1. Orbital Invariants of Admissible Representations

An example of a strong invariant of an admissible representation (by which I mean an invariant capable of splitting points in $\hat{G}_{adm}$) is the distributional character of a representation. Let $G$ be a reductive Lie group, $(\pi, \mathcal{H})$ an irreducible admissible representation of $G$ and $\mathcal{H}_K$ is associated Harish-Chandra $(\mathfrak{g}, K)$-module, and let $\Theta_\pi$ be its distributional character

$$\Theta_\pi(f) \equiv \text{Tr}_{\mathcal{H}} \left( \int f(g) \pi(g) \, dg \right), \quad \forall \, f \in C_c^\infty(G)$$

Weaker invariants are things like the annihilator of $\pi$ in $U(\mathfrak{g})$, minimal $K$-types, Gelfand-Kirillov dimensions.

There are also several invariants associated with certain homogeneous affine varieties associated with $\pi$. These I’ll describe below.

1.1. Asymptotic Expansion of Characters.

**Theorem 1.1** (Barbasch-Vogan 1980). Let $\theta_\pi$ be the lift of $\Theta_\pi$ via the exponential map to a neighborhood of the identity on $\mathfrak{g} = \text{Lie}(G)$. If $f \in C_c^\infty(G)$ and $t > 0$ define

$$f_t(X) = t^{-\dim \mathfrak{g}} f(t^{-1}X)$$

Then there is an integer $r$ and tempered distributions $\{D_i\}_{i=-r}^\infty$ on $\mathfrak{g}$, such that for $f \in C_c^\infty(G)$

$$\theta_\pi(f_t) \sim \sum_{i=-r}^\infty t^i D_i(f)$$

as $t \to 0^+$. Furthermore, the support of $\hat{D}_i$ is a union of nilpotent orbits in $\mathfrak{g}$.

1.2. Characteristic Cycles. Let $(\pi, \mathcal{H})$ be an irreducible admissible representation of a reductive Lie group $G$, and let $\mathcal{H}_K$ be its associated Harish-Chandra $(\mathfrak{g}, K)$-module.

**Theorem 1.2** (Matsumura). Let $R$ be a commutative Noetherian ring and $M \neq 0$ a finitely generated $R$-module. There exists a chain

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of submodules of $M$ such that for each $i$ we have $M_i/M_{i-1} \simeq R/P_i$, with $P_i$ a prime ideal of $R$.

**Remark 1.3** (worthy of a lemma). Let $R$ be a commutative Noetherian ring and let $Q$ be a minimal prime ideal of $R$. Suppose $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is a filtration of a finitely generated $A$-module $M$ such that $M_i/M_{i-1} = R/P_i$. Then the number of times $P_i = Q$ is independent of the choice of filtration.

**Definition 1.4.** Let $R$ be a commutative Noetherian ring and $M \neq 0$ a finitely generated $R$-module. Let $\{P_1, \ldots, P_r\}$ be the set of minimal ideals containing the annihilator of $\text{Ann}(M) = \{r \in R \mid rM = 0\}$. The characterisite cycle of $M$ is the formal sum

$$\text{Ch}(M) = \sum_{i=1}^r m(P_i, M) P_i$$
where \( m(P_i, M) \) is a positive integer defined as follows. Choose any finite filtration of \( M \) such that each subquotient \( M_j/M_{j-1} \) is of the form \( R/Q_j \) with \( Q_j \) a prime ideal of \( R \). \( m(P_i, M) \) is the number of times that \( Q_j = P_i \).

**Definition 1.5.** Suppose that \( X \) is a finitely generated Harish-Chandra module. Recall that \( K_C \) acts on \( X \). A **good filtration** \( \mathcal{F} \) of \( X \) is a (possibly infinite) increasing filtration

\[
0 = X_{-1} \subset X_0 \subset X_1 \subset X_2 \subset \ldots
\]

of \( X \), satisfying the conditions listed below. Write \( U_n \) for the \( n \)-th of the standard filtration of \( U[\mathfrak{g}] \). By the Poincaré-Birkhoff-Witt theorem, the associated graded ring \( \text{gr} (U[\mathfrak{g}]) \) is naturally isomorphic to \( S[\mathfrak{g}] \). The conditions on a good filtration of \( X \) are

1. \( X_m \) is finite-dimensional and \( K_C \)-invariant.
2. The union of all the \( X_m \) is \( X \).
3. The filtrations of \( X \) and \( U[\mathfrak{g}] \) are compatible:
   \[
   U_n X_m \subset X_{n+m}.
   \]
4. The associated graded \( (S(\mathfrak{g}), K_C) \)-module \( \text{gr}(X) \)
   \[
   \text{gr}_{\mathcal{F}}(X) = \bigoplus_{n=0}^{\infty} X_n/X_{n-1}
   \]
   is finitely generated.

