Amenable signed permutations

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Schubert Calculus

Giambelli Problem: Find polynomials that represent the cohomology classes of the Schubert varieties X_w in G/P.

Relative Version (Fulton-Pragacz, 1996): Find Chern class polynomials which represent the cohomology classes of *degeneracy loci* $\mathfrak{X}_w$ of vector bundles, when G is a classical Lie group.

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Intrinsic formulas

General w (T., 2009): $\exists$ polynomial formulas for $[\mathfrak{X}_w]$ which are *native to* G/P, for all $w \in W^P$.

New, intrinsic point of view in Schubert calculus.

In special cases, $\exists$ alternative intrinsic formulas:

In Lie type A

For $\varpi \in S_n$ *Grassmannian*:
Thom-Porteous (1970); Kempf-Laksov (1974).

More generally, for $\varpi \in S_n$ *vexillary*:
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Permutations

$$\varpi = (\varpi_1, \ldots, \varpi_n) \in S_n, \text{ where } \varpi_i = \varpi(i).$$

Code $\gamma = \gamma(\varpi)$ with $\gamma_i := \#\{j > i \mid \varpi_j < \varpi_i\}$.

Shape $\lambda = \lambda(\varpi)$ obtained by reordering the γ_i.

Example

$w = (2, 1, 5, 4, 3)$, $\gamma = (1, 0, 2, 1, 0)$, $\lambda = (2, 1, 1)$.

$S_n = \langle s_1, \ldots, s_{n-1} \rangle$. ϖ has a *right/left descent* at i if $\ell(\varpi s_i) < \ell(\varpi)$ (resp. $\ell(s_i \varpi) < \ell(\varpi)$).

ϖ is *Grassmannian* if $\ell(\varpi s_i) > \ell(\varpi)$, $\forall i \neq k$.

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Type A degeneracy loci

$E \to \mathfrak{X}$ is a vector bundle of rank n and $\varpi \in S_n$.

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E \quad 0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = E.$$

$$\mathfrak{X}_\varpi := \{x \in \mathfrak{X} \mid \dim(E_r(x) \cap F_s(x)) \geq d_\varpi(r,s), \ \forall\, r, s\}$$

where $d_\varpi(r,s) := \#\{i \leq r \mid \varpi_i > n - s\}$.

Assume: $\mathfrak{X}_\varpi$ has pure codimension $\ell(\varpi)$ in $\mathfrak{X}$.

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Type A vexillary degeneracy loci

Theorem (L.-S., Wachs, Macdonald, Fulton (1992))

Suppose that ϖ is a vexillary permutation of shape $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. Then there exist sequences $\mathfrak{f} = (f_1 \leq \cdots \leq f_\ell)$ and $\mathfrak{g} = (g_1 \geq \cdots \geq g_\ell)$ consisting of right and left descents of ϖ, respectively, such that

$$[\mathfrak{X}_\varpi] = \det(c_{\lambda_i+j-i}(E - E_{f_i} - F_{n-g_i}))_{1\leq i,j\leq \ell}$$

holds in $\mathrm{H}^*(\mathfrak{X})$.

Here $c_p(E - E' - E'')$ is defined by $c(E - E' - E'') := c(E)c(E')^{-1}c(E'')^{-1}$.

Raising Operator Form

$$[\mathfrak{X}_\varpi] = \prod_{i<j}(1 - R_{ij})\, c_\lambda(E - E_{\mathfrak{f}} - F_{n-\mathfrak{g}}) \quad (1)$$

For $i < j$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ an integer sequence,

$$R_{ij}\,\alpha := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$$

If $\mathfrak{c}_\alpha := \mathfrak{c}_{\alpha_1}\mathfrak{c}_{\alpha_2}\cdots$, then $R_{ij}\,\mathfrak{c}_\alpha := \mathfrak{c}_{R_{ij}\alpha}$.

(1) is the image of $\prod_{i<j}(1 - R_{ij})\,\mathfrak{c}_\lambda$ under the $\mathbb{Z}$-linear map sending $\mathfrak{c}_\alpha$ to $\prod_i c_{\alpha_i}(E - E_{f_i} - F_{n-g_i})$, for every integer sequence α.

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Amenable signed permutations

Question: What is the analogue of 'vexillary' in the other classical Lie types B, C, and D?

