A new method for computing Kazhdan-Lusztig polynomials

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Credits

- Based on the 2013 PhD thesis of Jennifer Koonz. Available through MathSciNet.
- Subsequent joint work with Koonz, in progress.

Kahzdan-Lusztig polynomials

W is a finite Weyl group.

For $x, y \in W$, get polynomial

$$p_{x,y}(q) \in \mathbb{N}[q].$$

Defined in 1979 paper of Kazhdan and Lusztig.

- Zero, unless $x \leq y$ in Bruhat order.
- Otherwise, constant term is 1.
- Any such polynomial is possible (Polo, 1999).
- Positivity in Weyl group case due to KL. Also affine Weyl group.
- General Coxeter case due to Elias-Williamson (2012).

Geometric interpretation

Take Schubert variety $X_y \subset G/B$, flag variety.

Ordinary cohomology of X_y has Poincare polynomial

$$\sum_{x \leq y} q^{2\ell(x)}.$$

But X_y may be singular and local intersection cohomology captures some of that story:

$$q^{2\ell(x)} p_{x,y}(q^2)$$

is the Poincare polynomial of local intersection cohomology of X_y at x.

So if X_y smooth, all nonzero $p_{x,y}$ are equal to 1.

Examples

$W = W(A_3)$ and $y = srts$ where r, s, t simple reflections with r, t commute.

Then $X_s \subset X_y$ is the singular locus.

$p_{1,y} = p_{s,y} = 1 + q$.

$W = W(E_8)$ and y is a certain element of length 75.

The coefficients of $p_{1,y}(q)$ are

1, 8, 35, 109, 271, 573, 1068, 1787, 2705, 3720, 4665, 5348, 5601, 5346, 4644, 3668, 2623, 1689, 978, 509, 236, 96, 34, 10, 2.

Algebraic definition

$\mathcal{H}$ Hecke algebra for W with standard basis T_w over $\mathbb{Z}[v, v^{-1}]$:

Definition

- $T_s T_w = T_{sw}$ if $\ell(sw) = \ell(w) + 1$.
- $T_s^2 = (v - v^{-1}) T_s + 1$

$T_s^{-1} = T_s + (v^{-1} - v)$

Involution $i(v) = v^{-1}$, $i(T_s) = T_s^{-1}$.

Theorem

There exists unique C_y which is:

- *invariant under involution i*
- $C_y = T_y + \sum_{x<y} v^{-1}\mathbb{N}[v^{-1}] T_x$

KL polynomial is defined by $v^{-\ell(y)+\ell(x)} p_{x,y}(v^2)$ being coefficient of T_x.

Note: degree $p_{x,y} < \frac{\ell(y)-\ell(x)}{2}$ when $x < y$.

Finding KL elements

First, $C_s = T_s + v^{-1}$ for simple reflection $s \in W$.

Let $W = W(A_2)$ with simple refls r, s.

$C_{rs} = C_r C_s = T_{rs} + v^{-1} T_r + v^{-1} T_s + v^{-2}$.

On the other hand,

$C_r C_s C_r = C_{rs} C_r =$
$T_{rsr} + v^{-1} T_r^2 + v^{-1} T_{sr} + v^{-2} T_r + v^{-1}(T_{rs} + v^{-1} T_r + v^{-1} T_s + v^{-2})$

Bad term: $v^{-1}((v - v^{-1})T_r + 1) = (1 - v^{-2})T_r + v^{-1}$.

Then $C_{rsr} = C_r C_s C_r - C_r$ by correcting the constant in v term.

So all KL polys here are 1.

Makes sense since X_{rsr} is smooth.

Better way

Let $W_{J'} \subset W_J$ be a parabolic subgroups.

Let d_J be the number of positive roots supported on J.

Definition

Let $M \subset W_J$ be the set of minimal length left coset reps of $W_{J'}$ in W_J. Define

$$C_{J',J} = \sum_{w \in M} v^{\ell(w) - d_J + d_{J'}} T_w.$$

$C_{J',J}$ is not invariant unless $J' = \emptyset$. But

Theorem (Koonz, S-)

If $C \in \mathcal{H}$ is invariant and satisfies $T_s C = vC$ for all $s \in J'$, then

$$C_{J',J} C$$

is invariant.

Finding KL elements

If $C = C_y$, then C_y satisfies the theorem for the left descent set of y. Call this J'.

Then we study

$$C_{J',J}C_y$$

for any J containing J', and correct lower order terms.

This will give the KL element for $w_{J',J}y$, where $w_{J',J}$ is longest minimal coset representative.