The \( \text{gr}(X) \) inherits compatible of the \( K_C \) and \( \mathfrak{g} \)-actions from \( X \), but because the action of the Lie algebra \( \mathfrak{t} = \text{Lie}(K_C) \) preserves the filtration, it follows that \( S(\mathfrak{g}) \mathfrak{t} \subset \text{Ann}(\text{gr}(X)) \) and so \( \text{gr}(X) \) can be regarded as a \( S(\mathfrak{g}/\mathfrak{t}) \simeq S(\mathfrak{p}) \)-module.

**Definition 1.6.** Let \( X \) be the Harish-Chandra module of an admissible representation of a reductive Lie group \( G \). Fix a good filtration of \( X \) and let \( \text{gr}(X) \) be the associated \( (S(\mathfrak{g})) \)

1.3. **Schmid-Vilonen.** Vogan and Barbasch conjectured and Schmid and Vilonen proved that the similarity between wave front cycles and characteristic cycles can actually be expressed as a precise equality. The leading term of the local character expansion can be thought of as a complex-linear combination of nilpotent \( G_{\mathbb{R}} \) orbits, allowing us to write

\[
\Theta_{\pi} = \sum_{\mathcal{O}_j \in WF(\pi)} b_j [\mathcal{O}_j] + \ldots
\]

Schmid and Vilonen refer to

\[
WF(\pi) = \sum_{\mathcal{O}_j \in WF(\pi)} b_j [\mathcal{O}_j]
\]

as the wave front cycle of \( \pi \) and prove that this wave front cycle coincides with the characteristic cycle of Vogan.

However, this coincidence is not nearly obvious as the notation suggests. First of all the “cycles” \([\mathcal{O}]\) that appear in the characteristic cycle are \( K_C \)-orbits in \( \mathfrak{p} \), while the cycles that appear in the wave front cycle are nilpotent \( G_{\mathbb{R}} \)-orbits in the real Lie algebra \( \mathfrak{g}_{\mathbb{R}} \). The basis for the characteristic cycle / wave front cycle correspondence is the following theorem of Sekiguchi

**Theorem 1.7** (Kostant-Sekiguchi Correspondence). Suppose that \( G \) is a real reductive Lie group with maximal compact subgroup \( K \). Then there is a natural one-to-one correspondence between the set of nilpotent \( G_{\mathbb{R}} \)-orbits in the real Lie algebra \( \mathfrak{g}_{\mathbb{R}} \) of \( G \) and the set of \( K_C \)-orbits in \( \mathfrak{p} \). Suppose that under this correspondence the orbit of \( \lambda_{\mathbb{R}} \in N^* \) corresponds to that of \( \lambda_{\mathfrak{t}} \in \mathfrak{p}^* \) and \( G_{\mathbb{C}} \) is any complex group with Lie algebra \( \mathfrak{g} = (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}} \), then

- The \( G_{\mathbb{C}} \)-orbits of \( i\lambda_{\mathbb{R}} \) and \( \lambda_{\mathfrak{t}} \) coincide.
• \( \dim \mathbb{R} G \cdot \lambda_{\mathbb{R}} = 2 \cdot \dim \mathbb{C} K_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} = \dim \mathbb{C} G_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} \)
• The maximal compact subgroups of the isotropy subgroups \( K(\lambda_{\mathbb{R}}) \) and \( G_{\mathbb{R}}(\lambda_{\mathbb{R}}) \) are isomorphic.

