# Fibers of Maps to Totally Nonnegative Spaces

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- see arXiv: 1903.01420 (joint work with James Davis & Ezra Miller)

Plan: Study fibers of maps $f_{(i_1,\ldots,i_d)}$ whose image is totally nonneg. real part of unipotent radical of Borel in semisimple, simply conn. algeb. group

# Running Example: Totally Nonnegative Part of a Space of Matrices

- $$x_i(t) = I_n + tE_{i,i+1} = \begin{array}{cc} & \text{column } i+1 \\ & \downarrow \\ & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & t & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \leftarrow \text{row } i \end{array}$$

$$\overset{\displaystyle x_i(t)}{\overset{\|}{\exp(te_i)}}$$

$\uparrow$ (general finite type, expon'd Chevalley generator)

$\uparrow$ (type A)

- $$f_{\underbrace{(i_1,\dots,i_d)}_{\text{reduced word}}} : \mathbb{R}^d_{\geq 0} \longrightarrow M_{n\times n} \cong \mathbb{R}^{n^2}$$

$$(t_1,\dots,t_d) \longmapsto x_{i_1}(t_1)\cdots x_{i_d}(t_d)$$

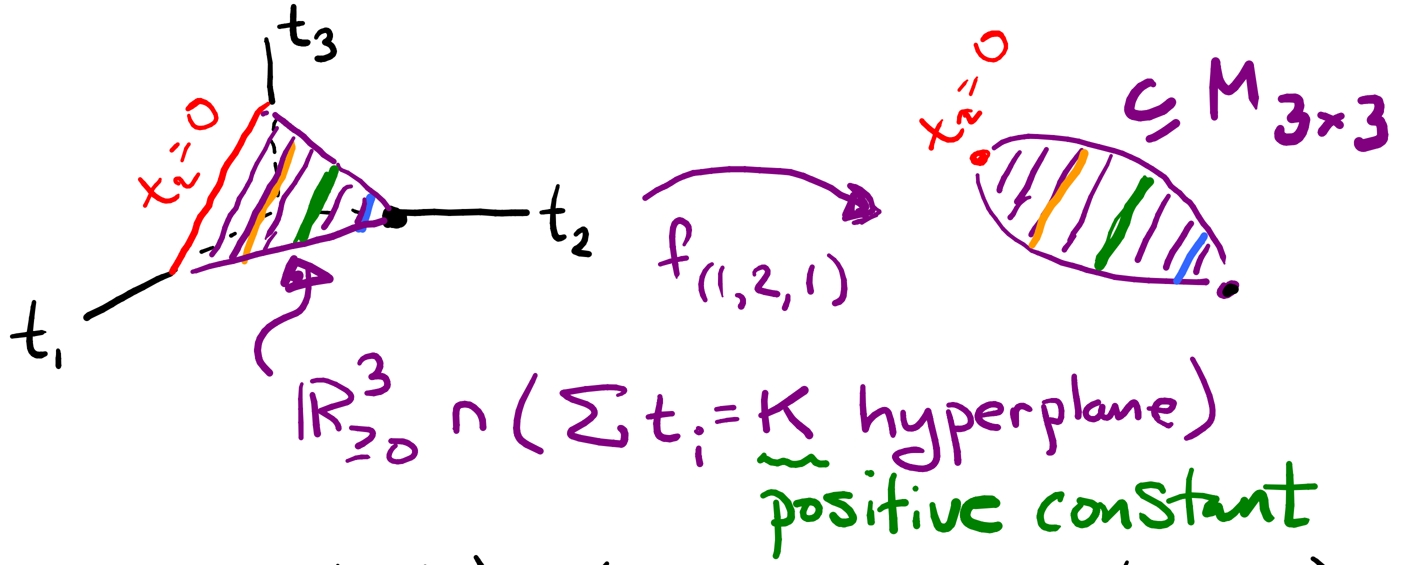
e.g. $$f_{(1,2,1)}(t_1,t_2,t_3) = x_1(t_1)x_2(t_2)x_1(t_3)$$

$$= \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$w_0$ case:

$$\left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \middle| \begin{array}{l}\text{tot.} \\ \text{nonneg}\end{array}\right\} = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

