MULTIPLICITY CONJECTURES

CHRISTOPHER A. FRANCISCO AND HEMA SRINIVASAN

1. Introduction

Let $S = k[x_1, x_2, \dots, x_n]$ be the polynomial ring in n variables over a field k and $m = (x_1, \dots, x_n)$ the irrelevant maximal ideal. Let I be an ideal of S minimally generated by homogenous polynomials $f_1, f_2, \dots, f_t$ in S. Then S/I has homogenous or graded resolution $\mathbf{F}$ over S given by

$$0 \to \bigoplus_{j=1}^{b_n} S(-d_{nj}) \xrightarrow{\delta_n} \cdots \to \bigoplus_{j=1}^{b_i} S(-d_{ij}) \xrightarrow{\delta_i} \cdots \to \bigoplus_{j=1}^{b_1=t} S(-d_{1j}) \xrightarrow{\delta_1} S.$$

The numbers d_{ij} come from the degrees of the homogeneous polynomials in the maps in the resolution. Thus, the numbers d_{1j}, $j = 1, \dots, n$ are simply the degrees of the generators f_j, $j = 1, \dots, n$ of the ideal I. Much information about S/I can be recovered from the shifts in the resolution. For instance, the height of the ideal I is the smallest positive integer t such that $\sum_{i=1}^n \sum_j (-1)^i d_{ij}^t \neq 0$. Thus,

$$\sum_{i=1}^n \sum_j (-1)^i d_{ij}^t = \begin{cases} 0 & 1 \le t \le h-1 \\ (-1)^h h! e(S/I) & t = h. \end{cases}$$

Such a resolution $\mathbf{F}$ is called minimal if $\delta(\mathbf{F}) \subset m\mathbf{F}$; that is, $\delta_i(F_i) \subset mF_{i-1}$, for all i. Let $\mathbf{F}_{\min}$ be the minimal homogenous resolution with the corresponding shifts d_{ij}. Let $m_i = \min_{j \ge 0}\{d_{ij}\}$ be the minimal shifts, and $M_i = \max_{j \ge 0}\{d_{ij}\}$ be the maximal shifts in the resolution. Then conjectures of Huneke-Herzog-Srinivasan state:

Conjecture 1.1. *Let h be the height of I.*

(a) *Suppose that S/I is Cohen-Macaulay (and thus h is also the projective dimension of S/I). Then the multiplicity e of S/I satisfies*

$$\frac{1}{h!} \prod_{i=1}^h m_i \le e \le \frac{1}{h!} \prod_{i=1}^h M_i.$$

(b) (Herzog-Srinivasan) *Even if S/I is not Cohen-Macaulay, the multiplicity e of S/I satisfies*

$$e \le \frac{1}{h!} \prod_{i=1}^h M_i.$$

(We use "height" and "codimension" interchangeably throughout.) One can replace the maximal and minimal shifts by the ones coming from a nonminimal resolution. The bounds then will be weaker than the one from the minimal resolution. When I is a monomial ideal, there is a natural nonminimal resolution called the Taylor resolution for S/I. We will call the upper bound coming from this the

Taylor bound for the multiplicity. The minimal bound does not change. Let L_i denote the maximal shifts in the Taylor resolution of S/I. So, we have another conjecture:

Conjecture 1.2. (Herzog-Srinivasan) *Suppose that h = height of I. Then the multiplicity e of S/I satisfies*

$$e \leq \frac{1}{h!}\prod_{i=1}^{h} L_i.$$

In this article, we will give a brief history with outlines of proofs of the various results arising from a study of these conjectures. The simplest case in which Conjecture 1.1 is not yet known is for height three Cohen-Macaulay ideals.

A resolution $\mathbf{F}$ is called a pure resolution if there is only one shift in every degree. In other words, S/I has a pure resolution if $m_i = M_i$ for all i. When S/I is Cohen-Macaulay and has a pure resolution of length h, a formula of Huneke and Miller [Huneke-Miller] computes the multiplicity as the product of the shifts divided by $h!$. Conjecture 1.1(a) was born while revisiting this formula with a different proof. It is not that difficult to verify the conjecture for the codimension two case which gave the conjecturers some strength. As the article shows, these simple numerical bounds for the multiplicity have not been easy to settle and are still largely open.

2. Quasipure resolutions

Let R be a standard graded algebra and I a homogeneous ideal. Let the minimal homogeneous resolution $\mathbf{F}$ of R/I over R be given by

$$0 \to \bigoplus_{j=1}^{b_n} R(-d_{nj}) \xrightarrow{\delta_n} \cdots \to \bigoplus_{j=1}^{b_i} R(-d_{ij}) \xrightarrow{\delta_i} \cdots \to \bigoplus_{j=1}^{b_1=t} R(-d_{1j}) \xrightarrow{\delta_1} R.$$

Lemma 2.1. [Peskine-Szpiro] *Let R, I and $\mathbf{F}$ be as above. Then if h is the height of I,*

$$\begin{array}{rcll} \sum_{i=1}^{n} b_j(-1)^i d_{ij}^k & = & -1, & k=0 \\ & = & 0, & 1 \leq k < h \\ & = & (-1)^n h! e & k = h \end{array}$$

Proof. Since the complex $\mathbf{F}$ is exact, the alternating sum of the ranks of the free modules must be zero. So, we get $\sum_{i=1}^{n} b_j(-1)^i d_{ij} k + 1 = 0$.

The Hilbert series of R/I is given by $\sum_{i=1}^{n} b_{ij}(-1)^i x^{d_{ij}} = (1-x)^h P(x)$, where $P(x)$ is the Hilbert Polynomial. By differentiating and evaluating at $x = 1$, we get,

$$\begin{array}{rcll} \sum_{i=1}^{n} b_j(-1)^i \binom{d_{ij}}{t} & = & 0, & 1 \leq t < h \\ & = & (-1)^n h! e & t = h \end{array}$$

Using the fact that $t!\binom{d_{ij}}{t}$ is a polynomial of degree t in d_{ij} and hence the d_{ij}^t can be written as sums of multiples of $\binom{d_{ij}}{k}, k < t$, we get the result. □

Recall that a resolution $\mathbf{F}$ is called a pure resolution if there is only one shift in every degree. Thus, R/I has a pure resolution if $m_i = M_i$ for all i. We will call a resolution *quasipure* if $m_i \geq M_{i-1}$ for all i. Now we will use the above equations to prove:

Theorem 2.2. [Herzog-Srinivasan1998] *Let R be a standard graded ring and I be a homogeneous ideal of R such that R/I is Cohen-Macaulay with a quasipure resolution. Let $M_i, 1 \leq i \leq n$, and $m_i, 1 \leq i \leq n$, be the maximal and minimal shifts respectively in the minimal homogeneous resolution of R/I. Then the multiplicity $e(R/I)$ satisfies $\prod m_i \leq n!e \leq \prod M_i$.*

Proof. Let $\mathbf{F}$, given by

$$0 \to \bigoplus_{j=1}^{b_n} R(-d_{nj}) \xrightarrow{\delta_n} \cdots \to \bigoplus_{j=1}^{b_i} R(-d_{ij}) \xrightarrow{\delta_i} \cdots \to \bigoplus_{j=1}^{b_1=t} R(-d_{1j}) \xrightarrow{\delta_1} R,$$

be the minimal resolution of R/I over R. Since R/I is Cohen-Macaulay, n is the height of I. Let m_i and M_i denote the minimal and maximal shifts respectively at the ith place in the resolution. By Peskine and Szpiro [Peskine-Szpiro], we see that

$$\begin{array}{rcll} \sum_{i=1}^n b_j(-1)^i d_{ij}^k & = & -1, & k=0 \\ & = & 0, & 1 \leq k < n \\ & = & (-1)^n h!e & k=n. \end{array}$$

Now consider the $n \times n$ matrix

$$T(n) = \begin{bmatrix} \sum_j d_{1j} & \cdots & \sum_j d_{ij} & \cdots & \sum_j d_{nj} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^k & \cdots & \sum_j d_{ij}^k & \cdots & \sum_j d_{nj}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^n & \cdots & \sum_j d_{ij}^n & \cdots & \sum_j d_{nj}^n \end{bmatrix}$$

We compute the determinant of $T(n)$ in two different ways. First, we perform the column operations of adding $(-1)^{n+i}$ times the ith column to the last column, for each i. Then using $\sum_{i=1}^n b_j(-1)^i d_{ij}^k = 0, 1 \leq k < h$, we get the determinant of T,

$$\det(T) = \det \begin{bmatrix} \sum_j d_{1j} & \cdots & \sum_j d_{ij} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^k & \cdots & \sum_j d_{ij}^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^n & \cdots & \sum_j d_{ij}^n & \cdots & n!e \end{bmatrix} = n!e \det Q,$$

where

$$Q = \begin{bmatrix} \sum_j d_{1j} & \cdots & \sum_j d_{ij} & \cdots & \sum_j d_{n-1,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^k & \cdots & \sum_j d_{ij}^k & \cdots & \sum_j d_{n-1,j}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^{n-1} & \cdots & \sum_j d_{ij}^{n-1} & \cdots & \sum_j d_{n-1,j}^{n-1} \end{bmatrix}.$$

