Motivic Chern classes and Hecke algebras

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K theory

X (smooth) projective algebraic variety defined over k, $char(k) = 0$. The K-theory

$$K(X) = \frac{\{[E] : E \to X \text{ vector bundle }\}}{[E] = [F] + [G]},$$

for any short exact sequence $0 \to F \to E \to G \to 0$. Addition and multiplication are given by

$$[E] + [F] := [E \oplus F]; \quad [E] \cdot [F] := [E \otimes F].$$

There is a pairing $\langle \cdot, \cdot \rangle : K(X) \times K(X) \to \mathbb{Z}$ defined by

$$\langle [E], [F] \rangle = \int_X E \otimes F = \sum (-1)^i \dim H^i(X; E \otimes F).$$

The Grothendieck group of varieties

Let X algebraic variety.

$$G_0(var/X) = \frac{\{[f : Y \to X] : Y - scheme\}}{[Y \to X] = [Z \to X] + [Y \setminus Z \to X]},$$

for $Z \subset Y$ a closed subvariety. For any $f : X_1 \to X_2$ have a push-forward:

$$\begin{array}{ccc} Y & \longrightarrow & X_1 \\ & \searrow & \big\downarrow f \\ & & X_2 \end{array}$$

$$f_! : G_0(var/X_1) \to G_0(var/X_2); \quad [g : Y \to X_1] \mapsto [f \circ g : Y \to X_2].$$

Motivic Chern classes

Theorem (Brasselet-Schürmann-Yokura, 2010)

There exists a unique natural transformation

$$MC_y : G_0(var/X) \to K(X)[y]$$

commuting with proper morphisms such that when X is smooth,

$$MC_y[id_X : X \to X] = \lambda_y(T^*X) := \sum [\wedge^i T^*(X)]y^i$$

is the Hirzeburch λ_y class of X.

Notation: if $Z \subset X$, denote by $MC_y(Z) := MC_y[Z \hookrightarrow X]$.
Initial goal: Calculate

$$MC_y(X_w^\circ) := MC_y[X_w^\circ \hookrightarrow \mathrm{Fl}(n)] \in K(\mathrm{Fl}(n)),$$

where X_w° is a Schubert cell in a flag manifold $\mathrm{Fl}(n)$.
(Feher-Rimányi-Weber, A.-M.-S.-S.)

Flag manifolds

$X = G/B = \mathrm{Fl}(n)$, the flag manifold, where G is a complex, simple Lie group, $B \subset G$ is Borel subgroup. Let $T := B \cap B^-$ be the maximal torus.

$$\mathrm{Fl}(n) = \{F_\bullet : F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n\}.$$

Let $W := N_G(T)/T = S_n$ be the Weyl group. For each $w \in W$ have Schubert cells and varieties

$$X_w^\circ := BwB/B; \quad X_w := \overline{X_w^\circ}.$$

Then $\dim X_w = \ell(w)$. Let

$$\mathcal{O}_w := [\mathcal{O}_{X_w}] \in K(\mathrm{Fl}(n)).$$

Then

$$K(\mathrm{Fl}(n)) = \oplus_w \mathbb{Z}\mathcal{O}_w.$$

the Schubert basis.

Examples

1. By the motivic property:

$$MC_y(\mathbb{A}^1 \subset \mathbb{P}^1) = \lambda_y(T^*_{\mathbb{P}^1}) - \lambda_y(T^*_{pt}) = (1+y)\mathcal{O}_{\mathbb{P}^1} - (1+2y)\mathcal{O}_{pt}.$$

2. The motivic class for the open cell in $\mathrm{Fl}(3)$ is:

$$\begin{aligned} MC_y(X(s_1s_2s_1)^\circ) =& (1+y)^3\mathcal{O}_{s_1s_2s_1} - (1+y)^2(1+2y)(\mathcal{O}_{s_1s_2} + \mathcal{O}_{s_2s_1}) + \\ & (1+y)(5y^2+4y+1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) \\ & - (8y^3+11y^2+5y+1)\mathcal{O}_{id} \end{aligned}$$

3. $\int_{G/B} MC_y(X_w^\circ) = MC_y[X_w^\circ \to pt] = MC_y[\mathbb{A}^1 \to pt]^{\ell(w)} = (-y)^{\ell(w)}$.
   In particular, the χ_y genus of $\mathrm{Fl}(n)$ is

$$\begin{aligned} \int_{\mathrm{Fl}(n)} MC_{-q}[id : \mathrm{Fl}(n) \to \mathrm{Fl}(n)] =& \sum_{w \in W} MC_{-q}[X_w^\circ \to pt] \\ =& \sum_{w \in W} q^{\ell(w)} \\ =& [n]_q! \end{aligned}$$

(the q-analogue of the factorial.)