# "Picture" of Map $f_{(i_1 \ldots i_d)}$



$$f_{(1,2,1)}(t_1,t_2,t_3) = \begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & t_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & t_3 \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Big\downarrow \; t_2 = 0$$

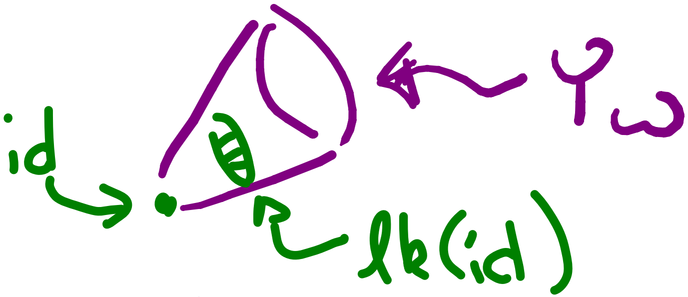
$$f_{(1,2,1)}(t_1,0,t_3) = \overbrace{\begin{pmatrix} 1 & t_1 & \\ & 1 & \\ & & 1 \end{pmatrix}}^{x_1(t_1)} \cdot \overbrace{\begin{pmatrix} 1 & t_3 & \\ & 1 & \\ & & 1 \end{pmatrix}}^{x_1(t_3)}$$

$$= \begin{pmatrix} 1 & t_1+t_3 & \\ & 1 & \\ & & 1 \end{pmatrix} = x_1(t_1+t_3)$$

simplex faces w/ same image "$x_1^2 = x_1$"

e.g. $\{x_1(t) \mid t>0\} = \{x_1(t_1)x_1(t_2) \mid t_1,t_2>0\}$

**Fomin-Shapiro Conjecture:** The Bruhat stratification of lk(id) in totally nonneg. real part of unipotent radical in Borel in algebraic group is regular CW complex homeomorphic to closed ball (w/ Bruhat order as face poset)



$$Y_w = \left[\overline{B^- w B^-} \cap \left(\begin{matrix}\text{unipotent}\\ \text{subgp of } B\end{matrix}\right)\right]_{\geq 0}$$

lower triangular opposite Borel $B^-$

permutation $w$

upper triang. w/ 1's on diagonal

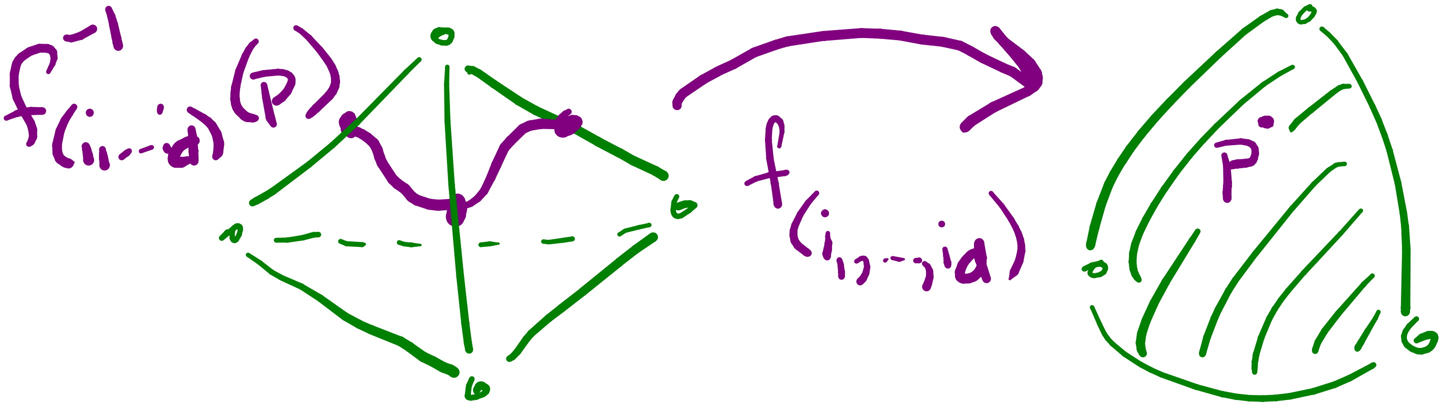
totally nonneg. part

**Theorem (H., 2014, Invent.):**
Fomin-Shapiro Conjecture holds.

**Special Case (Running example):**
Space of totally nonneg. upper triang. matrices with 1's on diag. s.t. superdiagonal sums to $k > 0$

**Concrete Realization:** Products $x_{i_1}(t_1) \cdots x_{i_d}(t_d)$ of elementary matrices, by results of Whitney, Loewner & (generalizing beyond type A) Lusztig.