On the other hand, since the resolution is quasipure, for any i, j,

$$m_i \leq d_{ij} \leq M_i \leq m_{i+1} \leq d_{i+1,j} \leq M_{i+1}.$$

If $V(a_1, \ldots, a_n)$ denotes the Vandermonde matrix

$$\begin{bmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^k & \cdots & a_i^k & \cdots & a_n^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1^n & \cdots & a_i^n & \cdots & a_n^n \end{bmatrix}$$

then $V(d_{1j_1}, \ldots, d_{nj_n})$ is positive for any $j_1, \ldots, j_n$. Hence decomposing the determinant T by Vandermonde determinants, we get

$$\det(T(n)) = \sum_{j_i < b_i} d_{1j_1} .. \cdots d_{nj_jn} V(d_{1jj_1}, \ldots, d_{nj_n}).$$

Now since the resolution is quasipure, $d_{ij_i} \geq d_{tj_t}, i > t$. So, all the Vandermonde determinants $V(d_{1j_1}, \ldots, d_{nj_n}) \geq 0$. Hence estimating d_{ij}, and summing up the Vandermonde determinants again, we get

$$\prod_{i=1}^{n} m_i \det L \leq \det T \leq \prod_{i=1}^{n} M_i \det L,$$

where

$$L = \begin{bmatrix} b_1 & \cdots & b_j & \cdots & b_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_j d_{1j}^k & \cdots & \sum_j d_{ij}^k & \cdots & \sum_j d_{ij}^k \\ \vdots & \vdots & \vdots & \vdots & \\ \sum_j d_{1j}^{n-1} & \cdots & \sum_j d_{ij}^{n-1} & \cdots & \sum_j d_{nj}^{n-1} \end{bmatrix}.$$

But the same column operations we performed on T, we can repeat now on L with the result that the last column becomes $\begin{bmatrix} (-1)^{n+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Hence the determinant of L is $\det L = (-1)^{2n+2} \det Q$. Putting these two together, we get the desired inequality, because $\det Q$, whatever it is, is certainly not zero for otherwise, the determinant of T would be zero, which is impossible. $\square$

As a corollary to this proof, we get a quick proof of the Huneke-Miller formula. The matrix Q in this case is a Vandermonde matrix once we divide each column by the corresponding Betti number b_i. Thus

$$\det Q = \prod_i d_i \prod_i b_i V(d_1, \ldots, d_n) = \prod_i d_i \prod_{i=1}^{n-1} b_i d_i V(d_1, \ldots, d_{n-1}) =$$

$$n!e \prod_{i=1}^{n-1} b_i \prod_{i=1}^{n-1} d_i V(d_1, \ldots, d_{n-1}).$$

Since $d_1, \ldots, d_n$ are all distinct and increasing, we see that $n!e = \prod_i d_i$.

The above proof also shows that it is necessary to have the Cohen-Macaulay hypothesis to get the lower bound. In the best possible case, let us say, we have an ideal of height n with a pure resolution of length $n+1$ with the shifts, $d_1 <$

$d_2 < \cdots < d_{n+1}$. Thus it fails to be Cohen-Macaulay only by having the length of its resolution just one more than its height, and it has a pure resolution. We follow the same proof as above and add one more row to Q by adding the d_i^0 as the first row; we will have an $(n+1) \times (n+1)$ essentially Vandermonde matrix. By taking its determinant in two ways, we get $n!e = \prod_{i=1}^{n} d_i - \prod_{i=1}^{n}(d_{n+1} - d_i)$. This is strictly less than the product of the shifts d_i. The upper bound, however, is clearly true since we get $n!e \leq \prod_{i=1}^{n} d_i$. Hence when R/I is not Cohen-Macaulay, the upper bound conjecture states that if R/I has height h and multiplicity e, then $h!e \leq \prod_{i=1}^{h} M_i$.

This upper bound has been proven for ideals with a q-linear resolution, which is a pure resolution, even when the ideals are not Cohen-Macaulay. The multiplicity of $e(R/I)$ and the equations are unaltered if we did not change the Hilbert function. Using this notion, and modifying the above proof, it can be shown that the conjectured bounds hold for algebras with almost pure resolution [Herzog-Srinivasan1998].

3. Low codimension

In this section, we survey what is known about Conjecture 1.1 in codimension two and three. The low codimension gives us some extra structure that allows us to write down minimal resolutions more explicitly. Our goal in this section is to describe what cases are known, give a few examples of those cases, and briefly survey the different methods of proof. Throughout, let $S = k[x_1, \ldots, x_n]$.

3.1. **Codimension two, Cohen-Macaulay.**

We begin with the Cohen-Macaulay codimension two case, where the Hilbert-Burch Theorem tells exactly what the minimal graded free resolution looks like. Suppose $I \subset S$ is a homogeneous ideal of codimension two such that S/I is Cohen-Macaulay. The minimal graded free resolution of S/I has the form

$$0 \to \bigoplus_{i=1}^{t-1} S(-b_i) \xrightarrow{\phi} \bigoplus_{i=1}^{t} S(-a_i) \to S \to S/I \to 0.$$

See, for example, [Eisenbud]. We assume that the a_i, the degrees of minimal generators of I, and the b_i, the degrees of minimal first syzygies on those generators, are weakly increasing, so $a_1 \leq \cdots \leq a_t$ and $b_1 \leq \cdots \leq b_{t-1}$. In terms of the a_i and b_i, we have $m_1 = a_1$, $M_1 = a_t$, $m_2 = b_1$, and $M_2 = b_{t-1}$.

To prove Conjecture 1.1 in this case, it is useful to have a way to express the m_i, M_i, and multiplicity in terms of the degrees of the syzygy matrix ϕ. For all i and j (where this formula makes sense), let $u_{ij} = b_i - a_j$. Then the (transpose of the) matrix (u_{ij}) gives the degrees of the entries of the syzygy matrix ϕ, where we allow zero to have any degree. Let $u_i = u_{ii}$, and let $v_i = u_{i,i+1}$. We have the following formulas from [Herzog-Trung-Valla].

Lemma 3.1. (Herzog-Trung-Valla) *With the notation above,*

(1) $a_1 = v_1 + \cdots + v_{t-1}$ *and* $a_t = u_1 + \cdots + u_{t-1}$.
(2) $b_1 = v_1 + \cdots + v_{t-1} + u_1$ *and* $b_{t-1} = u_1 + \cdots + u_{t-1} + v_{t-1}$.
(3) $e(S/I) = \displaystyle\sum_{i=1}^{t-1} u_i(v_i + \cdots + v_{t-1})$.

Example 3.2. Let $I = (a^3, a^2b, b^4) \subset R = k[a,b]$. The minimal graded free resolution of R/I is:

$$0 \to R(-4) \oplus R(-6) \xrightarrow{\begin{pmatrix} -b & 0 \\ a & -b^3 \\ 0 & a^2 \end{pmatrix}} R(-3)^2 \oplus R(-4) \xrightarrow{\begin{pmatrix} a^3 & a^2b & b^4 \end{pmatrix}} R \to R/I \to 0$$

In this case, we have $a_1 = a_2 = 3$, $a_3 = 4$, $b_1 = 4$, and $b_2 = 6$. Therefore $u_1 = b_1 - a_1 = 1$, $u_2 = b_2 - a_2 = 3$, $v_1 = b_1 - a_2 = 1$, and $v_2 = b_2 - a_3 = 2$.

The formulas of Lemma 3.1 are easy to verify: $a_1 = 3 = 1 + 2$, $a_3 = 4 = 1 + 3$, $b_1 = 4 = 1 + 2 + 1$, and $b_2 = 6 = 1 + 3 + 2$. Note that the Hilbert function of R/I is $(1, 2, 3, 2, 1)$, so $e(R/I) = 9$. The formula from Lemma 3.1 for the multiplicity gives

$$e(R/I) = u_1(v_1 + v_2) + u_2v_2 = 1(1 + 2) + 3(2) = 3 + 6 = 9.$$

Because

$$6 = \frac{1}{2}(3 \cdot 4) \leq e(R/I) = 9 \leq \frac{1}{2}(4 \cdot 6) = 12,$$

R/I satisfies Conjecture 1.1.

Making some careful computations using these formulas, Herzog and Srinivasan prove the following result in [Herzog-Srinivasan1998].

Theorem 3.3. (Herzog-Srinivasan) *Let $I \subset S = k[x_1, \dots, x_n]$ be an ideal of codimension two such that S/I is Cohen-Macaulay. Then S/I satisfies the bounds of Conjecture 1.1; that is, in the notation above,*

$$\frac{1}{2}a_1b_1 \leq e(R/I) \leq \frac{1}{2}a_tb_{t-1}.$$

The bound of Theorem 3.3 is sharp in the sense that if S/I has a pure resolution, the multiplicity is equal to the conjectured bounds. In Example 3.2, however, there is some room between the multiplicity and the bounds. Recently, Migliore, Nagel, and Römer proved a stronger version of Theorem 3.3 in [Migliore-Nagel-Römer]. Their argument is a refinement of the analysis in the proof of Theorem 3.3 in [Herzog-Srinivasan1998].

Theorem 3.4. (Migliore-Nagel-Römer) *Let $I \subset S = k[x_1, \dots, x_n]$ be an ideal of codimension two such that S/I is Cohen-Macaulay. Then*

$$e(S/I) \geq \frac{1}{2}m_1m_2 + \frac{1}{2}(M_2 - M_1)(M_2 - m_2 + M_1 - m_1)$$

and

$$e(S/I) \leq \frac{1}{2}M_1M_2 - \frac{1}{2}(m_2 - m_1)(M_2 - m_2 + M_1 - m_1).$$

In Example 3.2, Theorem 3.4 provides stronger bounds for the multiplicity. The new lower bound of 9 is sharp in this case, and the new upper bound is 10.5, which is better than the previous bound of 12.

We have an easy corollary to Theorem3.4 that says that the bounds of Theorem 3.3 are sharp if and only if S/I has a pure resolution.