1.4. Whittaker Cycles? There is a third way of attaching numbers to nilpotent orbits. Let \( O \) be a real nilpotent orbit occurring in the asymptotic expansion of the character of an admissible representation. Associated to \( O \) is a particular parabolic subgroup \( P = LN \) of \( G \), a particular character \( \chi_{O} \) of \( N \), and a generalized Gelfand-Graev representation \( \text{Ind}_{\mathbb{N}}^{G}(\chi_{O}) \). Set

\[
\begin{align*}
W_{\text{alg}}(\pi, O) &= \dim \text{Hom}_{\mathbb{R}, K}(V, \text{Ind}_{\mathbb{N}}^{G}(\chi_{O})) \\
W_{-\infty}(\pi, O) &= \dim \text{Hom}_{G}(V^{\infty}, \text{Ind}_{\mathbb{N}}^{G}(\chi_{O}))
\end{align*}
\]

be the dimensions of the space of generalized Whittaker modules of type \( O \), and write

\[
\begin{align*}
\mathcal{W}_{\text{alg}}(\pi) &= \sum_{O \in WF(\pi)} w_{\text{alg}}(\pi, O) [O] \\
\mathcal{W}_{-\infty}(\pi) &= \sum_{O \in WF(\pi)} w_{-\infty}(\pi, O) [O]
\end{align*}
\]

**Question 1.8.** Can one relate either of these Whittaker cycles to wave-front cycles (or equivalently, to associated cycles)?

**Theorem 1.9** (Matumoto). For large representations the dimension of the space algebraic Whittaker vectors coincides with the Bernstein degree of the representation.

**Theorem 1.10** (Kostant, Lynch). Let \( G = KAN \) be an Iwasawa decomposition of a real reductive Lie group. The principal series representations \( \text{ind}_{MAN}^{G}(\sigma \otimes e' \otimes 1) \) is large, and the Bernstein degree is \( \#W(\mathfrak{g}_{o}, a_{o}) \cdot \dim(\sigma) \), where \( \#W(\mathfrak{g}_{o}, a_{o}) \) is the dimension of the little Weyl group.

**Theorem 1.11** (Yamashita). If \( G \) is a connected simple Lie group of Hermitian type and let \( \pi \) be an irreducible unitary highest weight representation of \( G \). Then the multiplicities in the wave front cycle of \( \pi \) and in \( \mathcal{W}_{\text{alg}}(\pi) \) coincide.

**Theorem 1.12** (Nishiyama, Ochiai, Taniguchi, Yamashita and Kato). The associated cycle and the Bernstein degree of a large \((\mathfrak{g}, K)\)-module \( X_{\pi} \) are given by

\[
\begin{align*}
\mathcal{A}(\pi) &= \sum_{O_{R}} w_{-\infty}(\pi, O_{R}) \cdot [O] \\
\text{BernsteinDeg}(\pi) &= \frac{w_{G}}{l_{G}} \sum_{O_{R}} w_{-\infty}(\pi, O_{R})
\end{align*}
\]

where \( O_{R} \) runs over the principle nilpotent orbits in \( \mathfrak{g}_{R} \), \( O \) is the \( K_{\mathbb{C}} \)-orbit corresponding to \( O_{R} \) via the Kostant-Sekiguchi correspondence, \( l_{G} \) is the number of principal nilpotent \( G \)-orbits, and \( w_{G} \) is the order of the little Weyl group.

In these talks, we present the results of some exploratory calculations aimed at determining and correlating the characteristic cycles, multiplicities, and Whittaker vectors for a special class of small unitary representations. Although our purpose is purely representation-theoretical, and empirical at that, it also illustrates how representation-theoretical methods, in a sufficiently equivariant setting, might be used to answer rather difficult algebraic-geometric questions (in particular, the degree of a projective variety.).

2. The Class of Small Unitary Representations

2.1. Reductive Lie groups associated with simple Jordan algebras. Let \( N \) be a real simple Jordan algebra with unit \( e \) and norm \( \phi \). Let \( L \) be the subgroup of \( GL(N) \) that preserves \( \phi \) up to a scalar multiple. Let \( P \) be the semidirect product \( LN \), and \( G \) the group of rational transformations generated by \( P \) and \( \iota : x \rightarrow -x^{-1} \). Then \( G \) is a semisimple Lie group, and \( P = LN \) is the Levi decomposition of a parabolic subgroups such that
Lemma 2.1. Let \( m \) be the dimension of \( \mathfrak{m} \). Let \( n = \text{Lie}(N), \mathfrak{m} = \text{Lie}(M) \), etc. Let \( t_1 \) be a maximal toral subalgebra in the orthogonal complement of \( \mathfrak{m} \) in \( \mathfrak{t} \). It turns out that there are only three possibilities for the restricted root system \( \Sigma \) of \( t_1 \); namely, \( \Sigma \) is either of type \( A_{n-1}, D_n \), or \( C_n \). Moreover, there is a fairly uniform prescription for writing down the restricted root systems \( \Delta(\mathfrak{g}; t_1) \) and \( \Delta(\mathfrak{g}; t_1) \). This goes as follows.