**Conjecture (Davis-H-Miller):** $f^{-1}_{(i_1,\dots,i_d)}(p)$ is regular CW complex homeomorphic to interior dual block complex of subword complex $\Delta((i_1,\dots,i_d),w)$ for $p \in Y_w^o$.



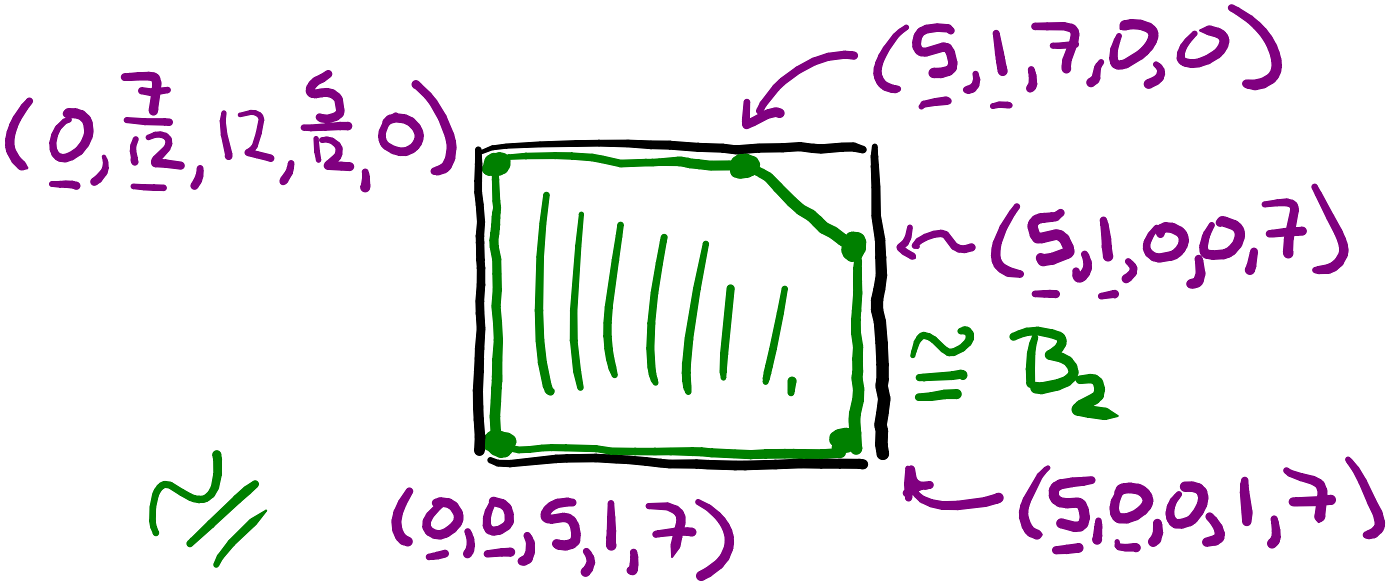
**Thm (DHM):** $f^{-1}_{(i_1,\dots,i_d)}(p)$ has cell decomposition & "correct" face poset.

**Thm (DHM):** Interior dual block complex of $\Delta((i_1,\dots,i_d),w)$ is contractible.

# Examples of Fibers:

(with realizations as suggested by various results towards proof of DHM conjecture)

e.g.



$f^{-1}_{(1,2,1,2,1)}(M)$ for $\underset{\underset{U^{o}_{1s_1s_2s_1}}{\cap}}{M} = x_1(5)x_2(1)x_1(7)$

# A Key Calculation

$$x_1(t_1)x_2(t_2)x_1(t_3) = \begin{pmatrix} 1 & t_1+t_3 & t_1t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\|$$

$$x_2(t_1')x_1(t_2')x_2(t_3') = \begin{pmatrix} 1 & t_2' & t_2't_3' \\ & 1 & t_1'+t_3' \\ & & 1 \end{pmatrix}$$

$$t_2' = t_1+t_3 \qquad t_2't_3' = t_1t_2 \qquad t_1'+t_3' = t_2$$

& more generally (simply laced)...