Corollary 3.5. (Migliore-Nagel-Römer) *Let I be a homogeneous ideal of codimension two in $S = k[x_1, \dots, x_n]$ such that S/I is Cohen-Macaulay. Then the following are equivalent:*

(1) S/I has a pure resolution.

(2) $e(S/I) = \frac{1}{2}m_1m_2$.
(3) $e(S/I) = \frac{1}{2}M_1M_2$.

Migliore, Nagel, and Römer also give a second proof of Theorem 3.4 in their paper [Migliore-Nagel-Römer]. They analyze the degree matrix in the minimal graded free resolution of S/I, and, using linkage and an induction argument on the number of minimal generators of I, arrive at the same result. The proof is similar to their argument in the Gorenstein codimension three case we shall discuss later in this section. See Section 2 of [Migliore-Nagel-Römer] for the details and a discussion of some ideas for an appropriate lower bound in the non-Cohen-Macaulay codimension two case.

3.2. **Codimension two, not Cohen-Macaulay.** We turn next to the case in which codim $I = 2$ but S/I is not Cohen-Macaulay. Of course, we cannot expect the lower bound of Conjecture 1.1 to hold in the non-Cohen-Macaulay case, so we consider only the upper bound. The first development on this problem came in work of Gold [Gold] in which she solved the codimension two lattice ideal case. We give a brief overview of her work here.

Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$. Given a vector $\mathbf{b} = (b_1, \ldots, b_n)$ with nonnegative integer entries, let $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$. A **lattice ideal** is an ideal

$$I_{\mathcal{L}} = (x^{\mathbf{a}_+} - x^{\mathbf{a}_-} \mid \mathbf{a} \in \mathcal{L}),$$

where $\mathbf{a}_+$ is a_i in coordinate i if $a_i > 0$ and zero otherwise, and $\mathbf{a}_-$ is $|a_i|$ in component i if $a_i < 0$ and zero otherwise.

Peeva and Sturmfels constructed a minimal (multigraded) free resolution for $I_{\mathcal{L}}$ when its codimension is two [Peeva-Sturmfels], and Gold exploits this structure to prove the upper bound of Conjecture 1.1 in this case. Essentially, Peeva and Sturmfels show that for each multidegree in which there is a minimal syzygy in the resolution, there exists a particular polytope. Moreover, each first syzygy corresponds to a line segment, each second syzygy corresponds to a triangle, and each third syzygy corresponds to a quadrangle, so the progression is systematic.

To prove the upper bound of Conjecture 1.1 for $I_{\mathcal{L}}$, Gold first proves it for certain four-generated subideals J of $I_{\mathcal{L}}$. She does a careful analysis of equalities derived from the fact that the syzygies in the resolution are homogeneous, which gives relations on the exponents of the terms of the minimal generators. One wants to show that the expression $M_1^{S/J}M_2^{S/J} - 2e(S/J)$ is nonnegative, and there are four cases to consider depending on which syzygies have maximal degree. Gold illustrates one in [Gold], and the others are similar.

Passing from the result for S/J to the bound for $S/I_{\mathcal{L}}$ is then easy. Because the minimal generators of J are also minimal generators of $I_{\mathcal{L}}$, it is obvious that $M_1^{S/J} \leq M_1^{S/I_{\mathcal{L}}}$. Additionally, Peeva and Sturmfels show that when $S/I_{\mathcal{L}}$ is not Cohen-Macaulay, the minimal resolution of $S/I_{\mathcal{L}}$ is comprised of a sum of resolutions of ideals S/J. Hence minimal syzygies in the resolution of S/J are minimal syzygies in the resolution of $S/I_{\mathcal{L}}$, and thus $M_2^{S/J} \leq M_2^{S/I_{\mathcal{L}}}$. Because $J \subset I_{\mathcal{L}}$, $e(S/I_{\mathcal{L}}) \leq e(S/J)$, the result follows.

Theorem 3.6. (Gold) *Let $I_{\mathcal{L}} \subset S = k[x_1, \ldots, x_n]$ be a codimension two lattice ideal. Then*

$$e(S/I_{\mathcal{L}}) \leq \frac{1}{2}M_1M_2.$$

Except for some cases discussed in Section 4, Gold's result was the only case of Conjecture 1.1 known in the non-Cohen-Macaulay case when she proved it. In 2003, however, Römer proved the upper bound of Conjecture 1.1 for all codimension two ideals I in which S/I is not Cohen-Macaulay, which completed the codimension two case. Römer's approach is to reduce the problem to dimension zero using the idea of almost regular elements. Let $R = S/I$. An element $y \in R_1$ is called **almost regular** if $(0 :_R y)_d = 0$ for all $d \gg 0$. Similarly, a sequence of elements $y_1, \ldots, y_s \in R_1$ is an **almost regular sequence** if y_i is almost regular for $R/(y_1, \ldots, y_{i-1})R$ for each i. It is easy to prove the following:

Lemma 3.7. *Let I be a homogeneous ideal of codimension two in S, and let $R = S/I$.*

(1) If $y_1, \ldots, y_{n-2} \in R_1$ is an almost regular sequence, then $R/(y_1, \ldots, y_{n-2})R$ has dimension zero.

(2) $e(R) \leq e(R/(y_1, \ldots, y_{n-2})R)$.

Let $\tilde{R} = R/(y_1, \ldots, y_{n-2})R$. Then, by Lemma 3.7, we have $\dim \tilde{R} = 0$ and $e(R) \leq e(\tilde{R})$. Since $\dim \tilde{R} = 0$, $\tilde{R}$ is Cohen-Macaulay. Moreover, $\tilde{R}$ is the polynomial ring $\tilde{S} = S/(y_1, \ldots, y_n)S$ mod the ideal $\tilde{I} = (I + (y_1, \ldots, y_n))/(y_1, \ldots, y_n)$. Consequently, because the upper bound holds for codimension two ideals in the Cohen-Macaulay case, $e(\tilde{R}) \leq \frac{1}{2}\tilde{M}_1\tilde{M}_2$, where the $\tilde{M}_i$ are the maximum degrees of minimal syzygies of $\tilde{S}/\tilde{I}$ over $\tilde{S}$.

Thus, if M_i represent the maximum degrees of minimal syzygies of S/I, it is enough to show that $\tilde{M}_i \leq M_i$ for $i = 1, 2$. If so, then

$$e(R) \leq e(\tilde{R}) \leq \frac{1}{2}\tilde{M}_1\tilde{M}_2 \leq \frac{1}{2}M_1M_2.$$

The inequality $\tilde{M}_1 \leq M_1$ is easy to see. To prove $\tilde{M}_2 \leq M_2$, Römer gives a clever argument analyzing some long exact sequences in Koszul homology. See Section 2 of [Römer] for the details.

In summary, we have the following theorem.

Theorem 3.8. (Herzog-Srinivasan, Gold, Römer) *Let I be a homogeneous ideal of codimension two in $S = k[x_1, \ldots, x_n]$. Then*

$$e(S/I) \leq \frac{1}{2}M_1M_2.$$

3.3. **Gorenstein codimension three.** In higher codimension, we have less structure to assist us. The conjectures are known for ideals that are either a complete intersection [Herzog-Srinivasan1998] or a powers of a complete intersection [Guardo-Van Tuyl], but Conjecture 1.1 is open even in the Cohen-Macaulay codimension three case. When S/I is Gorenstein, with codim $I = 3$, however, the bounds are known to hold. Even with the Buchsbaum-Eisenbud structure theorems, this case was partially open for a number of years before Migliore, Nagel, and Römer solved the lower bound in 2004.

Buchsbaum and Eisenbud proved a structure theorem for Gorenstein codimension three ideals in [Buchsbaum-Eisenbud]. Their result shows that these ideals are pfaffians of skew-symmetric matrices, and they have minimal graded free resolutions with a large amount of symmetry. Throughout the rest of the section, unless otherwise noted, let $I \subset S = k[x_1, \ldots, x_n]$ be a Gorenstein ideal of codimension three. Suppose its minimal generators have degrees $a_1 \leq \cdots \leq a_{2r+1}$; note that the Buchsbaum-Eisenbud result guarantees that I has an odd number of minimal generators. Then S/I has a minimal graded free resolution of the following form:

$$0 \to S(-c) \to \bigoplus_{i=1}^{2r+1} S(-(c-a_i)) \to \bigoplus_{i=1}^{2r+1} S(-a_i) \to S \to S/I \to 0$$

We can restate Conjecture 1.1 for the Gorenstein codimension three case using this notation:

$$\frac{1}{6}a_1(c-a_{2r+1})(c) \le e(S/I) \le \frac{1}{6}a_{2r+1}(c-a_1)(c)$$

First, we discuss Herzog and Srinivasan's proof of the upper bound in this case. A computation, using a result from [Peskine-Szpiro] relating the degrees in the shifts of the resolution of S/I to $e(S/I)$, yields the following lemma.

Lemma 3.9. (Herzog-Srinivasan) *Let $I \subset S$ be a Gorenstein codimension three ideal minimally generated by $2r+1$ elements. Then*

$$e(S/I) = \frac{1}{6}\sum_{i=1}^{2r+1} a_i(c-a_i)(c-2a_i).$$

Note that if $c \ge 2a_{2r+1}$, both bounds of Conjecture 1.1 follow immediately [Herzog-Srinivasan1998]. If not, we need more calculations. By looking at the sums of the degrees of the generators of the free modules at each step in the resolution, it is easy to see that

$$\sum_{i=1}^{2r+1} a_i = rc;$$

additionally, $c \ge a_i + a_{(2r+1)+2-i}$ for $i \ge 2$. After some computations using these facts, Herzog and Srinivasan prove the upper bound.

