On cyclic quiver parabolic Kostka-Shoji polynomials

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Lusztig's t-analog of weight multiplicity

G complex reductive group
$X^+(G) \subset X(G)$ (dominant) integral G-weights with respect to $B \subset G$

Two bases of $R(G)[t] = (\mathbb{Z}[t]X(G))^W \cong$ spherical AHA:

(1) $s_\lambda = \mathrm{ch}\, V(\lambda)$
(2) $P_\mu =$ Macdonald spherical function (Hall-Littlewood polynomial)

$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu}(t) P_\mu$

Theorem (Lusztig)

$K_{\lambda\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ for any $\lambda, \mu \in X^+(G)$.

$K_{\lambda\mu}(1) = \dim V(\lambda)_\mu$

"Dual" approach [Broer, R. Brylinski, Hesselink]

$\mathcal{L}_\mu = G \times^{B_-} \mathbb{C}_\mu$ line bundle on $X = G/B_-$

$\pi : T^*X \to X$

Definition (Hall-Littlewood series)

$\chi_\mu = \chi_{G\times\mathbb{C}^\times}(\pi^*\mathcal{L}_\mu) \in R(G)[[t]]$

Theorem

$\chi_\mu = \sum_{\lambda\geq\mu} K_{\lambda\mu}(t) s_\lambda$ for any $\lambda, \mu \in X^+(G)$.

Theorem [Broer]

For $\mu \in X^+(G)$, one has $H^p(T^*X, \pi^*\mathcal{L}_\mu) = 0$ for all $p > 0$.

Lascoux-Schützenberger formula

$G = GL_n$

For $\lambda, \mu \in X^+_{\text{pol}}(G)$ polynomial weights (partitions with at most n parts), the $K_{\lambda\mu}(t)$ are *Kostka-Foulkes polynomials*.

Theorem [Lascoux-Schützenberger]

$K_{\lambda\mu}(t) = \sum_T t^{\text{charge}(T)}$ where T runs over all semistandard Young tableaux of shape λ and weight μ.

Quiver generalization [OS]

$Q = (I, \Omega)$ quiver

- $\mathbf{i} = (i_1, \dots, i_\ell) \in I^\ell$
- $\mathbf{a} = (a_1, \dots, a_\ell) \in \mathbb{Z}_{\geq 0}^\ell$
- $d = d(\mathbf{i}, \mathbf{a}) = (d_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ given by $d_i = \sum_{i_k = i} a_k$

Definition [Lusztig]

$Z_{\mathbf{i},\mathbf{a}} =$ set of all pairs $(F_\bullet, x)$ where $x \in \mathrm{Rep}^d(Q)$ and

$$\oplus_{i \in I} \mathbb{C}^{d_i} = F_1 \supset F_2 \supset \cdots \supset F_\ell \supset F_{\ell+1} = 0$$

is a flag of I-graded subspaces such that $x(F_k) \subset F_{k+1}$ and F_k / F_{k+1} has dimension a_k at i_k and zero elsewhere.

$Z_{\mathbf{i},\mathbf{a}} \xrightarrow{\pi} X_{\mathbf{i},\mathbf{a}}$ is a $G = \prod_{i \in I} GL_{d_i}$-equivariant vector bundle over a product of partial flag varieties $X_{\mathbf{i},\mathbf{a}}$.

Quiver generalization [OS] (cont'd)

$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_\ell)$ where $\mu_k \in X(GL_{a_k})$ for all k

$\rightsquigarrow$ G-equivariant vector bundle $W_{\boldsymbol{\mu}}$ on $X_{\mathbf{i},\mathbf{a}}$

Definition [OS]

- Quiver Hall-Littlewood series

$$\chi_{\boldsymbol{\mu}}^{\mathbf{i},\mathbf{a}} = \chi_{G \times \mathbb{C}^\times}(Z_{\mathbf{i},\mathbf{a}}, \pi^* W_\mu) \in R(G)[[t]]$$

- Quiver Kostka-Shoji polynomials

$$\chi_{\boldsymbol{\mu}}^{\mathbf{i},\mathbf{a}} = \sum_\lambda K_{\lambda,\boldsymbol{\mu}}^{\mathbf{i},\mathbf{a}}(t) s_\lambda \qquad \Big(\lambda = (\lambda_i)_{i \in I} \in X_{\mathsf{pol}}^+(G)\Big)$$