Bott Samelson resolutions

Let $\omega = (i_1, \ldots, i_{k-1}, i_k)$ be a word and $\omega' = (i_1, \ldots, i_{k-1})$. The Bott-Samelson variety $Z := B_\omega$ can be defined inductively as a $\mathbb{P}^1$-bundle over $Z' := B_{\omega'}$, using the following fibre squares.

$$\begin{array}{ccccc}
 & & \overset{\theta}{\curvearrowright} & & \\
Z & \xrightarrow{\theta_1} & \mathfrak{X} & \xrightarrow{pr_1} & G/B = \mathrm{Fl}(n) \\
\sigma \uparrow \downarrow \pi & & \downarrow pr_2 & & \downarrow p_{i_k} \\
Z' & \xrightarrow{\theta'} & G/B = \mathrm{Fl}(n) & \xrightarrow{p_{i_k}} & G/P_{i_k} = \mathrm{Fl}(\widehat{i_k}; n)
\end{array}$$

If ω is reduced and $w := s_{i_1} \ldots s_{i_k}$, then $\theta : Z \to X_w$ is a resolution of singularities. For each subexpression $\underline{v} \subset \omega$ one defines a Bott stratum:

$$B^\circ_{\underline{v}} := \begin{cases} \pi^{-1}(B^\circ_{\underline{v}'}) \setminus \sigma(B^\circ_{\underline{v}'}) & \underline{v}' \not\subset \omega' \\ \sigma(B^\circ_{\underline{v}'}) & \underline{v}' \subset \omega'. \end{cases}$$

MC classes of Bott-Samelson varieties

1. Find $\theta_*(MC_y(B_\omega)) \in K(\mathrm{Fl}(n))[y]$.
2. Find $\theta_*(MC_y(B^\circ_{\underline{v}})) \in K(\mathrm{Fl}(n))[y]$ for any subexpression $\underline{v} \subset \omega$.

Let $w := s_{i_1} \circ \ldots \circ s_{i_k}$ be the Hecke product. Then in $G_0(var/\mathrm{Fl}(n))$:

$$[\theta : B_\omega \to \mathrm{Fl}(n)] = \sum_{v \le w} [\theta^{-1} X^\circ_v \to \mathrm{Fl}(n)] = \sum_{v \le w} [F_v \to pt] \boxtimes [X^\circ_v \to \mathrm{Fl}(n)],$$

where $F_v = \theta^{-1}(e_v)$. By motivic and product properties:

$$\theta_* MC_y(B_\omega) = \sum_{v \le w} MC_y[F_v \to pt] \cdot MC_y(X^\circ_v) = \sum_{v \le w} P_v(y) MC_y(X^\circ_v),$$

where $P_v(-q)$ is the Poincaré polynomial of the fibre F_v (by S. Gaussent stratification).

Demazure-Lusztig operators

Fix $1 \leq i \leq n-1$, and consider the projection: $p_i : G/B \to G/P_i$. The Demazure operator is $\partial_i := (p_i)^*(p_i)_* : K(G/B) \to K(G/B)$.

$$\partial_i \mathcal{O}_w = \begin{cases} \mathcal{O}_{ws_i} & ws_i > w \\ \mathcal{O}_w & ws_i < w. \end{cases}$$

The Demazure-Lusztig operators are:

$$\mathcal{T}_i = (1 + yT^*_{p_i})\partial_i - id; \quad \mathcal{T}_i^\vee = \partial_i(1 + yT^*_{p_i}) - id.$$

Lemma (Lusztig)

The operators $\mathcal{T}_i$ satisfy the following properties:

1. *(commutativity) E.g. in type A, $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ if $|i-j| \geq 2$;*
2. *(braid relations) E.g. in type A: $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$;*
3. *(quadratic relations): $(\mathcal{T}_i + y)(\mathcal{T}_i + id) = 0$.*

Same properties are satisfied by $\mathcal{T}_i^\vee$ and $\langle \mathcal{T}_i(a), b \rangle = \langle a, \mathcal{T}_i^\vee(b) \rangle$.

Motivic Chern classes

Theorem (A-M-S-S '19)

Let $w \in W$ and let i such that $ws_i > w$. Then

$$MC_y(X(ws_i)^\circ) = \mathcal{T}_i(MC_y(X(w)^\circ)).$$

In particular, $MC_y(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}_{id})$.

Corollary (M.-Withrow)

Let $\omega = (\omega', i_k)$ be any (possibly non-reduced) word. Then:

1. $\theta_* MC_y(B_\omega^\circ) = \mathcal{T}_{i_k} MC_y(B_{\omega'}^\circ)$;
2. $\theta_* MC_y(B_\omega) = (\mathcal{T}_{i_k} + 1) MC_y(B_{\omega'})$.