Theorem 3.10. (Herzog-Srinivasan) *Let $I \subset S$ be a Gorenstein codimension three ideal. Then S/I satisfies the upper bound of Conjecture 1.1; in the notation above,*

$$e(S/I) \le \frac{1}{6}a_{2r+1}(c-a_1)c.$$

Example 3.11. Let $R = k[a,b,c]$. Let F be an ideal generated by three random polynomials of degrees three, four, and five. (To produce a random polynomial of degree d in the ring R in Macaulay 2, for example, use the command `random(d,R)`.) Let g be a random polynomial of degree four, and let $I = F : g$. Then I is a Gorenstein codimension three ideal, and S/I has minimal graded free resolution

$$0 \to R(-8) \to R(-4) \oplus R(-5)^4 \oplus \to R(-3)^4 \oplus R(-4) \to R \to R/I \to 0.$$

The upper bound is

$$a_{2r+1}(c-a_1)c = \frac{1}{6}(4 \cdot 5 \cdot 8) = \frac{80}{3}.$$

Computing the multiplicity with Lemma 3.9, we have

$$e(R/I) = \frac{1}{6}\sum_{i=1}^{5} a_i(c-a_i)(c-2a_i) = \frac{1}{6}[4(3 \cdot 5 \cdot 2) + 4 \cdot 4 \cdot 0] = \frac{120}{6} = 20 < \frac{80}{3}.$$

In [Herzog-Srinivasan1998], Herzog and Srinivasan prove the lower bound in the codimension three Gorenstein case in three situations. First, they show it in the case that $c \geq 2a_{2r+1}$ as discussed above. With arguments similar to the ideas used in the proof of Theorem 3.10, Herzog and Srinivasan prove the lower bound when I has five generators. (Of course, when I has three generators, it is a complete intersection, so this is the next logical case to consider.) Finally, using a structure theorem for *monomial* Gorenstein ideals of codimension three from [Bruns-Herzog], they prove the lower bound when I is a Gorenstein ideal of codimension three, minimally generated by $2r+1$ monomials, with $a_1 = r$.

The lower bound of Conjecture 1.1 for arbitrary Gorenstein codimension three ideals remained open until 2004, when Migliore, Nagel, and Römer found a proof [Migliore-Nagel-Römer]. As in the codimension two Cohen-Macaulay case, Migliore, Nagel, and Römer proved a stronger version of Conjecture 1.1.

Theorem 3.12. (Migliore-Nagel-Römer) *Let $I \subset S = k[x_1, \dots, x_n]$ be a Gorenstein ideal of codimension three. Then*

$$e(S/I) \geq \frac{1}{6}m_1m_2m_3 + \frac{1}{3}(M_3 - M_2)(M_2 - m_2 + M_1 - m_1)$$

and

$$e(S/I) \leq \frac{1}{6}M_1M_2M_3 - \frac{1}{12}M_3(M_2 - m_2 + M_1 - m_1).$$

We get a corollary similar to the result in the codimension two Cohen-Macaulay case.

Corollary 3.13. (Migliore-Nagel-Römer) *Let I be a homogeneous, Gorenstein ideal of codimension three in $S = k[x_1, \dots, x_n]$. Then the following are equivalent:*

(1) S/I has a pure resolution.

(2) $e(S/I) = \frac{1}{6}m_1m_2m_3$.

(3) $e(S/I) = \frac{1}{6}M_1M_2M_3$.

Example 3.11 continued: Let R and I be as above in Example 3.11. We already computed that $e(R/I) = 20$. The upper bound of Conjecture 1.1 is $\frac{80}{3}$, and the lower bound is $\frac{1}{6}(3 \cdot 4 \cdot 8) = 16$. Thus R/I satisfies both bounds of Conjecture 1.1, but there is some slack here.

The bounds of Theorem 3.12 are tighter. Using those bounds, we have

$$16 + \frac{1}{3}(8-5)(5-4+4-3) = 18 \leq e(R/I) = 20 \leq \frac{76}{3} = \frac{80}{3} - \frac{1}{12}(8)(5-4+4-3).$$

We give a brief overview of the main ideas in the proof of Theorem 3.12. The argument proceeds by induction on the size of the square matrix in the Buchsbaum-Eisenbud resolution of the Gorenstein codimension three ideal, which we shall call the Buchsbaum-Eisenbud matrix. Suppose I has $2r+1$ minimal generators. The base case is when S/I is a complete intersection, and $r = 1$. Say the minimal generators have degrees $m_1 \leq y \leq M_1$. Since S/I is a complete intersection, we have immediately that $m_2 = m_1 + y$, $M_2 = y + M_1$, and $m_3 = M_3 = m_1 + y + M_1$. The bounds then follow from some straightforward computations; see the beginning of the proof of Theorem 1.4 in [Migliore-Nagel-Römer].

Now we assume that the bounds hold for some $r \geq 1$ and try to prove them for $r+1$. Start with a Gorenstein codimension three ideal I' with Buchsbaum-Eisenbud matrix of size $(2r+3) \times (2r+3)$. Migliore, Nagel, and Römer show that the matrix can be put in a convenient form so that the resulting degree matrix is symmetric about the diagonal opposite the main diagonal (that is, about the upward sloping diagonal of maximal length). Let I be the Gorenstein codimension three ideal that comes from removing the top and bottom rows and leftmost and rightmost columns of the Buchsbaum-Eisenbud matrix of I'. Then the bounds hold for I by induction.

There is a relatively simple relation between the multiplicities of R/I and R/I'. Using results from [Geramita-Migliore], Migliore, Nagel, and Römer compute this relation by considering some ideals of smaller codimension. The idea is to build two Gorenstein codimension three ideals with the same graded Betti numbers as I and I' in a particularly nice way. Since Conjecture 1.1 depends only on the graded Betti numbers, we can work with the new ideals instead. There exists a Cohen-Macaulay codimension two ideal J with resolution similar to I: Namely, if I has generators of degrees $a_1 \leq \cdots \leq a_{2r+1}$ and first syzygies of degrees $b_1 \leq \cdots \leq b_{2r+1}$, then S/J has minimal graded free resolution

$$0 \to \bigoplus_{i=1}^{r} S(-b_i) \to \bigoplus_{i=1}^{r+1} S(-a_i) \to S \to S/J \to 0.$$

Moreover, there is an ideal $\tilde{J}$ that is geometrically linked to J such that $J + \tilde{J}$ has the same graded Betti numbers (and hence multiplicity) as I. Similarly, there is a Cohen-Macaulay codimension two ideal J' corresponding to I'. Migliore, Nagel, and Römer find a relationship between $e(R/(J + \tilde{J})) = e(R/I)$ and $e(R/J)$ first, reducing to the case in which J and J' define curves. They can then relate the genera of J and J' and $e(R/J)$, which is enough to get a useful equation comparing $e(R/I)$ and $e(R/I')$. The bounds then follow from some long computations; see [Migliore-Nagel-Römer] for the details.

Remark 3.14. To summarize the results of this section, suppose that $I \subset S = k[x_1, \ldots, x_n]$ is a homogeneous ideal. If codim $I = 2$, or if S/I is Gorenstein of codimension three, then S/I satisfies Conjecture 1.1. Moreover, if codim $I = 2$ and S/I is Cohen-Macaulay, or if codim $I = 3$ and S/I is Gorenstein, then Theorem 3.4 gives stronger bounds for $e(S/I)$. In either of these two cases, $e(S/I)$ attains the lower or upper bound of Conjecture 1.1 if and only if S/I has a pure resolution. This provides a converse to Huneke and Miller's result in these cases. Finally, both bounds of Conjecture 1.1 are open even when codim $I = 3$ and S/I is Cohen-Macaulay.

4. Monomial ideals

It is natural to investigate Conjecture 1.1 in the case of monomial ideals since they are often easier with which to work than arbitrary ideals. While it is still too difficult to prove the conjectures for general monomial ideals, in some cases, we can exploit structure theorems on the resolutions of special kinds of monomial ideals. We investigate some of these ideals in this section.

4.1. **Stable ideals.** We begin by recalling the definition of a stable ideal. For a monomial u in $k[x_1, \ldots, x_n]$, let $\max(u)$ be the largest index of a variable dividing u. For example, $\max(x_2^3 x_4^8 x_6^2) = 6$. Eliahou and Kervaire made the following definition in [Eliahou-Kervaire].

Definition 4.1. Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$. We say that I is a **stable ideal** if for all monomials $u \in I$, $x_i u / x_{\max(u)} \in I$ for all $i \leq \max(u)$.

It suffices to check the condition for all monomials u in $G(I)$, the minimal monomial generating set of I. Consider, for instance, the ideal $J = (x_1^3, x_1^2x_2, x_1^2x_3^2)$. There is nothing to check with x_1^3. For the second generator, since $\max(x_1^2x_2) = 2$, we need only note that $x_1^3 \in J$. Finally, for the final generator, both $x_1^2x_2x_3$ and $x_1^3x_3$ are in J, so J is stable.

There are a number of reasons that stable ideals are interesting. Suppose $S = k[x_1, \dots, x_n]$ and that k is a field of characteristic zero. Then the generic initial ideal of any homogeneous ideal I, written $\mathrm{gin}(I)$, is a stable ideal (in fact, a strongly stable ideal). Taking the generic initial ideal is a useful tool that appears in a number of results and conjectures in commutative algebra and algebraic geometry. Additionally, lexicographic ideals, which play an important role in extremal results about Hilbert functions and graded Betti numbers (see, for example, Section 5), are stable ideals. Finally, stable ideals have convenient combinatorial properties that make them easy with which to work. Eliahou and Kervaire computed the minimal free resolution of an arbitrary stable ideal in [Eliahou-Kervaire], and from that resolution, one can easily compute the graded Betti numbers of a stable ideal.