Prior attempts at a definition of vexillary signed permutations: Billey-Lam (1998) & Anderson-Fulton (2012).

Both of these definitions are *wrong*.

Reason: According to either of them, the Grassmannian signed permutations are not all vexillary.

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Amenable signed permutations: Type C

$$w = (w_1, \ldots, w_n) \in W_n = \langle s_0, s_1, \ldots, s_{n-1} \rangle.$$

A-code $\gamma = \gamma(w)$ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$.

Definition (T., 2019)

w is leading *if the A-code $\widetilde{\gamma}$ of the extended sequence $(0, w_1, \ldots, w_n)$ is unimodal.*

$$\{\text{Leading elements}\} \supset \{\text{Grassmannian elements}\}$$

Definition (T., 2019)

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Let $k \geq 0$ be the first right descent of w.

List the values $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_m < 0 < u_{m+1} < \cdots < u_{n-k}.$$

A simple transposition s_i, $i \geq 1$ is called

w-*negative*, if $\{i, i+1\} \subset \{-u_1, \ldots, -u_m\}$;

w-*positive*, if $\{i, i+1\} \subset \{u_{m+1}, \ldots, u_{n-k}\}$.

Let σ^- (resp. σ^+) be the longest subword of $s_{n-1} \cdots s_1$ (resp. $s_1 \cdots s_{n-1}$) consisting of w-negative (resp. w-positive) transpositions s_i.

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Definition

A modification *of* w *is an element* ϖw, *where* $\varpi \in S_n$, $\ell(\varpi w) = \ell(w) - \ell(\varpi)$, *and* ϖ *has a reduced decomposition of the form* $R_1 \cdots R_{n-1}$, *where each* R_j *is a (possibly empty) subword of* $\sigma^- \sigma^+$, *and all simple transpositions in* R_p *are also contained in* R_{p+1}, $\forall\, p < n-1$.

Theorem (T., 2019)

If $w \in S_n$, *then* w *is amenable if and only if* w *is vexillary.*

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Type C degeneracy loci

$E \to \mathfrak{X}$ is a symplectic bundle of rank $2n$ and $w \in W_n$.

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{2n} = E;$$
$$0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = E.$$

$E_{n+i} = E_{n-i}^{\perp}$ and $F_{n+i} = F_{n-i}^{\perp}$ for each $i \geq 0$.

$\mathfrak{X}_w := \{x \in \mathfrak{X} \mid \dim(E_r(x) \cap F_s(x)) \geq d'_w(r,s),$ for $r \in [1,n], s \in [1,2n]\}$.

Assume: $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $\mathfrak{X}$.

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Suppose that w is amenable. Then there exist sequences $\mathfrak{f} = (f_1 \geq \cdots \geq f_\ell \geq 0)$ and $\mathfrak{g} = (g_1 \leq \cdots \leq g_\ell)$ such that f_i (resp. $|g_i|$) is a right (resp. left) descent of w, for all $i \in [1, \ell]$, and we have

$$[\mathfrak{X}_w] = R^D \, c_\lambda(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}})$$

in $\mathrm{H}^*(\mathfrak{X})$.

The operator R^D and the shape λ

Fix $w \in W_n$ amenable, k and u as before.

$$D := \{(i,j) \mid i < j \text{ and } u_i + u_j < 0\}.$$

$$R^D := \prod_{i<j}(1 - R_{ij}) \prod_{(i,j)\in D}(1 + R_{ij})^{-1}.$$

Definition (T., 2017)

The shape of w is the partition $\lambda := \mu + \nu$, where $\mu := (-u_1, \ldots, -u_m)$ and ν is the partition with $\nu_j := \#\{i \mid \gamma_i \geq j\}$ for each $j \geq 1$.

Example

Let $w := (2,4,6,5,-1,-3) \in W_6$, a leading element.

We have $k = 3$, $\gamma = (2,2,3,2,1,0)$, $\mu = (3,1)$, $\nu = (5,4,1)$, $\lambda = (8,5,1)$,

$u = (-3,-1,5)$, $D = \{(1,2)\}$, $\mathfrak{f} = (5,4,3)$, $\mathfrak{g} = (-2,0,5)$.

The cohomology class $[\mathfrak{X}_w]$ is equal to

$$\frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\, c_{8,5,1}(E - E_{6-\mathfrak{f}} - F_{6+\mathfrak{g}}).$$

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