B_{n-1}-orbits on the flag variety, I

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Let $G_n = GL(n)$ or $SO(n)$ over $\mathbb{C}$.

G_{n-1} is a subgroup of $G = G_n$.

For the $GL(n)$ case, G_{n-1} is embedded in G_n as the upper left corner, with 1 in nn entry. G_{n-1} is a factor of the symmetric subgroup $G_{n-1} \times G_1$ of G_n.

For the $SO(n)$ case, G_{n-1} is the identity component of the fixed set of an involution.

Let $B \subset G_n$ and $B_{n-1} \subset G_{n-1}$ be Borel subgroups, with $B \cap G_{n-1} = B_{n-1}$.

Then B_{n-1} acts with finitely many orbits on G_n/B

Goal of this work is to understand B_{n-1}-orbits on G_n/B, closure relations, and how to move from one orbit to another.

Motivation is to understand $(\mathfrak{g}, B_{n-1})$-modules via the Beilinson-Bernstein correspondence.

Further, in study of complex Gelfand-Zeitlin system, one considers a chain of subalgebras

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n$$

where $\mathfrak{g}_i$ is the Lie algebra of G_i.

The study of this problem depends on a partial version.

Quantization of this problem leads to Gelfand-Zeitlin modules, and $(\mathfrak{g}_n, B_{n-1})$-modules provide examples of a variant of Gelfand-Zeitlin modules.

Previous work

For the GL(n) case, Hashimoto [2004] gives a parametrization of orbits, but only gives a partial description of the closure relations. We give a formally different combinatorial description, and our approach is very different.

Gandini and Pezzini [2018] have general results on spherical varieties which pertain to this case, but are not as specific.

Consider the chain

$$G_1 \subset G_2 \subset \ldots \subset G_i \subset \ldots \subset G_{n-1} = K \subset G_n = G,$$

where in the $GL(n)$ case, $G_i = GL(i)$ is the "big" part of the symmetric subgroup.

In the $SO(n)$ case, or $G_i = SO(i)$ is the identity component of a symmetric subgroup.

There is a corresponding chain of Borel subgroups:

$$B_1 \subset B_2 \subset \ldots \subset B_i \subset \ldots \subset B_{n-1} \subset B_n = B,$$

where $B_i := G_i \cap B$ is a Borel subgroup of G_i.

Let $\mathcal{B}_M$ be the flag variety of a connected group M. Let $\mathcal{B}_n = G_n/B_n$ be the flag variety of G_n. We use these chains to describe B_{n-1}-orbits on $\mathcal{B}_n$ via an inductive argument.

K-ORBITS ON $\mathcal{B}_n$
Let $K = G_{n-1}$

Theorem 1:
Let Q_K be a K-orbit on G_n/B_n. Then G_n has parabolic subgroup P with Levi factor L such that

(i) $K \cap P$ is a parabolic subgroup of K, and $B_n \cap L$ is a Borel subgroup of L,

(ii) $(L, L \cap K)$ has the same semisimple type as the pair (G_i, G_{i-1}) for some $i \leq n$,

(iii) Consider the projection $\pi : G_n/B_n \to G_n/P$. Then $\pi(Q_K)$ is a closed K-orbit in $K/K \cap P$. Further, $\pi : Q_K \to K/K \cap P$ is a fibre bundle with fibre isomorphic to the open $L \cap K$-orbit in $\mathcal{B}_L$.

We use this result to describe B_{n-1} orbits in Q_K. To explain this, it is useful to describe the parabolics P that arise in this theorem, which we call *special* parabolics.

Notation for Flags and Partial Flags:

Let $v_1, \ldots, v_n$ be a basis of $\mathbb{C}^n$.

Let $V_i = \text{span}\{v_1, \ldots, v_i\}$, and denote the flag

$$V_1 \subset V_2 \subset \ldots \subset V_n$$

by

$$v_1 \subset v_2 \subset \ldots \subset v_n$$

More generally, we denote the partial flag

$$V_{i_1} \subset V_{i_2} \subset \ldots \subset V_{i_k}$$

by

$$\{v_1, \ldots, v_{i_1}\} \subset \{v_{i_1+1}, \ldots, v_{i_2}\} \subset \ldots$$

$$\subset \{v_{i_1+\ldots+i_{k-1}+1}, \ldots, v_{i_k}\}$$

K-ORBITS IN GL(n) CASE:

The closed orbits are the K-orbits Q_i through the points:

$$e_1 \subset \ldots \subset e_{i-1} \subset \underbrace{e_n}_{i} \subset e_{i+1} \subset \ldots \subset e_{n-1}.$$

These points are stabilized by a Borel subgroup of K, and that is the associated parabolic to the orbit. In this case, the fibre is trivial.