$$x_i(t_1)x_{i+1}(t_2)x_i(t_3) = x_{i+1}(t_1')x_i(t_2')x_{i+1}(t_3')$$

$$t_2' = t_1+t_3 \qquad t_3' = \frac{t_1t_2}{t_1+t_3} \qquad t_1' = \frac{t_2t_3}{t_1+t_3}$$

& similar changes of bases in finite type

# Role of 0-Hecke Algebra in Stratification for $\mathrm{im}(f_{(i_1,\dots,i_d)})$

(1) $x_i(t_1)x_i(t_2)=x_i(t_1+t_2)$

$\rightsquigarrow$ suppress parameters

$x_ix_i=x_i$

(2) $x_i(t_1)x_{i+1}(t_2)x_i(t_3)=x_{i+1}\left(\frac{t_2t_3}{t_1+t_3}\right)x_i(t_1+t_3)x_{i+1}\left(\frac{t_1t_2}{t_1+t_3}\right)$

$\rightsquigarrow$ (type A) for $t_1,t_2,t_3>0$

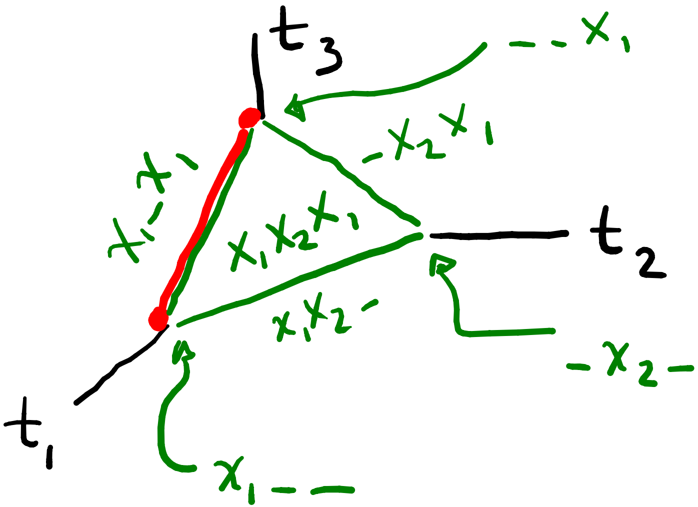
$x_ix_{i+1}x_i=x_{i+1}x_ix_{i+1}$

(& analogous relations outside type A)

**Upshot:** $\mathrm{im}(f_1)=\mathrm{im}(f_2) \Leftrightarrow \underbrace{x(f_1)=x(f_2)}_{\text{equal as 0-Hecke algebra elements}}$