Special cases of quiver Kostka-Shoji polynomials

- *Jordan quiver*: (parabolic) Kostka-Foulkes polynomials
- *Cyclic quiver (affine type A)*: Kostka polynomials for complex reflection groups defined by Shoji (limit symbol case); intersection cohomology of enhanced nilpotent cone [Achar-Henderson]
- *Directed path (type A)*: truncated Littlewood-Richardson coefficients [W. Craig]

Higher vanishing conjecture

Conjecture [OS]

If $\boldsymbol{\mu}$ concatenates to a dominant G-weight, then $H^p(Z_{\mathbf{i},\mathbf{a}}, \pi^* W_{\boldsymbol{\mu}}) = 0$ for all $p > 0$.

Known cases

- Jordan quiver, $a_k \equiv 1$ [Broer]
- Cyclic quiver, $a_k \equiv 1$ [Panyushev, Finkelberg-Ionov]
- Any quiver, sufficiently dominant $\boldsymbol{\mu}$ [Panyushev]

Combinatorial positivity

We say that <u>*combinatorial positivity*</u> holds for $(\mathbf{i}, \mathbf{a}, \boldsymbol{\mu})$ if

$$K^{\mathbf{i},\mathbf{a}}_{\lambda,\boldsymbol{\mu}}(t) \in \mathbb{Z}_{\geq 0}[t]$$

for all $\lambda \in X^+(G)$.

For the *Jordan quiver*, combinatorial positivity is known when $\boldsymbol{\mu}$ is a sequence of rectangles [Shimozono]. The parabolic Kostka polynomials count graded multiplicites in:

- Kirillov-Reshetikhin modules
- (twisted) functions on nilpotent orbit closures

Cyclic quiver, rectangles at a single vertex

$I = \mathbb{Z}/r\mathbb{Z} \qquad \Omega = \{(i, i+1) : i \in I\}$

$$\left\{ \begin{array}{c|cccc|cccc|c} \mathbf{i} & 0 & \cdots & r-2 & r-1 & 0 & \cdots & r-2 & r-1 & \\ \mathbf{a} & \eta_1 & \cdots & \eta_1 & \eta_1 & \eta_2 & \cdots & \eta_2 & \eta_2 & \cdots \\ \boldsymbol{\mu} & 0 & \cdots & 0 & \nu_1^{\eta_1} & 0 & \cdots & 0 & \nu_2^{\eta_2} & \end{array} \right.$$

$\eta = (\eta_1, \dots, \eta_s)$ arbitrary heights
$\nu = (\nu_1 \geq \cdots \geq \nu_s)$ decreasing widths

Theorem [OS]

Combinatorial positivity holds for $(\mathbf{i}, \mathbf{a}, \boldsymbol{\mu})$ above. For any $\lambda \in X^+(G)$,

$$K_{\lambda,\boldsymbol{\mu}}^{\mathbf{i},\mathbf{a}}(t) = \sum_T t^{\mathsf{charge}(T)}$$

where $T = (T_i)_{i \in I}$ runs over "Littlewood-Richardson multitableaux" of shape $\lambda = (\lambda_i)_{i \in I}$.

Related results, observations, and conjectures

1. If the *rectangles are columns*, i.e., $\nu = (1, \ldots, 1)$ and η dominant: quiver Kostka-Shoji polynomials give irreducible multiplicites in graded induction from S_n to $\Gamma_n = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$ of Garsia-Procesi S_n-module R_η $(n = |\eta|)$.

2. In the setting of (1), quiver Hall-Littlewood functions arise from $q = 0$ specialization of Haiman's wreath Macdonald function $H_\eta(q, t)$.

3. For cyclic quivers and *any* $(\mathbf{i}, \mathbf{a}, \boldsymbol{\mu})$ satisfying dominance, we conjecture a positive combinatorial "charge" formula for $K^{\mathbf{i},\mathbf{a}}_{\lambda,\boldsymbol{\mu}}(t)$ over "catabolizable tableaux."