The non-closed orbits are the orbits $Q_{i,j} = K \cdot \mathcal{F}_{i,j}$, where $\mathcal{F}_{i,j} =$

$$e_1 \subset \ldots \subset \underbrace{e_i + e_n}_{i} \subset \ldots \subset e_{j-1} \subset \underbrace{e_n}_{j} \subset e_j \subset \ldots e_{n-1}.$$

Note that $\dim(Q_{i,j}) = \dim(Q_i) + j - i$, and $Q_{1,n}$ is the open K-orbit.

For the $Q_{i,j}$, the associated special parabolic $P_{i,j}$ is the stabilizer of the partial flag

$$\mathcal{P}_{i,j} = e_1 \subset \ldots \subset \{e_i, \ldots, e_{j-1}, e_n\} \subset e_j \subset \ldots \subset e_{n-1}.$$

and $P_{i,j}$ has Levi decomposition LU with
$L = GL(j - i + 1) \times GL(1)^{n-(j-i)-1}$ and
$L \cap K = GL(j - i) \times GL(1)^{n-(j-i)-1}$

K-ORBITS FOR SO(n) CASE:

For $SO(2n+1)$, the K-orbits form a linear chain.
The associated special parabolics are standard, i.e., they contain B
and have Levi factor $L = SO(2(n-i)+1)$ times a product of $GL(1)'s$
and $L \cap K = SO(2(n-i))$ times a product of Levi factors.
This includes the case $n = i$, where the associated parabolic is B.

For $G = SO(2n)$, the K-orbits almost form a linear chain.
The unique closed $O(2n)$-orbit is a union of two $SO(2n)$-orbits.
One is stabilized by B and the other is stabilized by a small variant of B.
Otherwise the description is roughly the same, except the odd orthogonal group factors are replaced by even factors.

DESCRIPTION OF B_{n-1}-ORBITS IN A K-ORBIT Q_K:

Let P be special parabolic associated to Q_K, and let $\pi : \mathcal{B}_n \to G_n/P$ be the associated projection.

Let $Q \subset Q_K$ be a B_{n-1}-orbit. Then

$\pi(Q) = B_{n-1}w(K \cap P) \subset \pi(Q_K) = K/K \cap P$ for unique $w \in W_K/W_{K\cap L}$

For $x \in Q$, $Q_K \cap \pi^{-1}\pi(x)$ is isomorphic to the open $K \cap L$-orbit in $\mathcal{B}_L$.

Thus, the description of B_{n-1}-orbits reduces to the data:

$(P, w, \mathcal{O}_t)$, where P is special, $w \in W_K/W_{K\cap L}$, and $\mathcal{O}_t$ is a $w^{-1}B_{n-1}w \cap L$-orbit in the open $K \cap L$-orbit in $\mathcal{B}_L$.

To understand these orbits, we need:

Lemma 1 : $w^{-1}B_{n-1}w \cap L \cong B_{n-1} \cap L.$

Lemma 2 : There is a point in the open K-orbit in G_n/B with stabilizer B_{n-2}.

For the $GL(n)$-case, it follows that the orbit data $\mathcal{O}_t$ from the previous slide are parametrized by:

B_{m-1}-orbits on $GL(m-1)/B_{m-2}$,
where $L = GL(m)$ times a product of $GL(1)'s$.

These are the same as B_{m-2}-orbits on $GL(m-1)/B_{m-1}$, which is a lower dimensional case of the problem we want to solve.

For the $SO(n)$-cases, we reduce to a lower dimensional case in a similar way.

Main idea: we can use induction to classify the B_{n-1}-orbits on $\mathcal{B}_n$.

Remark: there are a lot of them.

$G_n = GL(2)$, B_1 has 3 orbits on $\mathcal{B}_2$.

$G_n = GL(3)$, B_2 has 13 orbits on $\mathcal{B}_3$.

$G_n = GL(4)$, B_3 has 73 orbits on $\mathcal{B}_4$.

