Multiplicities of Schubert Varieties

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Preliminary Definitions

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- B acts on G/B by left multiplication.
- The orbit BwB/B where w is a permutation matrix is called a **Schubert Cell**.

Schubert Varieties

The **Schubert variety** X_w is the closure of the Schubert cell BwB/B.

Schubert Varieties

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Theorem (Lakshmibai, Sandhya 1990)

The Schubert variety X_w is smooth if and only if w avoids the permutations 3412 and 4231

Pattern avoidance

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- A permutation w is said to contain a permutation v if, when written in one-line notation, w contains a subsequence in the same relative order as v. Otherwise, we say that w avoids v.
- For instance, **56**3**4**2**1** contains the permutation 4231.
- So X_{563421} is not smooth.

Multiplicity

- The Hilbert-Samuel multiplicity of a local ring $(R, \mathfrak{m}, \mathbb{C})$ is the degree of the projectiive tangent cone Proj$(\mathrm{gr}_m R)$ as a subvariety of the projective tangent space Proj$(\mathrm{sym}^* \mathfrak{m}/\mathfrak{m}^2)$.
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- For a scheme X and a point p, the multiplicity of X at p is the multiplicity of the local ring $(\mathcal{O}_{X_p}, \mathfrak{m}_p, \mathbb{C})$.
- A variety is smooth if and only if it has multiplicity one at all points.

A first attempt at characterizing Schubert varieties of multiplicity at most two

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- Answer: No
- The permutation 354612 embeds in 4657312, but X_{354612} has multiplicity three while $X_{4657312}$ has multiplicity two.

Schubert points

- The points of X_w that correspond to permutations are called **Schubert points.** For a permutation x, we denote the Schubert point by e_x. Moreover e_x is a Schubert point of X_w precisely when $x \leq w$ in Bruhat order.

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- Every point on a Schubert variety is in the B-orbit of some Schubert point, and the B-action gives an isomorphism between local neighborhoods.
- So if we want to study local properties of Schubert varieties, it suffices to focus on Schubert points.

The Rothe Diagram

To calculate the local equations for X_w, we will need to construct the **Rothe Diagram** for w. We will proceed by example for $w = 819372564$.

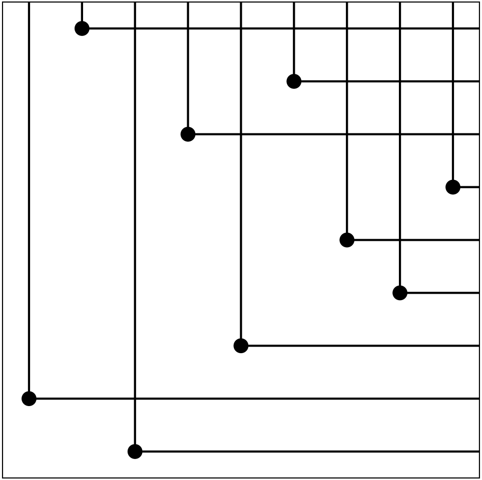
The Rothe Diagram

Start with the permutation matrix for w.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

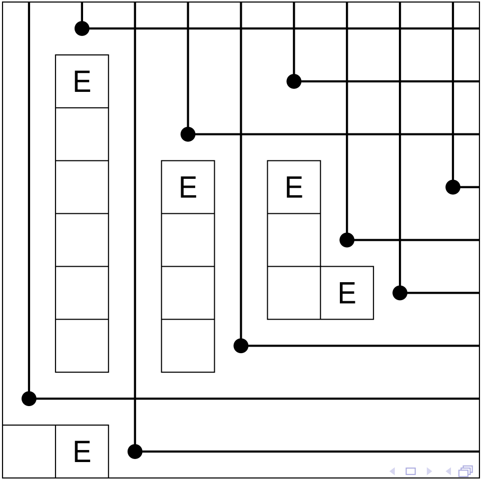
The Rothe Diagram

Our diagram starts with a 9x9 grid with dots in place of each 1 in the permutation matrix. Draw a hook that extends north and east of each dot.



The Rothe Diagram

The **Rothe Diagram** consists of the positions not in any hook, designated by squares. The **essential set** consists of the northeast corners of the connected components of the diagram, designated by E's.



The Kazhdan-Lusztig ideal

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- For a permutation $x \in S_n$, let let $Z^{(x)}$ be the $n \times n$ matrix where the entries at $(x(i), i)$ are 1 for all i; the entries at $(x(i), a)$ and (b, i) are 0 for $a > i$ and $b < x(i)$; and the remaining entries are variables.