Theorem 4.2. (Eliahou-Kervaire) *Let I be a stable ideal in $S = k[x_1, \dots, x_n]$. Then*

$$\beta_{i,i+j}^{S/I} = \sum_{\substack{u \in G(I) \\ \deg u = j+1}} \binom{\max(u) - 1}{i - 1}.$$

We use this formula to explore Herzog and Srinivasan's work on Conjecture 1.1 in [Herzog-Srinivasan1998] in the stable ideal case. Our approach is to present the upper bound in the Cohen-Macaulay case in detail and then sketch the other (more involved) cases. Assume first that I is stable, and S/I is Cohen-Macaulay.

Theorem 4.3. (Herzog-Srinivasan) *Let I be a stable ideal of codimension c, and let S/I be Cohen-Macaulay. Then*

$$\frac{1}{c!} \prod_{i=1}^{c} m_i \le e(S/I) \le \frac{1}{c!} \prod_{i=1}^{c} M_i.$$

We can reduce to the case in which S/I has depth zero, and since we suppose S/I is Cohen-Macaulay, we have that S/I is Artinian and $c = n$. One can see the technique Herzog and Srinivasan use to prove the upper bound in an example.

Example 4.4. Let

$$I = (a^3, a^2b, a^2c, ab^2, abc^2, ac^3, b^5, b^4c, b^3c^2, b^2c^3, bc^4, c^5) \subset R = k[a, b, c].$$

Then I is a stable ideal, and R/I is Artinian. To display the graded Betti numbers of R/I, we use the notation of the computer algebra system Macaulay 2 [Grayson-Stillman]. The rows and columns of the diagram below are numbered beginning with zero, and one finds β_{ij} in column i and row $j - i$. The graded Betti diagram of R/I is:

R/I:	total:	1	12	19	8
	0:	1	.	.	.
	1:	.	.	.	.
	2:	.	4	4	1
	3:	.	2	4	2
	4:	.	6	11	5

The Betti diagram is a particularly nice way to visualize where the maximum degree syzygies occur at each step in the resolution, which is why we use it in this case. Here, the highest degree of a syzygy at each step occurs in row four, so $M_1 = 5$, $M_2 = 6$, and $M_3 = 7$. Because

$$e(R/I) = 21 \leq \frac{1}{3!}(5 \cdot 6 \cdot 7) = 35,$$

R/I satisfies the upper bound of Conjecture 1.1.

The form of the resolution in Example 4.4 is no accident. Suppose I is stable, S/I is Artinian, and the maximum degree of a minimal generator of I is d. Then $M_1 = d$, $M_2 = d+1$, and in general, $M_i = d+i-1$ for $1 \leq i \leq n$. An easy computation shows that

$$e(S/I) \leq e(S/(x_1, \ldots, x_n)^d) = \frac{1}{d!}\prod_{i=1}^{n} M_i,$$

which proves the upper bound.

The lower bound is a bit trickier, and we only sketch the main idea here. It follows immediately from Theorem 4.2 that for each i, $m_i = \min\{\deg(u) : u \in G(I) \mid \max(u) = i\} + i - 1$. Additionally, we have the short exact sequence

$$0 \to S/(I : x_n) \to S/I \to S/(I, x_n) \to 0.$$

Because we are assuming that S/I has finite length, it follows that the multiplicities of the modules in the short exact sequence are equal to their lengths. A careful comparison of the m_i for S/I, $S/(I, x_n)$ (thought of as a module over $k[x_1, \ldots, x_{n-1}]$), and $S/(I : x_n)$ along with inducting on the length gives the lower bound of Theorem 4.3. See [Herzog-Srinivasan1998] for the details.

Next, we consider the case in which I is stable, but S/I is not Cohen-Macaulay. Initially, note that the lower bound may fail if S/I is not Cohen-Macaulay even when I is stable.

Example 4.5. Let $I = (a^2, ab) \subset R = k[a, b]$. Clearly, I is stable, but R/I is not Cohen-Macaulay. The minimal graded free resolution of R/I is

$$0 \to R(-3) \to R(-2)^2 \to R \to R/I \to 0,$$

and $e(R/I) = 1$. But $\frac{1}{1!}(2) \not\leq 1$.

Thus we shall consider only the upper bound in the non-Cohen-Macaulay case. Herzog and Srinivasan prove the following result.

Theorem 4.6. (Herzog-Srinivasan) *If $I \subset S$ is stable of codimension c, then*

$$e(S/I) \leq \frac{1}{c!}\prod_{i=1}^{c} M_i.$$

The proof is somewhat more complicated than in the upper bound Cohen-Macaulay case, and we omit it. Roughly, one uses the same short exact sequence as we used above in the lower bound case along with Noetherian induction.

4.2. **Squarefree strongly stable ideals.** Using similar methods, we can prove Conjecture 1.1 for the related class of squarefree strongly stable ideals. We recall the definition of these ideals.

Definition 4.7. Let I be a monomial ideal in $S = k[x_1, \ldots, x_n]$. We say that I is **squarefree strongly stable** if I is generated by squarefree monomials, and for each $u \in G(I)$, $x_i(u/x_j) \in I$ for all $i < j$ such that x_j divides u, and x_i does not divide u.

This is essentially the same as the condition for being strongly stable except that we only require the new monomial be in I when it is squarefree.

Example 4.8. Let $I = (x_1x_2, x_1x_3, x_2x_3, x_1x_4x_5) \subset R = k[x_1, \ldots, x_5]$. We check the squarefree strongly stable condition for the four minimal generators. The first two obviously meet the condition, and x_2x_3 does also since x_1x_2 and x_1x_3 are in I. Consider $x_1x_4x_5$. The squarefree monomials we can get via the shifting operation are $x_1x_3x_5$, $x_1x_2x_5$, $x_1x_3x_4$, $x_1x_2x_4$, and $x_1x_2x_3$. All of these are in I, and thus I is squarefree strongly stable.

Squarefree monomial ideals are of particular interest in combinatorics because they are the Stanley-Reisner ideals of simplicial complexes. Squarefree strongly stable ideals were introduced in [Aramova-Herzog-Hibi1998], and this class has convenient properties similar to those of stable ideals. In particular, an explicit minimal graded free resolution of such an ideal is known. Charalambous and Evans described the resolution for squarefree lexsegment ideals, a special type of squarefree strongly stable ideal, in [Charalambous-Evans], where they considered a more general class of ideals that they called lex-seg with holes ideals. Aramova, Herzog, and Hibi generalized this work to all squarefree strongly stable ideals in [Aramova-Herzog-Hibi1998], and Gasharov, Hibi, and Peeva proved an even more general result in [Gasharov-Hibi-Peeva].

Let $I \subset S$ be a squarefree strongly stable ideal. The minimal graded free resolution of S/I is a subcomplex of an Eliahou-Kervaire resolution, and the bases for the free modules F_i are straightforward to describe. F_i has as basis elements symbols $f(\sigma, u)$ such that: $u \in G(I)$, $\sigma \subset [n] = \{1, \ldots, n\}$, $|\sigma| = i-1$, $\max(\sigma) < \max(u)$, and if $i \in \sigma$, then x_i does not divide u. The degree of a basis element $f(\sigma, u)$ is $\deg u + i - 1$. Therefore, for u to give rise to a basis element $f(\sigma, u)$ of F_i, we must have $\max(u) - \deg u \geq i - 1$. Hence

$$m_i = \min\{\deg u : u \in G(I), \max(u) - \deg u \geq i-1\} + i - 1$$

and

$$M_i = \max\{\deg u : u \in G(I), \max(u) - \deg u \geq i-1\} + i - 1.$$

Thus we have a combinatorial description of the m_i and M_i for squarefree strongly stable ideals.

Example 4.9. As in Example 4.8, let $I = (x_1x_2, x_1x_3, x_2x_3, x_1x_4x_5) \subset R = k[x_1, \ldots, x_5]$. We compute m_2 and M_2 using the formulas above. We have

$$m_2 = \min\{\deg x_1x_3, \deg x_2x_3, \deg x_1x_4x_5\} + 2 - 1 = 2 + 2 - 1 = 3;$$

note we consider $\deg u$ only for those minimal generators u with $\max(u) - \deg u \geq 2 - 1 = 1$. For the maximum, we have

$$M_2 = \max\{\deg x_1x_3, \deg x_2x_3, \deg x_1x_4x_5\} + 2 - 1 = 3 + 2 - 1 = 4.$$

One can show that $m_1 = 2$, $m_3 = 5$, $M_1 = 3$, $M_3 = 5$, and $e(R/I) = 2$. Note also that codim $I = 2$. Since R/I is not Cohen-Macaulay, we do not necessarily expect the lower bound of Conjecture 1.1 to hold, and it does not. However,

$$e(R/I) = 2 < \frac{1}{2!}(3 \cdot 4) = 6,$$

so the upper bound does hold.

We have the following theorem from [Herzog-Srinivasan1998].

Theorem 4.10. (Herzog-Srinivasan) *Let $I \subset S = k[x_1, \ldots, x_n]$ be a squarefree strongly stable ideal of codimension c. If S/I is Cohen-Macaulay, then*

$$\frac{1}{c!}\prod_{i=1}^{c} m_i \leq e(S/I) \leq \frac{1}{c!}\prod_{i=1}^{c} M_i.$$

If S/I is not Cohen-Macaulay, then the upper bound still holds.

