next $n-d-1$ entries is t and the nth entry is t^{-1}. It is easy to see that $\tilde{L}/\tilde{Q}(\lambda'_d) \simeq \mathrm{Gr}(d,n)$ and $L/Q((\lambda'_d)_v) \simeq \mathrm{Gr}(d,n)^3$.

7.3. Examples from Representation Theory

There are many interesting examples arising from representation theory. We remark that the examples in this section are inspired by examples worked out in [5]. Let V be a faithful N-dimensional representation of $SL(n)$. Since $SL(n)$ is a simple Lie group, we have an embedding

$$f : SL(n) \hookrightarrow SL(V).$$

Fix the standard objects $\tilde{H}, \tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}_+$ for $\tilde{G} = SL(n)$. Let

$$V = \bigoplus_{\mu \in \tilde{\mathfrak{h}}^*} V_\mu$$

denote the weight space decomposition of V and let $m(\mu)$ denote the dimension of V_μ (Clearly, $\sum_\mu m(\mu) = N$). Choosing a maximal torus H in $SL(V)$ reduces to choosing a basis for the weight spaces V_μ. Fix

$$\mathcal{B}^\mu = \{e_1^\mu, e_2^\mu, \ldots, e_{m(\mu)}^\mu\}$$

a basis for V_μ and choose $H \subseteq SL(V)$ to be the maximal torus which is diagonal with respect to the basis $\mathcal{B} := \bigcup_{\mu \in \tilde{\mathfrak{h}}^*} \mathcal{B}^\mu$ of V. It is easy to see that $\tilde{H} = H \cap SL(n)$ and that the embedding $f_* : \tilde{\mathfrak{h}} \hookrightarrow \mathfrak{h}$ is given by

$$f_*(\mathbf{t}) = \sum_\mu \sum_{i=1}^{m(\mu)} \mu(\mathbf{t}) e_i^\mu.$$

The root system of $SL(V)$ is given by the set

$$R = \{\varepsilon_k^\mu - \varepsilon_l^\nu \mid (\mu, k) \neq (\nu, l)\}$$

where $\varepsilon_k^\mu\big(\sum_{(\nu,l)} e_l^\nu\big) := e_k^\mu$. Choosing a positive Weyl chamber in $\mathfrak{h}$ reduces to fixing an ordering of the basis $\mathcal{B}$. Fix an ordering $\mu \prec \nu$ on the weights of V and an ordering $k \leq l$ on each set $\mathcal{B}^\mu$. So $(\mu, k) \leq (\nu, l)$ if $\mu \prec \nu$ and $(\mu, k) \leq (\mu, l)$ if $k \leq l$. Then the positive Weyl chamber is

$$\mathfrak{h}_+ = \{\mathbf{t'} \in \mathfrak{h} \mid t_i' \in \mathbb{R} \text{ and } \varepsilon_k^\mu - \varepsilon_l^\nu(\mathbf{t'}) \geq 0 \ \forall \ (\mu, k) \leq (\nu, l)\}$$

and the set of positive roots is

$$R^+ = \{\varepsilon_k^\mu - \varepsilon_l^\nu \mid (\mu, k) \leq (\nu, l)\}.$$

Let $\tau \in OPS(\tilde{H})$ be dominant with respect to $SL(n)$. Then τ is dominant with respect to $SL(V)$ if and only if

$$\varepsilon_k^\mu - \varepsilon_l^\nu(\dot{\tau}) = \mu(\dot{\tau}) - \nu(\dot{\tau}) \geq 0 \quad \forall \ \mu \prec \nu.$$

Hence the relative Weyl set $W_{\mathrm{rel}} \subseteq W \simeq S_N$ is only dependant on the choice of ordering $\mu \prec \nu$. We now look at some specific examples in more detail.