**Thm (Lusztig):** If $(i_1,\dots,i_d)$ is reduced, then $f_{(i_1,\dots,i_d)}$ is homeomorphism on $\mathbb{R}^d_{>0}$

# Faces of (Preimage) Simplex as Subexpressions in 0-Hecke Algebra



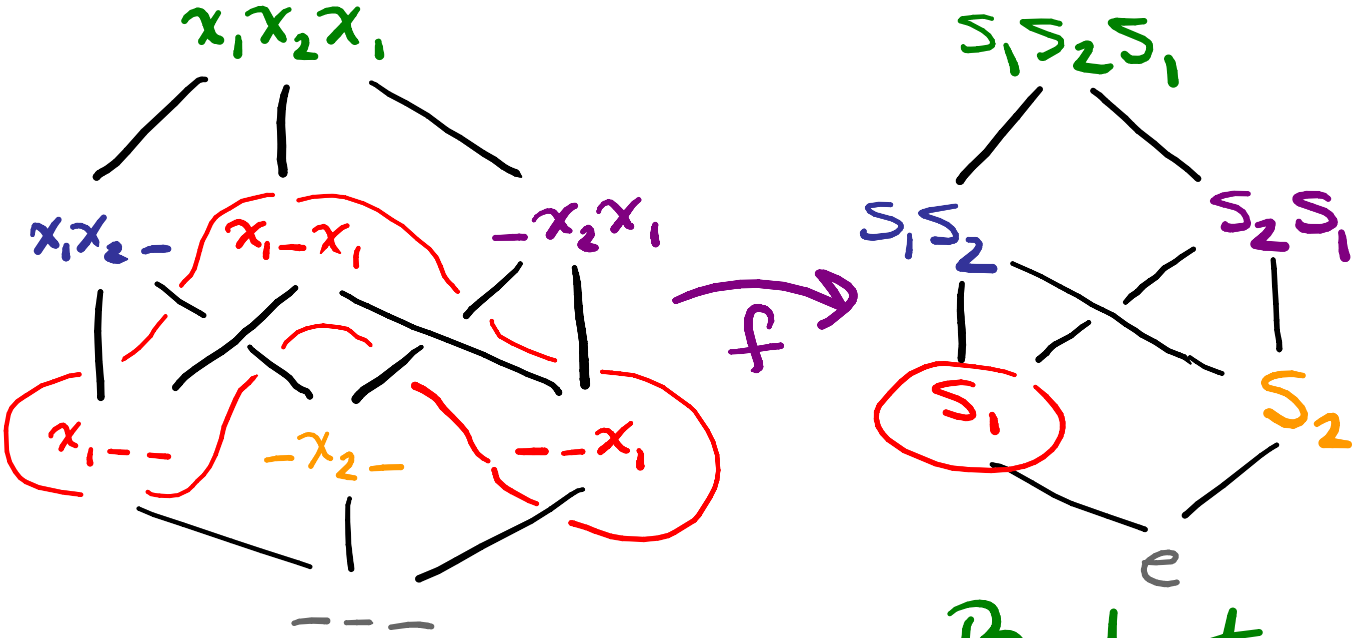
- let $Y_w^o$ = open cell in $\mathrm{im}(f_{(i_1,\dots,i_d)})$ indexed by $w \in W$

- let $\delta(x_{i_1} \dots x_{i_d})$ denote (unsigned) 0-Hecke algebra product (a.k.a. "Demazure product")

e.g. $\delta(\underbrace{x_1x_2x_1}_{x_2x_1x_2}x_2x_1) = \delta(x_2x_1x_2x_2x_1) = s_1s_2s_1$

