Exceptional group G_2 and set partitions

Proof of a conjecture by Mihailovs'

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Invariant theory

Let G be a reductive algebraic group and V a (finite dimensional) representation. Then we form the sequence of vector spaces whose n-th term is the invariant subspace in the tensor power $\otimes^n V$.
Let C be the (Kashiwara) crystal of V. Then we form the sequence of sets whose n-th term is the set of invariant words in the tensor power $\otimes^n C$.
Then we get a sequence by taking the dimension of the vector space or the cardinality of the set.

Tableaux

This gives some well-known combinatorial sequences

- $\mathrm{SL}(2)$, Dyck paths and Riordan paths
- $\mathrm{SL}(n)$, semistandard tableaux
- $\mathrm{Sp}(2n)$, oscillating tableaux and matchings
- $\mathrm{Spin}(2n+1)$, fans of Dyck paths

Example

For V the defining representation of $\mathrm{SL}(2)$ this gives

0	1	2	3	4	5	6	7	8
1	0	1	0	2	0	5	0	14

Exceptional group G_2

Example

For V the fundamental representation of G_2 this gives

0	1	2	3	4	5	6	7	8
1	0	1	1	4	10	35	120	455

This is sequence A059710. This enumerates nonpositive planar trivalent graphs.

Recurrence relation

The sequence is determined by the recurrence relation

$$14(n+1)(n+2)a(n)+(n+2)(19n+75)a(n+1)$$
$$+2(n+2)(2n+11)a(n+2)-(n+8)(n+9)a(n+3)=0.$$

together with the initial conditions $a(0)=1$, $a(1)=0$, $a(2)=1$.
This recurrence relation for this sequence can be found by computing initial terms and then fitting a recurrence relation. This was first done by Mihailov.

Laurent polynomials

The n-th term is the constant term of the Laurent polynomial $W\,K^n$ where

$$K = (1 + x + y + x\,y + x^{-1} + y^{-1} + (xy)^{-1})$$

and W is the Laurent polynomial

$$\begin{aligned} W = x^{-2}y^{-3}(&x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} \\ &+ x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2) \end{aligned}$$

This gives a direct proof of the recurrence relation.

Binomial transform

Given the sequence a with n-th term $a(n)$, the *binomial transform* is the sequence whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} a(i)$$

The binomial transform arises naturally for sequences a_V since we have

$$a_{V \oplus \mathbb{C}} = \mathcal{B} a_V$$

Binomial transform

The binomial transform is a known sequence

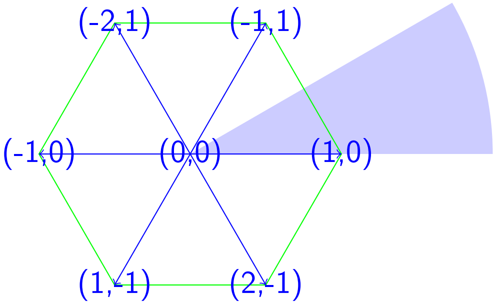
0	1	2	3	4	5	6	7	8
1	1	2	5	15	51	191	772	3320

This is sequence A108307. This sequence enumerates

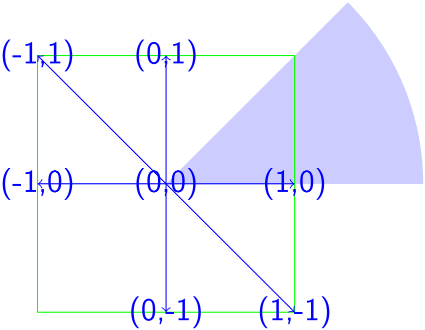
- hesitating tableaux of height two
- set partitions with no enhanced 3-crossing
- 2-regular set partitions with no 3-crossing

The recurrence relation for this sequence is known. Then Mihailov's recurrence relation can be deduced from this using creative telescoping.

Crystal paths



Hesitating tableaux



The binomial transform of A108307 is also a known sequence

0	1	2	3	4	5	6	7	8
1	2	5	15	52	202	859	3930	19095

This is sequence A108304. This sequence enumerates

- vacillating tableaux of height two
- set partitions with no 3-crossing

Generating functions

Let $G(t)$ be the generating function of a sequence. Then the generating functions of the binomial transform and the inverse binomial transform are

$$\frac{1}{1-t}G\left(\frac{t}{1-t}\right) \qquad \frac{1}{1+t}G\left(\frac{t}{1+t}\right)$$

The differential equation for the generating function is also known. A differential equation equivalent to Mihailov's recurrence relation can be deduced from this.