7.3.1. The representation $\mathrm{sym}^2(\mathbb{C}^3)$ of $SL(3)$

Let $n = 3$ and let $V = \mathrm{sym}^2(\mathbb{C}^3)$ be the second symmetric power of the standard representation of $SL(3)$. The representation V is 6 dimensional with weights

$$\{(\varepsilon_i + \varepsilon_j) \mid 1 \leq i \leq j \leq 3\}.$$

Since all the weight spaces are one dimensional, we only need to fix an ordering on the weights of V to determine a positive Weyl chamber. Fix the following ordering on the weights of V:

$$\begin{array}{lll} \mu_1 = 2\varepsilon_1 & \mu_2 = \varepsilon_1 + \varepsilon_3 & \mu_3 = 2\varepsilon_2 \\ \mu_4 = \varepsilon_2 + \varepsilon_3 & \mu_5 = 2\varepsilon_3 & \mu_6 = \varepsilon_1 + \varepsilon_3 \end{array}$$

If $\mathbf{t} = \mathrm{diag}(t_1, t_2, t_3) \in \tilde{\mathfrak{h}}$, then

$$f_*(\mathbf{t}) = \sum_{i=1}^{6} \mu_i(\mathbf{t}) e^{\mu_i} = \mathrm{diag}(2t_1, t_1 + t_2, 2t_2, t_2 + t_3, 2t_3, t_1 + t_3) \in \mathfrak{h}.$$

The condition that $\mathbf{t} \in \tilde{\mathfrak{h}}_+$ is that $t_1 \geq t_2 \geq t_3$ and $t_i \in \mathbb{R}$. Hence $f_*(\tilde{\mathfrak{h}}_+) \cap \mathfrak{h}_+ = \{0\}$ and the relative Weyl set W_{rel} does not contain the identity. If $W \simeq S_6$ is the Weyl group of $SL(V)$, then $W_{\mathrm{rel}} = \{s_5 s_4, s_5 s_4 s_3\}$ where s_i is the simple transposition $(i, i+1)$.

Choose the dominant $\tau \in OPS(\tilde{H})$ defined by $\tau(t) := \mathrm{diag}(t^2, t^{-1}, t^{-1})$. Then τ_v is dominant for $SL(V)$ where $v = s_5 s_4 s_3$. We have that

$$\dot{\tau} = \mathrm{diag}(2, -1, -1) \text{ and } \dot{\tau}_v = (4, 1, 1, -2, -2, -2).$$

It is easy to see that

$$SL(3)/P^{SL(3)}(\tau) \simeq \mathrm{Gr}(1,3) \text{ and } SL(V)/P^{SL(V)}(\tau_v) \simeq \mathrm{F}\ell(\{1,3\}, 6).$$

The twisted embedding

$$\phi : \mathrm{Gr}(1,3) \hookrightarrow \mathrm{F}\ell(\{1,3\}, 6)$$

is quite mysterious. Naively, one might suspect that $\mathrm{Im}\phi$ is equal some fiber over the projection $\psi : \mathrm{F}\ell(\{1,3\}, 6) \twoheadrightarrow \mathrm{Gr}(3,6)$. However this is not true since it would imply that $f(SL(3))$ stabilizes a 3 dimensional subspace of V, and V is an irreducible representation.

Consider the Levi-factors

$$\tilde{L} = \{(g_1, g_2) \in GL(1) \times GL(2) \mid \det g_1 \cdot \det g_2 = 1\}$$

and

$$L = \{(g_1, g_2, g_3) \in GL(1) \times GL(2) \times GL(3) \mid \prod_{i=1,2,3} \det g_i = 1\}.$$

Unfortunately, the twisted embedding $\tilde{L} \hookrightarrow L$ is also mysterious which makes applying Theorem 5.3.8 in a practical way unclear.