↑ since $x_2x_2 = x_2$

# Map of Face Posets Induced by Map $f_{(i_1,\ldots,i_d)}$ of Spaces

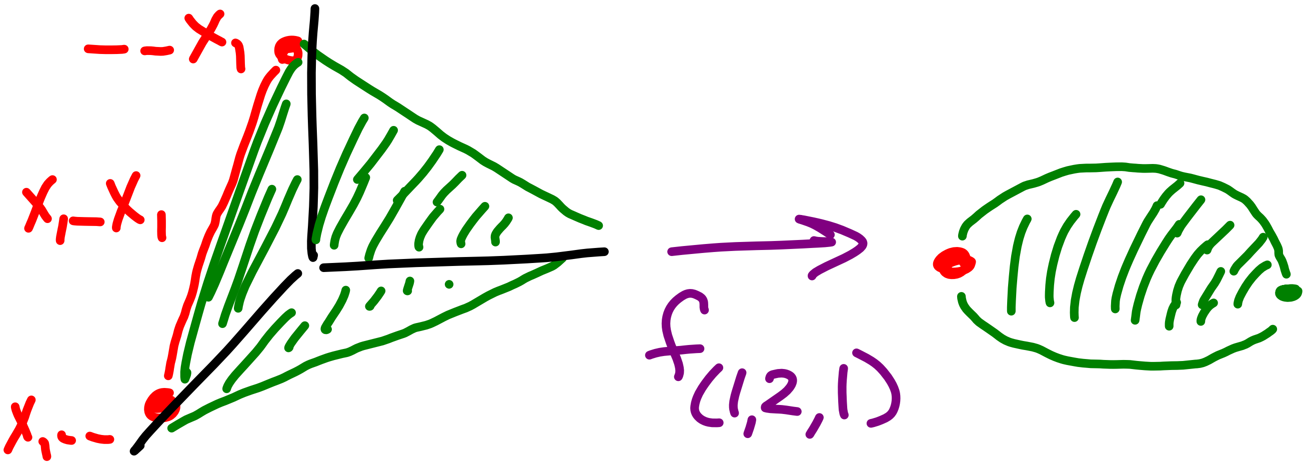


Boolean lattice $B_n$ = Face poset of simplex

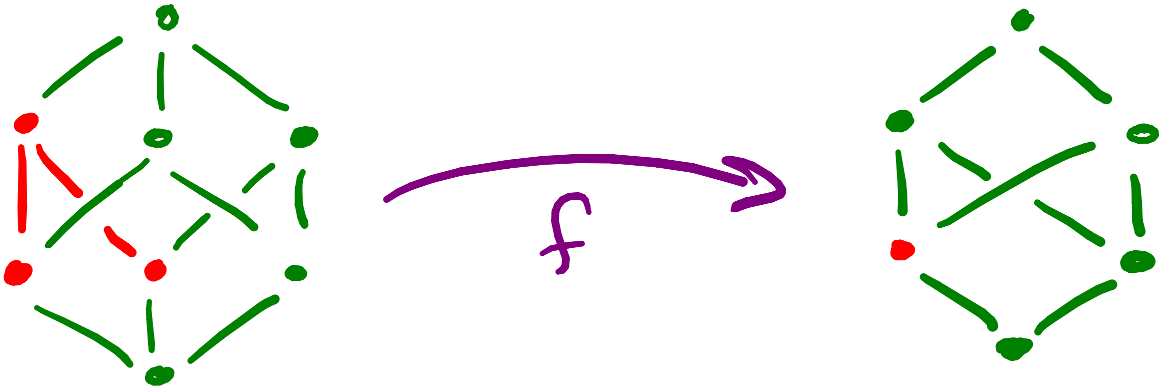
Bruhat order = $F(\mathrm{im}(f_{(i_1,\ldots,i_d)}))$

$$f : x_{i_{j_1}} \cdots x_{i_{j_r}} \longmapsto \delta(x_{i_{j_1}} \cdots x_{i_{j_r}})$$

# Combinatorics of Fibers



Induced map of face posets:



Thm (Armstrong-H., 2011): For each $u \in W$, $f^{-1}_{\geq}(u) = \{x \in B_n \mid f(x) \geq u\}$ is dual (i.e. upside-down) to face poset for subword complex $\Delta((i_1, \dots, i_d), u)$.

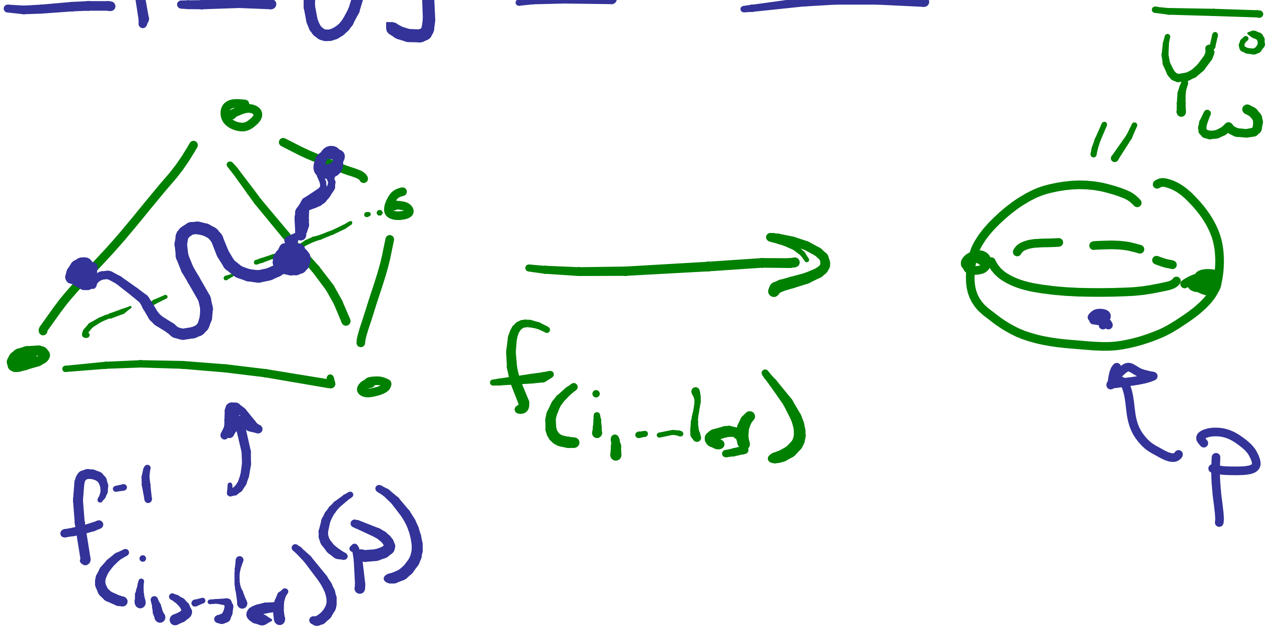
Thm (DHM, 2018): $f_{=}^{-1}(u)$ is face poset of interior dual block complex for subword complex $\Delta((i_1,\dots,i_d),u)$