Define $\mathcal{D}_\omega := (\mathcal{T}_{i_1} + 1) \cdot \ldots \cdot (\mathcal{T}_{i_k} + 1)$. (This depends on the word.) Then:

$$\theta_*(MC_y(B_\omega)) = [\mathcal{O}_{pt}].\mathcal{D}_{\omega^{-1}} = [\mathcal{O}_{pt}].(\sum_{v \le w(\omega)} P_v(y)\mathcal{T}_{v^{-1}}).$$

Examples

Consider $\mathrm{Fl}(3)$ and the Bott-Samelson variety $\pi : BS(121) \to \mathrm{Fl}(3)$. One calculates ($y = -q$):

$$\begin{aligned}\pi_*(\lambda_y(T^*_{BS(1,2,1)}) = & MC_y(X(s_1s_2s_1)^\circ) + MC_y(X(s_1s_2)^\circ) + MC_y(X(s_2s_1)^\circ) \\ & + MC_y(X(s_2)^\circ) + (1-y)MC_y(X(s_1)^\circ) \\ & + (1-y)MC_y(X(id)).\end{aligned}$$

The image of a Bott stratum for a non-reduced expression:

$$\pi_* MC_y(B^\circ_{101}) = -(1+y)MC_y(X(s_1)^\circ) - yMC_y(X(id)).$$

This suggests:

- the preimage (in B°_{101}) of e_{s_1} is $\mathbb{C}^*$;
- the preimage (in B°_{101}) of e_{id} is $\mathbb{C}$.

Connections to Hecke algebra

The Hecke algebra is a $\mathbb{Z}[q^{1/2}, q^{-1/2}]-$algebra freely generated by elements T_w, $w \in W$ satisfying the relations from previous Lemma. There is an involution:

$$\overline{q^{1/2}} = q^{-1/2}; \quad \overline{T_w} = T_{w^{-1}}^{-1}.$$

The element $D_i' := q^{-1/2}(T_i + 1)$ is satisfies $\overline{D_i'} = D_i'$.

Theorem (Deodhar '90 , Brubaker-Bump-Licata '16)

For a reduced decomposition $w = s_{i_1} \cdots s_{i_k}$, let $D_{\underline{w}}' = D_{i_1}' \cdots D_{i_k}'$. Then

$$D_{\underline{w}}' = q^{-\ell(w)/2} \sum_{v \leq w} P_{\underline{w},v}(q) T_v.$$

A more refined geometric interpretation is:

$$\pi_*(MC_y(B_{\underline{w}})) = [\mathcal{O}_{pt}].\mathcal{D}_w = \sum_{v \leq w} P_{\underline{w},v}(y) MC_y(X(v)^\circ).$$

A motivic Demazure formula

We work in equivariant K-theory. For a weight λ, let

$$\mathcal{L}_\lambda := G \times^B \mathbb{C}_\lambda.$$

Let $A : K_G(G/B) \to K_G(G/B)$ be any $K_G(pt)$-linear operator. Recall $K_T(pt) = R(T) \simeq K_G(G/B) \simeq K_T(G/B)^W$. Define

$$\widetilde{A} : K_T(pt) \to K_T(pt); \quad \widetilde{A}(e^\lambda) = A(\mathcal{L}_\lambda)_{|id}.$$

The Demazure formula is

$$\begin{aligned}\chi(G/B; \mathcal{L}_\lambda \otimes \mathcal{O}_w) &= \langle \mathcal{L}_\lambda, \mathcal{O}_w \rangle \\ &= \langle \mathcal{L}_\lambda, \partial_{w^{-1}}(\mathcal{O}_{id}) \rangle \\ &= \langle \partial_w(\mathcal{L}_\lambda), \mathcal{O}_{id} \rangle \\ &= \widetilde{\partial}_w(e^\lambda).\end{aligned}$$

A motivic Demazure formula

Define:

$$MC'_y(X(w)^\circ) := \prod_{\alpha>0}(1+ye^{\alpha})\frac{MC_y(X(w)^\circ)}{\lambda_y(T^*_X)}.$$

Theorem (M.-Su '19)

The following hold:

1. $\chi(X, \mathcal{L}_\lambda \otimes MC_y(X(w)^\circ)) = \widetilde{\mathcal{T}_w^\vee}(e^\lambda)$.
2. $\chi\left(X, \mathcal{L}_\lambda \otimes MC'_y(X(w)^\circ)\right) = \widetilde{\mathcal{T}_w}(e^\lambda)$.

The element $\widetilde{\mathcal{T}_w}(e^\lambda)$ is the Iwahori-Whittaker function for the principal series representation (Brubaker-Bump-Licata; Lee-Lenart-Liu). In particular, we obtain the Casselman-Shalika formula:

$$\sum_{w\in W}\widetilde{\mathcal{T}_w}(e^\lambda) = \sum_{w\in W}\chi(G/B; MC'_y(X(w)^\circ)\otimes\mathcal{L}_\lambda)$$

$$= \chi(G/B; \prod_{\alpha>0}(1+ye^\alpha)\otimes\mathcal{L}_\lambda) = \prod_{\alpha>0}(1+ye^\alpha)\widetilde{\partial_{w_0}}(e^\lambda).$$

THANK YOU!