Consider the "straightened" embedding $\bar{\phi} : \mathrm{Gr}(1,3) \hookrightarrow \mathrm{F}\ell(\{1,3\},6)$ where we identify $\mathrm{Gr}(1,3)$ with the fiber $\psi^{-1}(E) \subseteq \mathrm{F}\ell(\{1,3\},6)$ over some fixed 3 dimensional vector space $E \subseteq V$. Then $\mathrm{Gr}(1,3)$ is equal to the Schubert variety $\bar{X}_{(3,1,2,4,5,6)}(F_\bullet) \subseteq \mathrm{F}\ell(\{1,3\},6)$ where $F_3 = E$. Hence, for any $w \in S_6(\{1,3\})$ such that

$$\dim X_w = \dim \mathrm{F}\ell(\{1,3\},6) - \dim \mathrm{Gr}(1,3) = 9,$$

we have that $\bar{\phi}^*([X_w]) = c[pt] \in H^*(\mathrm{Gr}(1,3))$ where c is given by

$$[X_w] \cdot [X_{(3,1,2,4,5,6)}] = c[pt] \in H^*(\mathrm{F}\ell(\{1,3\},6)).$$

Therefore the structure constant $c = 1$ if $w = (4,5,6,1,2,3)$, the Poincaré dual of (312456), and $c = 0$ otherwise. This fact is supported by Theorem 5.3.8. Consider the one parameter subgroup $\lambda := \mathrm{diag}(t,t,t^{-2}) \in OPS(\tilde{H})$. The OPS λ generates a single inequality which is violated by all $w \in S_6(\{1,3\})$ such that $\dim X_w = 9$ and $w \neq (4,5,6,1,2,3)$.

Question 7.3.1. *Does the cohomology class* $[\mathrm{Im}\phi] = [X_{(3,1,2,4,5,6)}]$ *in* $H^*(\mathrm{F}\ell(\{1,3\},6))$ *in the above example? If so, then the twisted embedding* $\phi : \mathrm{Gr}(1,3) \hookrightarrow \mathrm{F}\ell(\{1,3\},6)$ *is invariant at the level of cohomology.*

Remark 7.3.2. *The "straightened" embedding in the above example corresponds to choosing $\tilde{G}$ to be a Levi subgroup of $G = SL(V)$. Let $E = \mathrm{Span}\{e^{\mu_1}, e^{\mu_2}, e^{\mu_3}\}$ in the above example. Define $\tilde{G} \subseteq SL(V)$ to be the Levi factor of the stabilizer subgroup of the vector space E. It is easy to see that $\tilde{G} \simeq \{(g_1, g_2) \in GL(3)^2 \mid \det g_1 \cdot \det g_2 = 1\}$. The embedding $\bar{\phi} : \mathrm{Gr}(1,3) \hookrightarrow \mathrm{F}\ell(\{1,3\},6)$ corresponds to the embedding*

$$\tilde{G}/P^{\tilde{G}}(\tau) \hookrightarrow SL(V)/P^{SL(V)}(\tau)$$

where $\tau(t) := \mathrm{diag}(t^3, 1, 1, t^{-1}, t^{-1}, t^{-1})$.

We now consider a different example with $G = SL(V)$ where $V = \mathrm{sym}^2(\mathbb{C}^3)$. Choose the $\tau \in OPS(\tilde{H})$ defined by $\tau(t) := \mathrm{diag}(t, t, t^{-2})$. Then τ_v is dominant for $SL(V)$ where $v = s_4 s_5$. We have that

$$\dot{\tau} = \mathrm{diag}(1, 1, -2) \text{ and } \dot{\tau}_v = (2, 2, 2, -1, -1, -4).$$

It is easy to see that

$$SL(3)/P^{SL(3)}(\tau) \simeq \mathrm{Gr}(2,3) \text{ and } SL(V)/P^{SL(V)}(\tau_v) \simeq \mathrm{F}\ell(\{3,5\},6).$$

Once again, the twisted embedding

$$\phi : \mathrm{Gr}(2,3) \hookrightarrow \mathrm{F}\ell(\{3,5\},6)$$

is quite mysterious and we consider the corresponding "straightened" embedding

$$\bar{\phi} : \mathrm{Gr}(2,3) \simeq \bar{X}_{(1,2,3,5,6,4)}(F_\bullet) \hookrightarrow \mathrm{F}\ell(\{3,5\},6).$$