We sketch the idea of the proof of the upper bound in the Cohen-Macaulay case because it involves some interesting interplay between commutative algebra and combinatorics. For a monomial u, let $\min(u)$ be the minimum i such that x_i divides u. We reduce to the case in which $v = x_{n-d+1} \cdots x_n$ has maximal $\min(u)$ among all monomials $u \in G(I)$. As a result, all the squarefree monomials of degree d are in I.

Next, let $A = S/(I, x_1^2, \ldots, x_n^2)$. (This is sometimes called an indicator algebra.) A monomial is zero in this quotient if and only if it is either in I or divisible by a square. Therefore the only surviving monomials are those that correspond to faces of Δ, where I is the Stanley-Reisner ideal of Δ. Hence the Hilbert function of A gives the f-vector $(f_{-1}, f_0, \ldots,)$ of Δ, where f_i is the number of faces of Δ of dimension i. In particular, $\dim_k A_i = f_{i-1}$. Using the correspondence between the f_i and the h_i, the coefficients in the numerator of the rational function expression of the Hilbert series in lowest terms, it is not hard to show that the number of maximal faces of Δ is equal to the multiplicity of S/I, which is the sum of the h_i. See, for example, [Stanley].

Example 4.11. Let $I = (x_1x_2x_3) \subset R = k[x_1, x_2, x_3]$. This corresponds to the simplicial complex Δ consisting of the vertices v_1, v_2, and v_3, and the three line segments connecting the vertices, but not the interior of the triangle. The Hilbert series of R/I, in lowest terms, is

$$\frac{1 + t + t^2}{(1-t)^2},$$

so $e(R/I) = 1 + 1 + 1 = 3$. This is the same as the number of maximal faces (in this case, the three edges).

Since I contains all squarefree monomials of degree d, $A_d = 0$. Additionally, we have $A_{d-1} \neq 0$, for otherwise, $x_{n-d+2} \cdots x_n \in I$, which is impossible since v is a minimal generator of I. Thus $e(R/I) = \dim_k A_{d-1} \leq \binom{n}{d-1}$, the number of squarefree monomials in S of degree $d-1$. Now, using the formula for the M_i given above, we have $M_i = d + i - 1$ for $i = 1, \ldots, n-d+1$; note that $n - d + 1$ is the codimension of I (see [Herzog-Srinivasan1998] for a formula for the codimension).

Thus

$$e(S/I) \leq \binom{n}{d-1} = \frac{1}{(n-d+1)!} \prod_{i=1}^{n-d+1} M_i,$$

and the upper bound holds.

The proof of the non-Cohen-Macaulay upper bound case is almost exactly the same as for stable ideals, using the short exact sequence

$$0 \to S/(I : x_n) \to S/I \to S/(I, x_n) \to 0.$$

The lower bound requires a bit more computation; see the end of Section 4 of [Herzog-Srinivasan1998].

4.3. **Componentwise linear ideals.** We move now to a larger class of ideals that generalizes the previous examples in this section. Stable ideals and squarefree strongly stable ideals have a property known as componentwise linearity. For a homogeneous ideal I and a degree d, let $I_{<d>}$ be the ideal generated by the degree d elements of I. Herzog and Hibi made the following definition in [Herzog-Hibi].

Definition 4.12. Let I be a homogeneous ideal. We say that I is **componentwise linear** if $I_{<d>}$ has a d-linear resolution for each d.

Example 4.13. Let $I = (c^2, abc, a^2b^2) \subset R = k[a, b, c]$. Then I is not stable or squarefree strongly stable, but we claim I is componentwise linear. Clearly $I_{<2>}$ is 2-linear. Note that we have the 3-linear minimal graded free resolution

$$0 \to R(-5) \to R(-4)^4 \to R(-3)^4 \to R \to R/I_{<3>} \to 0.$$

Since the regularity of R/I is three, it follows that $R/I_{<d>}$ has a d-linear resolution for all d, and thus I is componentwise linear.

Componentwise linear ideals are especially important in combinatorics, providing a nice duality result. A theorem of Eagon and Reiner says that a Stanley-Reisner ideal I_Δ associated to a simplicial complex Δ has a linear resolution if and only if the Alexander dual Δ^* is Cohen-Macaulay. Herzog and Hibi and Herzog, Reiner, and Welker generalized this result by showing that I_Δ is componentwise linear if and only if Δ^* is sequentially Cohen-Macaulay, a less restrictive condition than Cohen-Macaulayness [Herzog-Hibi, Herzog-Reiner-Welker]. In addition, there is a good algebraic characterization of componentwise linear ideals as well [Aramova-Herzog-Hibi2000]:

Theorem 4.14. (Aramova-Herzog-Hibi) *Let I be a homogeneous ideal in $S = k[x_1, \dots, x_n]$, where char $k = 0$. Let $gin(I)$ be the reverse-lex generic initial ideal of J. Then I is componentwise linear if and only if*

$$\beta_{ij}(S/I) = \beta_{ij}(S/gin(I))$$

for all i and j.

In [Römer], Römer combined Theorem 4.14 and the results of Herzog and Srinivasan on stable ideals to prove the next result.

Theorem 4.15. (Römer) *Let I be a componentwise linear ideal of codimension c in $S = k[x_1, \dots, x_n]$, where k is a field of characteristic zero. Then*

$$e(S/I) \leq \frac{1}{c!} \prod_{i=1}^{c} M_i.$$

Proof. Because I is componentwise linear, $\beta_{ij}^{S/I} = \beta_{ij}^{S/\mathrm{gin}(I)}$ for all i and j. Taking the gin preserves the Hilbert function, and thus we have

$$e(S/I) = e(S/\mathrm{gin}(I)) \leq \prod_{i=1}^{c} M_i^{S/\mathrm{gin}(I)} = \prod_{i=1}^{c} M_i^{S/I},$$

where the inequality follows from Theorem 4.6. □

Remark 4.16. Under the hypotheses of Theorem 4.15, if we also assume that S/I is Cohen-Macaulay, the same argument gives the lower bound of Conjecture 1.1 for S/I (since $S/\mathrm{gin}(I)$ is also Cohen-Macaulay).

Knowing the bounds for componentwise linear ideals yields the conjecture for a number of interesting special cases. Some examples include ideals with a linear resolution (with the appropriate Cohen-Macaulay assumption), the **a**-stable ideals of Gasharov-Hibi-Peeva [Gasharov-Hibi-Peeva], and at most $n+1$ fat points in general position in $\mathbb{P}^n$ [Francisco2004b].

4.4. **Taylor bounds.** For a monomial ideal, there is a natural resolution called the Taylor resolution. It is not minimal except when the ideal is a complete intersection. However, the equations

$$\begin{array}{rcll} \sum_{i=1}^n b_j(-1)^i d_{ij}^k & = & -1, & k=0 \\ & = & 0, & 1 \leq k < h \\ & = & (-1)^n h! e & k = h \end{array}$$

hold even if the resolution is not minimal. Hence we can ask for the bounds from this nonminimal resolution. To establish notations, we denote the cardinality of a set A by $|A|$. Let $[n]$ denote the set of first n positive integers. Let I be a monomial ideal generated by the monomials $f_1, \ldots, f_n$. If σ is a subset of $[n]$, let f_σ denote the lcm of $f_i, i \in \sigma$ and $|\sigma|$ denote the size of σ. Then the shifts in the Taylor resolution of R/I at the jth place are $\deg(\mathrm{LCM}\, f_\sigma, |\sigma| = j)$. The maximal shifts in the ith place, is $L_j = \max\{\deg(\mathrm{LCM}\, f_\sigma, |\sigma| = j))\}$.

Then the conjecture of Herzog and Srinivasan is:

Conjecture 1.2: (Herzog-Srinivasan) *Suppose that I is a monomial ideal of height h and is minimally generated by $f_1, \ldots, f_n$. Let L_i be the maximal shifts in the Taylor resolution of R/I. Then the multiplicity e of R/I satisfies*

$$\frac{1}{h!} \prod_{i=1}^{i=h} L_i \geq e.$$

If I is a complete intersection ideal, clearly the Taylor resolution is the same as the Koszul resolution and all the bounds hold. If I is a monomial ideal of height h whose minimal generating set contains a regular sequence of length h, then the Taylor bound holds. For if K is the ideal generated by the regular sequence that forms part of the minimal generating set of I, then $e(R/I) \leq e(R/K)$ and $L_i(I) \geq L_i(K)$. So we get that $e(R/I) \leq \prod_{i=1}^{i=h} L_i$. Herzog and Srinivasan prove:

Theorem 4.17. *(Corollary 4.3* [Herzog-Srinivasan2004]*) For a monomial ideal of height 2, the Taylor bound holds.*

Theorem 4.18. *(Theorem 5.3* [Herzog-Srinivasan2004]*) Let I be an almost complete intersection monomial ideal. Then the Taylor bound holds for I.*

As a first reduction, they show that if Taylor bound holds for all squarefree monomial ideals, then it holds for all monomial ideals [Herzog-Srinivasan2004]. Thus the problem is one for squarefree monomial ideals. If I is a squarefree monomial ideal, then its multiplicity can also be computed as the number of primes of height h in an irredundant primary decomposition of I. Then they estimate this number from the primary decomposition of I to arrive at the Taylor bound.