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- Let $Z_{ij}^{(x)}$ be the southwest submatrix of $Z^{(x)}$ with northeast corner (i, j).
- The **Kazhdan-Lusztig ideal** $\mathcal{I}_{x,w}$ is generated by the size $1 + r_w(p,q)$ minors of Z_{ij}^{x} over all i, j.

The Kazhdan-Lusztig ideal

Theorem

Let $\mathcal{N}_{x,w} := Spec(\mathbb{C}[z^{(x)}]/\mathcal{I}_{x,w})$. Then $\mathcal{N}_{x,w} \times \mathbb{A}^{l(x)}$ is isomorphic to an affine neighborhood of X_w at e_x.

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- In particular, a local property $\mathcal{P}$ holds at e_x on X_w if and only if $\mathcal{P}$ holds at the origin 0 on $\mathcal{N}_{x,w}$.

Schubert varieties of multiplicity at most two

- We can simplify our computations by restricting our attention to e_{id}. This is because $\mathrm{mult}_{e_u}(X_w) \geq \mathrm{mult}_{e_v}(X_w)$ when $u \leq v \leq w$ in Bruhat order.

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- $\mathcal{I}_{id,w}$ has a known set of minimal generators corresponding to diagram boxes when X_w is LCI.
- If X_w is LCI, then there are strong restrictions on where the essential set boxes may appear.

The shifted diagram

- These constraints are sufficient to provide a characterization of Schubert varieties of multiplicity at most two based on the Rothe diagram.

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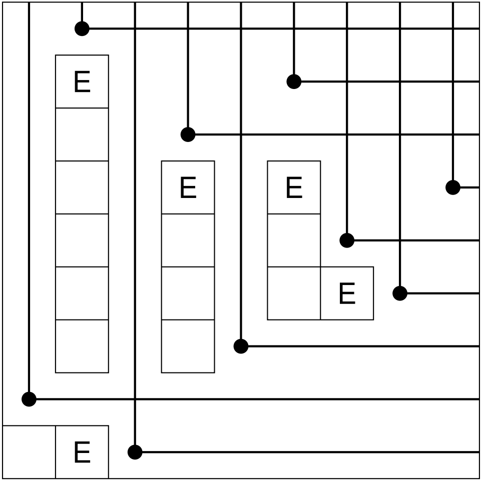
- These constraints are sufficient to provide a characterization of Schubert varieties of multiplicity at most two based on the Rothe diagram.
- For an entry (a, b) in the Rothe diagram, shift every entry $(p, q) \neq (a, b)$ in the Rothe diagram southwest by $r_w(p, q)$. The resulting diagram is the **shifted diagram** for w at (a, b).

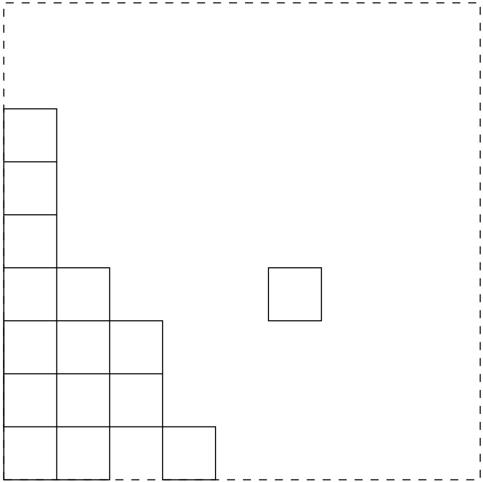
The shifted diagram

The Rothe diagram for $w = 819372564$



The shifted diagram

The shifted diagram for $w = 819372564$ at $(6, 6)$



A characterization for Schubert varieties of multiplicity at most two

- If the shifted diagram at (a, b) contains an entry $r_w(a, b)$ southwest of (a, b), then (a, b) is called a **double** box.
- If the shifted diagram at (a, b) contains a hook of length $r_w(a, b) + 1$ with vertex $r_w(a, b)$ southwest of (a, b), then (a, b) is called a **triple** box.

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Theorem (M.)

If X_w is an LCI Schubert variety, then it has multiplicity at least three if and only if it contains a triple box or two double boxes. Moreover, it suffices to check only the southwest corners of each connected component and essential set boxes that are not defined by inclusions.

Thank you!