It is easy to see that the Schubert variety $\bar{X}_{(1,2,3,5,6,4)}(F_\bullet) = \psi^{-1}(F_3)$ where ψ denotes the projection $\psi : \mathrm{F}\ell(\{3,5\},6) \twoheadrightarrow \mathrm{Gr}(3,6)$. For any $w \in S_6(\{3,5\})$ such that

$$\dim X_w = \dim \mathrm{F}\ell(\{3,5\},6) - \dim \mathrm{Gr}(2,3) = 9,$$

we have that $\bar{\phi}^*([X_w]) = [pt] \in H^*(\mathrm{Gr}(1,3))$ if $w = (4,5,6,1,2,3)$, the Poincaré dual of $(1,2,3,5,6,4)$, and $\bar{\phi}^*([X_w]) = 0$ otherwise. We ask the same question as in Question 7.3.1.

Question 7.3.3. *Does the cohomology class* $[\mathrm{Im}\phi] = [X_{(1,2,3,5,6,4)}]$ *in* $H^*(\mathrm{F}\ell(\{3,5\},6))$*?*

7.3.2. The adjoint representation

Finally, we give an example where the weight spaces of V are not all one dimensional. Let $n = 3$ and let $V = \mathfrak{sl}(3)$ denote the adjoint representation of $SL(3)$. Note that V is an 8 dimensional representation with weights

$$\{0, \pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\}.$$

The only two dimensional weight space is the 0 weight (ie. the Cartan subalgebra of $\mathfrak{sl}(3)$). Let

$$e_1^0 = \mathrm{diag}(1,-1,0) \text{ and } e_2^0 = \mathrm{diag}(0,1,-1)$$

be a basis for V_0. Fix the following ordering on the weights of V:

$$\mu_1 = \varepsilon_1 - \varepsilon_2 \qquad \mu_2 = \varepsilon_1 - \varepsilon_3 \qquad \mu_3 = \varepsilon_2 - \varepsilon_3 \qquad \mu_4 = 0$$
$$\mu_5 = -\varepsilon_1 + \varepsilon_2 \quad \mu_6 = -\varepsilon_1 + \varepsilon_3 \quad \mu_7 = -\varepsilon_2 + \varepsilon_3$$

If $\mathbf{t} = \mathrm{diag}(t_1, t_2, t_3) \in \tilde{\mathfrak{h}}$, then

$$f_*(\mathbf{t}) = \sum_{i=1}^{7} \mu_i(\mathbf{t}) e^{\mu_i} = \mathrm{diag}(t_1 - t_2, t_1 - t_3, t_2 - t_3, 0, 0, t_2 - t_1, t_3 - t_1, t_3 - t_2) \in \mathfrak{h}.$$

As with the representation $\mathrm{sym}^2\mathbb{C}^3$, we have that $f_*(\tilde{\mathfrak{h}}_+) \cap \mathfrak{h}_+ = \{0\}$ and the relative Weyl set W_{rel} does not contain the identity. We have that

$$W_{\mathrm{rel}} = \{s_1s_7, s_1s_6s_7, s_1s_2s_7, s_1s_2s_6s_7\} \subseteq W = S_8$$

where s_i is the simple transposition $(i, i+1)$. The since the analysis of the next example is similar to that of the examples in the previous section, we only construct the embedding.

Choose the dominant $\tau \in OPS(\tilde{H})$ defined by $\tau(t) := \mathrm{diag}(t^2, t^{-1}, t^{-1})$. Then τ_v is dominant for $SL(V)$ where $v = s_1s_6s_7$. We have that

$$\dot{\tau} = \mathrm{diag}(2, -1, -1) \text{ and } \dot{\tau}_v = \mathrm{diag}(3, 3, 0, 0, 0, 0, -3, -3).$$

The flag varieties

$$SL(3)/P^{SL(3)}(\tau) \simeq \mathrm{Gr}(1, 3) \text{ and } SL(V)/P^{SL(V)}(\tau_v) \simeq \mathrm{F}\ell(\{2, 6\}, 8).$$

Hence we have the twisted embedding $\phi : \mathrm{Gr}(1, 3) \hookrightarrow \mathrm{F}\ell(\{2, 6\}, 8)$.

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