In fact, in light of the first reduction to squarefree monomial ideals, the conjecture can be stated as a problem in combinatorics. An antichain $\mathcal{A}$ on n vertices is a collection of subsets of $[n]$ such that none of the sets in $\mathcal{A}$ contains another set in $\mathcal{A}$. A subset $B \subseteq [n]$ is a minimal vertex cover of $\mathcal{A}$ if $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and for any proper subset $C \subset B$, there exists $A' \in \mathcal{A}$ such that $C \cap A' = \emptyset$. Let $M(\mathcal{A}) = B_1, \ldots, B_t$ be all the distinct set of minimal vertex covers of $\mathcal{A}$ and let $L_i = \max\{|\cup_{k\in\sigma} B_k|, \sigma \subset [t], |\sigma| = i\}$ If $h = h(\mathcal{A})$ denotes the least cardinality of an element of $\mathcal{A}$, and $e(\mathcal{A})$ equals the number of subsets in $\mathcal{A}$ of cardinality h, then the conjecture states that $h!e(\mathcal{A}) \leq L_1 L_2 \cdots L_h$.

In addition, this method of stating the problem has a further advantage. Since it can be easily proved that $M(M(\mathcal{A})) = \mathcal{A}$ for any antichain A, a theorem for $\mathcal{A}$ has a dual theorem for $M(\mathcal{A})$. Recall that the sup height is the maximal height of a minimal prime of I. Thus the theorems in codimension two tell us that if a squarefree monomial ideal I in a polynomial ring of dimension n, then the number of generators of I in degree two is bounded above by $(n/2)$sup height$I$.

A simple reduction one can do to prove the conjecture is:

Theorem 4.19. *Suppose the Taylor bound holds for monomial ideals of height h generated by $f_1, f_2, \ldots, f_t$ such that no $t-1$ of them will generate an ideal of height h. Then the Taylor bound holds for all monomial ideals of height h. That is, the conjectured bound for the antichains $\mathcal{A}$ holds if it holds for those antichains $\mathcal{A}$ such that every proper subset $\mathcal{B}$ of $M(\mathcal{A})$ has a minimal vertex cover of cardinality $h-1$, where $h = h(\mathcal{A})$.*

Proof. We proceed by induction on the number of generators, the case of one generator being trivial. Let I be a monomial ideal of height h minimally generated by $f_1, \ldots, f_n$. Let $J_i = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n), 1 \leq i \leq n$. If one of the J_i has height h, then since the generators of J are among the generators of I, $L_t(R/J_i) \leq L_t(R/I)$. However, since $J_i \subset I$ and is of the same height, $e(R/I) \leq e(R/J_i)$. Now, J_i has one fewer generator and hence by induction the bound holds for J_i. Thus, we get

$$e(R/I) \leq e(R/J_i) \leq \prod_{t=1}^{h} L_t(R/J_i) \leq \prod_{t=1}^{h} L_t(R/I)$$

as desired. If none of the $J_i, 1 \leq i \leq n$, has height h, then the result is true by the assumption. □

5. Resolutions for a given Hilbert function and truncation

In this section, we take a different approach to the conjectures, surveying work from [Francisco2004a]. We devote the first part of the section to investigating Conjecture 1.1 by analyzing what sets of graded Betti numbers occur for a fixed Hilbert function. One can reduce Conjecture 1.1 for all modules with a given Hilbert function to considering only a few arrangements of graded Betti numbers, often getting the bounds for all modules with that Hilbert function from analyzing just those sets. In the final portion of this section, we discuss the upper bound in the Cohen-Macaulay case, describing how to prove the result for ideals with generators in high degrees relative to their regularity.

5.1. **Multiplicity and resolutions for a given Hilbert function.** We begin with a quick overview of the theory of resolutions for a given Hilbert function. In his 1890 paper [Hilbert], Hilbert showed that one can compute the Hilbert function of a module from its graded free resolution. Recently, researchers have been interested in going the other direction: Given a Hilbert function, what minimal graded free resolutions (that is, sets of graded Betti numbers) occur for modules with that Hilbert function? There may be a number of possibilities; for example, (a^2, b^2) and (a^2, ab, b^3) have the same Hilbert function, but they do not even have the same number of generators.

One way to study this question is to impose a partial order on the sets of graded Betti numbers that occur for modules with a fixed Hilbert function. Suppose S/I and S/J are graded modules with the same Hilbert function. We say that $\beta^{S/I} \leq \beta^{S/J}$ if and only if $\beta_{ij}^{S/I} \leq \beta_{ij}^{S/J}$ for all i and j. This is a strong condition, and in particular, there may be many incomparable resolutions.

In the early 1990s, Bigatti [Bigatti] and Hulett [Hulett] proved independently in characteristic zero that if $L \subset S$ is a lexicographic ideal, and $I \subset S$ is an ideal with the same Hilbert function, then $\beta^{S/I} \leq \beta^{S/L}$. Pardue generalized the result to positive characteristic [Pardue]. Thus there is always a uniquely maximal element in the partially ordered set, though there are often multiple minimal elements. Since lexicographic ideals are stable, they satisfy the bounds of Conjecture 1.1.

Hence the top of the partially ordered set satisfies the bounds, but what about the other sets of graded Betti numbers? To consider this question, let I be a homogeneous ideal of codimension c in $S = k[x_1, \ldots, x_n]$. Suppose S/I satisfies the bounds of Conjecture 1.1. (If S/I is not Cohen-Macaulay, then by this assumption we mean that S/I satisfies the upper bound, and one makes the obvious adjustments in the discussion that follows.) Then we have

$$\frac{1}{c!}\prod_{i=1}^{c} m_i^{S/I} \leq e(S/I) \leq \frac{1}{c!}\prod_{i=1}^{c} M_i^{S/I}.$$

Let J be a homogeneous ideal in S satisfying the following conditions: S/J has the same Hilbert function as S/I, and $\beta^{S/I} \leq \beta^{S/J}$. (Since I and J are both ideals in S with the same Hilbert series, they have the same codimension.) Then because $\beta_{ij}^{S/I} \leq \beta_{ij}^{S/J}$ for all i and j,

$$m_i^{S/J} \leq m_i^{S/I} \quad \text{and} \quad M_i^{S/I} \leq M_i^{S/J};$$

the resolution of S/J has all the terms of the resolution of S/I plus possibly more, so the minimum shifts can be lower, and the maximum shifts can be higher. In addition, because I and J have the same Hilbert function, and the Hilbert function determines the multiplicity, $e(S/I) = e(S/J)$. Putting these facts together, we have

$$\frac{1}{c!}\prod_{i=1}^{c} m_i^{S/J} \leq \frac{1}{c!}\prod_{i=1}^{c} m_i^{S/I} \leq e(S/I) = e(S/J) \leq \frac{1}{c!}\prod_{i=1}^{c} M_i^{S/I} \leq \frac{1}{c!}\prod_{i=1}^{c} M_i^{S/J}.$$

Therefore S/J also satisfies the bounds of Conjecture 1.1. We have proven the following proposition.

Proposition 5.1. *Let I and J be homogeneous ideals in $S = k[x_1, \ldots, x_n]$. Suppose that I and J have the same Hilbert function, $\beta^{S/I} \leq \beta^{S/J}$, and S/I satisfies the bounds in Conjecture 1.1. Then Conjecture 1.1 also holds for S/J.*

As a consequence of Proposition 5.1, we need only check Conjecture 1.1 for the (finitely many) minimal elements in the partially ordered set for each Hilbert function. Suppose we fix a Hilbert function and find all the minimal elements in the partial order. If all these minimal sets of graded Betti numbers satisfy the bounds of Conjecture 1.1, then we can use Proposition 5.1 to "lift" the result to all resolutions above the minimal elements. This proves the bounds for all modules with the given Hilbert function. We illustrate this technique in an example.

Example 5.2. Let $R = k[a, b, c]$, and let $L \subset R$ be the lexicographic ideal such that R/L has Hilbert function $H = (1, 3, 6, 9, 9, 6, 2)$. Let I be the ideal (a^3, b^4, c^4, b^2c^2). Then L and I have the same Hilbert function. The Betti diagrams of R/L and R/I are below:

R/L:	total:	1	16	27	12	R/I:	total:	1	4	5	2
	0:	1	.	.	.		0:	1	.	.	.
	1:	.	.	.	.		1:	.	.	.	.
	2:	.	1	.	.		2:	.	1	.	.
	3:	.	3	5	2		3:	.	3	.	.
	4:	.	5	9	4		4:	.	.	2	.
	5:	.	5	9	4		5:	.	.	3	.
	6:	.	2	4	2		6:	.	.	.	2

Note that by making all potentially possible cancellations in the Betti diagram of R/L (that is, removing an even and odd syzygy of the same degree), we obtain the Betti diagram of R/I, the unique minimal element in the partial order. Therefore $\beta^{R/I} \leq \beta^{R/J}$ for all ideals $J \subset R$ with the same Hilbert function as I; the resolution of R/I is the unique minimal element in the partial order on resolutions with the fixed Hilbert function. The bounds on R/I from Conjecture 1.1 are

$$27 = \frac{1}{3!}(3)(6)(9) \leq e(R/I) = 36 \leq \frac{1}{3!}(4)(7)(9) = 42,$$

and thus R/I satisfies the conjecture. (This is actually immediate since R/I has a quasipure resolution.) By Proposition 5.1, the bounds of Conjecture 1.1 hold for all modules with the same Hilbert function as I.

Macaulay 2 tests with this technique yield the following theorem [Francisco2004a].