Thm (DHM, 2018): Interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ is contractible.

Pf: Discrete Morse theory

Combining: DHM Conjecture would imply $f_{(i_1,\dots,i_d)}^{-1}(p) \cong$ interior dual block complex of $\Delta((i_1,\dots,i_d),u)$ for $p \in Y_u^{\circ}$, hence $f_{(i_1,\dots,i_d)}^{-1}(p)$ contractible.

# Topology of Fibers



**Thm** (Davis-H-Miller): Each fiber $f^{-1}_{(i_1,\ldots,i_d)}(p)$ admits a cell decomposition induced by the natural cell decomposition of the simplex $\Delta_{d-1}$.

**Proof**: Parametrization + continuity lemmas

**Parametrization:** Given $(i_1,\dots,i_d)$, consider rightmost subword that is reduced word for $w=\delta(i_1,\dots,i_d)$

e.g. $(1,2,1,2,1,2,1)$

Parametrize $f^{-1}_{(1,2,1,2,1,2,1)}(p)$ in $Y^{o}_{s_1s_2s_1}$

$$x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4)x_1(t_5)x_2(t_6)x_1(t_7)$$

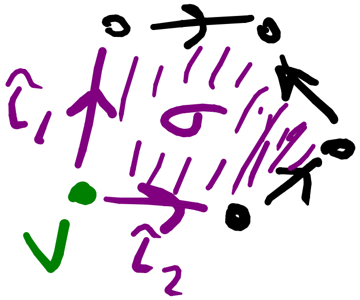
"free" parameters

"dependent" parameters

**Thm** (Davis-H-Miller):

$$[0,1)^{\dim(\sigma)} \cong \bigcup_{v\le\hat{\tau}\le\bar{\sigma}} \hat{\tau}$$

↑ source vertex of $\bar{\sigma}$



# Topology of Fibers: Key Lemmas

Defn (Davis-H-Miller): The letter $x_{i_1}$ is redundant in $x_{i_1} \cdots x_{i_d}$ if $\delta(i_1,\ldots,i_d) = \delta(i_2,\ldots,i_d)$

Lemma (Davis-H-Miller): $x_{i_1}$ is non-redundant in $x_{i_1} \cdots x_{i_d}$

$\Longleftrightarrow$ $f^{-1}_{(i_1 \ldots i_d)}(p)$ for $p \in Y^\circ_{\delta(i_1 \ldots i_d)}$

$$\overset{\shortparallel}{\{(t_1,\ldots,t_d) \mid x_{i_1}(t_1) \cdots x_{i_d}(t_d) = p\}}$$

has unique value $k_1$ for $t_1$.

$\Longleftrightarrow$ $f^{-1}_{(i_1 \ldots i_d)}(p) \cong f^{-1}_{(i_2 \ldots i_d)}(x_{i_1}(-k_1)p)$

Lemma (Davis-H-Miller): Given $(t_1,\dots,t_d) \in f^{-1}_{(i_1,\dots,i_d)}(p)$ with $t_1 > 0$ and $x_{i_1}$ redundant, then $\exists\, (t_1',\dots,t_d') \in f^{-1}_{(i_1,\dots,i_d)}(p)$ for every $t_1' \in [0, t_1]$.

e.g. $M = \begin{pmatrix} 1 & \pi & e \\ & 1 & 14 \\ & & 1 \end{pmatrix}$ then $f^{-1}_{(1,2,1,2)}(M)$

$$= \{(t_1,t_2,t_3,t_4) \mid x_1(t_1)x_2(t_2)x_1(t_3)x_2(t_4) = M\}$$

achieves every $t_1 \in [0, \frac{e}{14}]$.