$G_n = GL(5)$, B_4 has 501 orbits on $\mathcal{B}_5$.

SUMMARIZE:

Theorem :
B_{n-1}-orbits on $\mathcal{B}_n$ correspond to triples $(P, w, \mathcal{O}_t)$ where (i) P is a special parabolic,
(ii) $w \in W_K/W_{K\cap L}$, and
(iii) $\mathcal{O}_t$ is parametrized by B_{m-1}-orbits on $\mathcal{B}_m$ for some $m < n$ determined by the parabolic P.

NOTATION: Given orbit datum $(P, w, \mathcal{O}_t)$, let $Q_{P,w,\mathcal{O}_t}$ denote the corresponding orbit.

Mark will explain a much better description.

Note that $Q_{P,w,\mathcal{O}_t}$ fibres over $B_{n-1}wK\cap P/K\cap P$ with fibre isomorphic to $\mathcal{O}_t$

MONOID ACTION

Let H be a reductive group with flag variety $\mathcal{B}_H$.
Let α be a simple root for H,
and let $\mathcal{P}_\alpha$ be the variety of parabolics of type α.

Let $p : \mathcal{B}_H \to \mathcal{P}_\alpha$ be the associated $\mathbb{P}_1$-bundle.

Let $R \subset H$ be a subgroup which acts on $\mathcal{B}_H$ with finitely many orbits.

Then for $Q \subset \mathcal{B}_H$ a R-orbit, and $x \in Q$
there is a unique R-orbit $m(s_\alpha)(Q)$ whose intersection with $p^{-1}p(x)$ is open in $p^{-1}p(x)$.

The operations $m(s_\alpha)$ generate a monoid action, and these operators satisfy braid relations.

For K-orbits, one can describe the type of an orbit for a simple root, and we can do something similar here.

Let $K_\Delta := \{(x,x) \in G_{n-1} \times G_n : x \in G_{n-1}\}$

Note: K_Δ-orbits on $\mathcal{B}_{n-1} \times \mathcal{B}_n$ correspond to B_{n-1}-orbits on $\mathcal{B}_n$.

This is by a general fact: let A be a group with subgroups B, C, and D.

Then C-orbits on $C/D \times A/B$ correspond to D-orbits on A/B.

Since the monoid $M(G_{n-1} \times G_n)$ acts on K_Δ-orbits on $\mathcal{B}_{n-1} \times \mathcal{B}_n$,
$M(G_{n-1} \times G_n)$ acts on B_{n-1}-orbits on $\mathcal{B}_n$.

Thus, one gets monoidal operations coming both from simple roots of G_{n-1} and from simple roots of G_n.

We generate a *weak order* on the B_{n-1}-orbits on $\mathcal{B}_n$ by taking the order relation generated by requiring $Q \leq m(s_\alpha)(Q)$.

This order relation will play a role in Mark's talk.

We prove some compatibility results for the monoid action with the fibre bundle structure in our theorem:

(i) Let Q be a B_{n-1}-orbit in $\mathcal{B}_n$ contained in a K-orbit Q_K in $\mathcal{B}_n$.
If α is a simple root of G_{n-1}, then $m(s_\alpha)(Q) \subset Q_K$.
$m(s_\alpha)(Q_{P,w,\mathcal{O}_t})$ is computed explicitly, and is given by a monoid action on at most one of the base data w or the fibre data $\mathcal{O}_t$.

(ii) Let Q be a B_{n-1}-orbit in $\mathcal{B}_n$ contained in a K-orbit Q_K in $\mathcal{B}_n$.
If α is a simple root of G_n, then $m(s_\alpha)(Q) \subset m(s_\alpha)(Q_K)$.
If $m(s_\alpha)(Q_K) = Q_K$, and $Q_{P,w,\mathcal{O}_t} \subset Q_K$, then $m(s_\alpha)(Q_{P,w,\mathcal{O}_t})$ is computed explicitly, and is given by a monoid action on at most one of the base data w or the fibre data $\mathcal{O}_t$.

Mark will explain the combinatorial aspects of our results.

We hope to use these to:

(1) Understand the category of $(\mathfrak{g}, B_{n-1})$-modules with fixed central character.

(2) Use category of $(\mathfrak{g}, B_{n-1})$-modules to construct new Gelfand-Zeitlin modules.