Theorem 5.3. *Let I be a homogeneous ideal in $R = k[x_1, x_2, x_3]$ such that R/I is zero in degree 10 and higher. Then R/I satisfies the upper bound of Conjecture 1.1.*

Remark 5.4. In testing Theorem 5.3, one finds only 197 Hilbert functions for which there is a potential minimal resolution that violates the upper bound of Conjecture 1.1. One can exclude each of those potential counterexamples with various easy arguments; see [Francisco2004a]. The existence of these potential counterexamples is important, however, because it shows that there is no purely numerical proof for Conjecture 1.1.

5.2. **Truncation.** We turn now to the technique of truncation. For the rest of this section, assume that S/I is Cohen-Macaulay, and we shall consider only the upper bound of Conjecture 1.1. The idea behind truncation is that once we know the multiplicity of S/I, there is a lot of extraneous information in the resolution. We need only compute the maximum degree of a shift at each step in the resolution, so the syzygies in lower degrees are not so important. Truncation allows us to focus only on the most significant portion of the resolution.

Let $I_{\geq d}$ be the ideal in S consisting of all elements of I of degree d or higher. Instead of working with S/I, we shall truncate and work with modules of the form $S/I_{\geq d}$. It is easy to see that $e(S/I) \leq e(S/I_{\geq d})$ for all d. The next lemma explains how the graded Betti numbers of S/I are related to those of $S/I_{\geq d}$.

Lemma 5.5. *Let I be a homogeneous ideal in $S = k[x_1, \ldots, x_n]$, and let d be a positive integer. Then for each integer $l \geq 0$,*

$$\beta_{i,i+d+l}^{S/I} = \beta_{i,i+d+l}^{S/I_{\geq d}}.$$

That is, rows d and higher of the Betti diagrams of S/I and $S/I_{\geq d}$ are the same.

Moreover, if I has its highest degree minimal generator in degree $\geq d$, and S/I is Cohen-Macaulay, then $M_i^{S/I} = M_i^{S/I_{\geq d}}$.

The proof is a relatively straightforward exercise in homological algebra, considering a long exact sequence in Tor induced by the short exact sequence

$$0 \longrightarrow I/I_{\geq d} \longrightarrow S/I_{\geq d} \longrightarrow S/I \longrightarrow 0.$$

Example 5.8, an example related to Theorem 5.7, illustrates the truncation process, and the reader may wish to look at it now. Note that truncation allows us to reduce the upper bound portion of Conjecture 1.1 to the case of ideals whose minimal generators are all in a single degree.

Proposition 5.6. *Let I be an Artinian homogeneous ideal in $S = k[x_1, \ldots, x_n]$. Let d be the highest degree in which I has a minimal generator. If $S/I_{\geq d}$ satisfies the upper bound of Conjecture 1.1, then so does S/I.*

Proof. We know $e(S/I) \leq e(S/I_{\geq d})$. We are assuming that $S/I_{\geq d}$ satisfies the upper bound of Conjecture 1.1, so

$$e(S/I) \leq e(S/I_{\geq d}) \leq \frac{1}{n!} \prod_{i=1}^{n} M_i^{S/I_{\geq d}}.$$

By Lemma 5.5, $M_i^{S/I} = M_i^{S/I_{\geq d}}$ for each i. Hence S/I satisfies the upper bound of Conjecture 1.1. □

It follows easily that in the Artinian case, we need only consider ideals with all their minimal generators in a single degree. The Artinian case also implies the Cohen-Macaulay case; see [Francisco2004a].

We use Proposition 5.6 in conjunction with Theorem 2.2 on quasipure resolutions. The proof of Theorem 2.2 is numerical: Any potential quasipure resolution below that of a lex ideal satisfies the bounds; there need not exist a module with that resolution. Combining this with the work in this section gives the following.

Theorem 5.7. *Let I be a homogeneous ideal of codimension c in S such that S/I is Cohen-Macaulay of regularity d. Suppose I contains a minimal generator of degree d or $d + 1$. Then S/I satisfies the upper bound of Conjecture 1.1.*

Proof. We may assume that I is Artinian and that $c = n$ (for if there is a Cohen-Macaulay module with the given Betti numbers, there is an Artinian module with the same Betti numbers). Note that if S/I has regularity d, then degree $d+1$ is the highest degree in which I can have a minimal generator. The resolution of $S/I_{\geq d}$ is concentrated in rows $d-1$ and d of the Betti diagram, the bottom two rows. Therefore $S/I_{\geq d}$ has a quasipure resolution, and it satisfies the bounds of Conjecture 1.1. Lemma 5.5 combined with $e(S/I) \leq e(S/I_{\geq d})$ gives

$$e(S/I) \leq e(S/I_{\geq d}) \leq \prod_{i=1}^{c} M_i^{S/I_{\geq d}} = \prod_{i=1}^{c} M_i^{S/I}.$$

Hence S/I satisfies the upper bound of Conjecture 1.1. □

This result gives, for example, an easy proof of the upper bound for stable Cohen-Macaulay ideals. There are examples of potential Betti diagrams with three nonzero rows (instead of just two) that do not satisfy the upper bound of Conjecture 1.1, which shows that no further reduction like the one in Theorem 5.7 is possible.

We illustrate Theorem 5.7 with an example.

Example 5.8. Let $I = (a^3, b^4, c^4, ab^2, a^2bc^3) \subset R = k[a,b,c]$. Then R/I has regularity six, and I has a minimal generator in degree six. We resolve R/I and $R/I_{\geq 6}$ below.

R/I:	total:	1	5	8	4
	0:	1	.	.	.
	1:	.	.	.	.
	2:	.	2	.	.
	3:	.	2	2	.
	4:	.	.	.	.
	5:	.	1	5	3
	6:	.	.	1	1

$R/I_{\geq 6}$:	total:	1	27	46	20
	0:	1	.	.	.
	1:	.	.	.	.
	2:	.	.	.	.
	3:	.	.	.	.
	4:	.	.	.	.
	5:	.	27	45	19
	6:	.	.	1	1

Note that R/I does not have a quasipure resolution, so Theorem 2.2 does not apply. However, the truncation $R/I_{\geq 6}$ does have a quasipure resolution, and R/I and $R/I_{\geq 6}$ have the same maximum shifts at each step in the resolution. Also, $e(R/I) = 31 \leq 57 = e(R/I_{\geq 6})$, and thus R/I satisfies the upper bound of Conjecture 1.1 because $R/I_{\geq 6}$ does.

6. Zero dimensional schemes

L. Gold, H. Schenck and H. Srinivasan consider the case of zero dimensional schemes in [Gold-Schenck-Srinivasan]. Let $X \subseteq \mathbb{P}^n$ be a set of points which form a complete intersection. If $Y \subset X$ is a set of points and $Z = X \setminus Y$ is the set of points obtained by deleting Y from X, then Z is called residual to Y, and their ideals are given by $I_Z = (I_X : I_Y)$. Let $R = k[x_0, \ldots, x_n]$ denote the homogenous coordinate ring of the projective space. This means a resolution of R/I_Z can be obtained from the mapping cone of the resolutions of R/I_X and R/I_Y. In particular, if $\mathbf{F}_X, \mathbf{F}_Y$ denote the minimal resolutions of R/I_X and R/I_Y respectively, and $\phi : \mathbf{F}_X \to \mathbf{F}_Y$ is the complex map induced by the inclusion of Y in X, then the dual of the mapping cone of ϕ is a resolution of R/I_Z. The minimal resolution of R/I_Z is a direct summand of this nonminimal resolution, naturally. The multiplicity of Z is simply the multiplicity of X minus the multiplicity of Y as it counts the number of points in the set. The main result in this context is:

Theorem 6.1. [Gold-Schenck-Srinivasan] *Let $X \subseteq \mathbb{P}^n$ be a set of points forming a complete intersection and $Y \subset X$ be collinear or $|Y| = 3$. Then the multiplicity of R/I_Z satisfies the conjectured lower and upper bounds.*

The computations in the proofs are still somewhat complicated, and we refer the reader to the original paper cited above. It is worth noting that if the minimal resolution of R/I_Z is the dual of the mapping cone with the obvious truncation at the end, the computations are manageable. In other cases, the fact that the number of points in Y are small or special enough to lie on a line simplifies the computations in the resolution of R/I_Z just enough to estimate the multiplicity and get the bounds.

Remark 6.2. Since we wrote the original draft of this paper, there has been a considerable amount of work on the multiplicity conjectures, and we mention a sample of that work here. Papers of Novik and Swartz and Kubitzke and Welker have attacked the conjectures from innovative combinatorial perspectives. Zanello has proposed and, in some cases, proven improved bounds in the case of codimension three level algebras, and he, along with Migliore and Nagel, have proven stronger bounds in the Gorenstein codimension three case. These bounds are particularly interesting since they are expressed in terms of the Hilbert function, not the Betti numbers. In another direction, Migliore, Nagel, and Römer have attacked the conjectures using linkage theory, and they have proposed extensions of the conjectures to modules. Additionally, several authors, including Migliore, Nagel, and Römer and Herzog and Zheng, have conjectured that the bounds of the multiplicity conjectures are sharp if and only if S/I is Cohen-Macaulay with a pure resolution. Herzog and Zheng have proven this sharpness result in several cases, and they have also shown that the conjectures behave well after quotienting an ideal by a regular sequence. Finally, Miró-Roig has proven the conjectures for determinantal ideals. We encourage the reader to explore these papers on the arXiv.

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Christopher A. Francisco, Department of Mathematics, University of Missouri, Mathematical Sciences Building, Columbia, MO 65211

E-mail address: `chrisf@math.missouri.edu`

Hema Srinivasan, Department of Mathematics, University of Missouri, Mathematical Sciences Building, Columbia, MO 65211

E-mail address: `hema@math.missouri.edu`