For example, instead of $C_rC_sC_r$, we use

$$C_{\{r\},\{r,s\}}C_r,$$

and this is the KL element for $rsr = (rs)r$

Summary: to find KL element, it is more efficient to consider these generalized factorizations that use longest coset representatives.

Geometric meaning

Usual factorizations give rise to Bott-Samelson resolutions $Z_{\underline{w}}$ of X_w. Inductively:

$$Z_{s\underline{y}} = P_s \times^B Z_{\underline{y}} \text{ if } sy > y.$$

Fibers are $P_s/B \simeq \mathbb{P}^1$.

These factorizations give rise to generalized resolutions. If $sy < y$ for all $s \in J'$ and $sy > y$ for all $s \in J - J'$, then inductively:

$$Z_{w_{J',J}\underline{y}} = P_J \times^{P_{J'}} Z_{\underline{y}}$$

Fibers are P_J/P'_J, partial flag manifolds.

Fact

Such a resolution of X_y is small if and only if the corresponding product of the $C_{w_{J',J}}$ is equal to C_y.

These resolutions also have the analog of the paving mentioned in Mihalcea's talk and of Deodhar's result on using subwords of the factorization.

In particular, the Euler characteristic of the fiber above X_x is the number of subwords that multiply to x.

So we don't actually multiply in the Hecke algebra, just track subwords.

Generalized Bott-Samelson resolutions

These resolutions have shown up before.

- Scott Larson's talk yesterday
- **Type A, X_y smooth**

  There exists factorization with $X_y = Z_{\underline{y}}$ (Wolpert, Ryan). Presents X_y as tower of $\mathbb{P}^k$-bundles.
- **Simply-laced: X_y smooth**

  There exists factorization with $X_y = Z_{\underline{y}}$ (Billey-Postnikov)
- Appears in the work of Gelfand-MacPherson, Zelevinsky, Sankaran-Vanchinathan
- Used by Polo for his result:

  Every possible polynomial is a type A Kazhdan-Lusztig polynomial

Which resolution to choose?

There are many choices for resolutions. Which ones are best?

Let $I(x) = \Phi^- \cap x(\Phi^+)$ be the inversion set for x.

For $\alpha \in \Phi^+$ and S a set of positive roots, we define the **height of α relative to S** to be the maximal way to write α as a sum of elements of S.

Usual height if S includes the simple roots.

Definition

We say the factorization

$$y = w_{J',J}x$$

preserves heights if all $\alpha \in I(x)$ have same modified height whether computed with $S = I(x)$ or $S = I(y)$.

We seek y where we can completely factor and preserve heights at each step

Preserving heights

Fact

Resolution is small implies the factorization preserves heights.

In A_3 every element can be completely factored and preserve heights at each step. In A_4, for $y = 45312$ has no such factorizations.

Fact

But not every factorization that preserves heights is small.

Billey-Warrington studied the cases (in this language) where the *usual* Bott-Samelson preserves heights. This just means y is 321 avoiding. They classified the non-small cases, called ‘hexagon permutations’.

The non-small cases in A_7 are on Williamson’s list of X_y with torsion in intersection cohomology.

Torsion in A_7 and D_4

Analogous result for generalized Bott-Samelson resolutions.

Theorem

Among the factorizations in A_7 and D_4 that preserves height, but do not give small resolutions, are exactly the cases Williamson found that have torsion in IC.

- There is a pattern in $A_6 = W(S_7)$ which preserves heights and is not small. If we avoid this pattern, then there are 38 elements in $S_8 = W(A_7)$ which preserve height and are not small.
- The maximal torsion points are maximal non-small points.
- They are all codimension 8 and all semi-small. So we get fiber of dimension 4, with Poincare polynomials

$$1 + 7v + 13v^2 + 7v^3 + v^4,$$

$$1 + 8v + 15v^2 + 8v^3 + v^4,$$

$$1 + 9v + 18v^2 + 9v^3 + v^4, \textit{etc}$$

(fibers suggest smoothness, but need not even be irreducible)

Computing KL polynomials

- We used these resolutions to compute KL polynomials. Wrote program in C.
- Take resolution that is closest to preserving heights and compute the corresponding product of the $C_{w_{J',J}}$.
- Correct maximal terms where resolution is not small.
- Optimizations:
  - Efficient encoding of Weyl group elements
  - Hash tables for Weyl groups elements and polynomials (using uthash)
  - Parabolic speed-up which is basically the same thing as noticed by Lubeck and Scott-Sprowl in their work on the affine Weyl group.
  - Don't forget 'gcc -03'.
- Considerably faster than Coxeter (which is now part of atlas) and Greg Warrington's program for S_n.