Summary

This gives three sequences related by binomial transforms.

	0	1	2	3	4	5	6	7	8	9
A059710	1	0	1	1	4	10	35	120	455	1792
A108307	1	1	2	5	15	51	191	772	3320	15032
A108304	1	2	5	15	52	202	859	3930	19095	97566

These arise from the G_2 representations V, $V \oplus \mathbb{C}$, $V \oplus 2\mathbb{C}$.

This connects the invariant theory of G_2 with 3-noncrossing set partitions.

Iterated binomial transform

- If the original sequence arises from a representation V then the iterated binomial transform arises from $V \oplus k\,\mathbb{C}$.
- The k-th binomial transform of the sequence a is the sequence whose n-th term is

$$\sum_{i=0}^{n} \binom{n}{i} k^i \, a(i)$$

- Let $G(t)$ be the generating function of a sequence. Then the generating function of the k-th binomial transform is

$$\frac{1}{1-k\,t} G\left(\frac{t}{1-k\,t}\right)$$

Quadrant sequences

Take the sequences associated to $U \oplus U^* \oplus k\,\mathbb{C}$ for U and U^* the two fundamental representations of $\mathrm{SL}(3)$.
For $k = 0, 1, 2, 3$ this gives

	0	1	2	3	4	5	6	7
A151366	1	0	2	2	12	30	130	462
A236408	1	1	3	9	33	131	561	2535
A001181	1	2	6	22	92	422	2074	10754
A216947	1	3	11	47	225	1173	6529	38265

These are four sequences connected by the binomial transform.
The third sequence enumerates Baxter permutations.
The OEIS entries do not mention this or the connection with invariant theory.

Laurent polynomials

The n-th term is the constant term of the Laurent polynomial $W\,K^n$ where

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x}$$

and W is the Laurent polynomial

$$W = 1 - \frac{x^2}{y} + x^3 - x^2y^2 + y^3 - \frac{y^2}{x}.$$

Recurrence relation

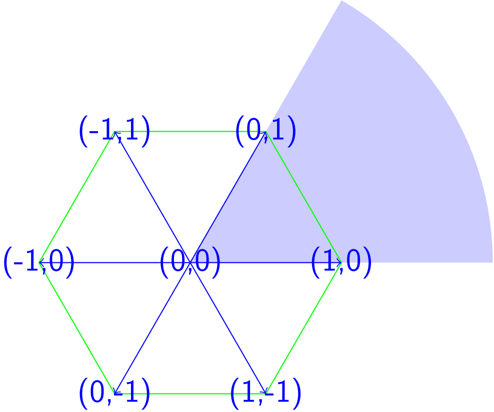
The k-th sequence satisfies the recurrence relation

$$
\begin{aligned}
&(-9+k)(-1+k)k^2(1+n)(2+n)a(n)+ \\
&2k(2+n)(36-56k+8k^2+9n-15kn+2k^2n)a(n+1)+ \\
&(162-510k+114k^2+81n-254kn+54k^2n+9n^2-30kn^2+6k^2n^2)a(n+2 \\
&\quad +2(-153+70k-56n+24kn-5n^2+2kn^2)a(n+3) \\
&\qquad +(7+n)(8+n)a(n+4)=0
\end{aligned}
$$

This can also be regarded as a recurrence relation for a sequence of polynomials.

Branching rules

There is an inclusion $\mathrm{SL}(3) \subset G_2$. Restricting $V \oplus k\,\mathbb{C}$ gives $U \oplus U^* \oplus (k+1)\,\mathbb{C}$.



Branching rules

Extend sequence to sequence of functions on dominant weights.
Then the A_2 sequence is determined by the G_2 sequence using the branching rules for $A_2 \to G_2$.
For example, the number of Baxter permutations is the number of hesitating tableaux whose final shape has one row.

Generating functions

For V a representation of $\mathrm{SL}(2)$ the generating function is algebraic.
For V a representation of a group of rank two the generating function is holonomic. We have given a recurrence relation for all the sequences.
The generating functions have the further property that the differential operator factors with first factor first order. This implies that there is an exact formula for the generating function in terms of hypergeometric functions.