There is a orthonormal basis \( \{\gamma_1, \ldots, \gamma_n\} \) of \( t_1^* \) such that the simple roots of \( t_1 \) in \( \mathfrak{t} \) are
\[
\begin{align*}
\frac{1}{2}\gamma_i + \frac{1}{2}\gamma_{i+1} & \quad | \quad i = 1, \ldots, n-1, \quad \text{if } \Sigma = A_{n-1} \\
\frac{1}{2}\gamma_i + \frac{1}{2}\gamma_{i+1} & \quad | \quad i = 1, \ldots, n-1, \quad \text{if } \Sigma = C_n \\
\frac{1}{2}\gamma_i & \quad | \quad i = 1, \ldots, n-1, \quad \text{if } \Sigma = D_n
\end{align*}
\]
and that the roots of \( t_1 \) in \( \mathfrak{g} \) are \( \{\pm \frac{1}{2}\gamma_i, \pm \frac{1}{2}\gamma_j, \pm \gamma_i\} \) in all cases. Moreover, it turns out, that for a given \( G \), both the short roots \( \pm \frac{1}{2}\gamma_i \), \( \pm \frac{1}{2}\gamma_j \), and the long roots \( \gamma_i \), have fixed multiplicities. Accordingly, we define integers \( d \) and \( e \) by
\[
\begin{align*}
d &= \text{common multiplicity of short roots } \pm \frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j \text{ in } \mathfrak{t} \\
e &= \text{common multiplicity of long roots } \pm \gamma_i \text{ in } \mathfrak{t}
\end{align*}
\]
and the multiplicities of the (restricted) short and long roots of \( t_1 \) in \( \mathfrak{g} \) are then, respectively, \( 2d \) and \( e + 1 \). Let \( n \) be the dimension of \( t_1 \) (which coincides with the real rank of \( \mathfrak{g} \)). It so happens that there is also a convenient formula for \( m = \text{dim}(n) \).
\[
m = n (d(n-1) + e + 1)
\]

Lemma 2.1. Let \( t = t_1 + t_0 \) be an extension of \( t_1 \) to a CSA for \( \mathfrak{t} \) (with \( t_0 \subset \mathfrak{m} \)). For \( \alpha \in t^* \cong t_1^* \oplus t_2^* \), we will write \( \alpha = (\alpha; \mu) \), with \( \alpha \in t_1^* \), \( \mu \in t_2^* \). Then the positive system \( \Sigma^+ \) corresponding to the choice (1) of simple roots in \( \Sigma \) can be extended to a positive system \( \Delta^+(\mathfrak{t}; t) \) for \( \Delta(\mathfrak{t}; t) \), in such a way that a root \( (\alpha; \mu) \) is positive in \( \Delta^+ \) if \( \alpha \in \Sigma^+ \).

Remark 2.2. Write \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) for the Cartan decomposition of \( \mathfrak{g} \). It turns out that the weights \( \{ (\pm \gamma_1; 0), (\pm \gamma_2; 0), \ldots, (\pm \gamma_n; 0) \} \) are the extremal weights of representation of \( K \) carried by \( \mathfrak{p} \). Indeed, for the positive system \( \Delta^+ \) arising from positive system (1) of \( \Sigma \), the highest weight of the representation of \( K \) on \( s \) is \( (\gamma_1; 0) \) and the weights
\[
\lambda_i = (\gamma_1; 0) + (\gamma_2; 0) + \cdots + (\gamma_i; 0)
\]
are dominant integral for \( \Delta^+ \).

Indeed, let \( t \) be an arbitrary Cartan subalgebra of \( \mathfrak{t} \), and \( \Delta^+ \) a positive system for \( \Delta(\mathfrak{t}; t) \). Then set \( \gamma_\perp \) equal to the highest root of the representation of \( K \) on \( \mathfrak{p} \) and let \( W \cdot \gamma_\perp \) be the Weyl orbit of \( \gamma_\perp \). Then there will be a unique root \( \gamma_\perp \) in \( W \cdot \gamma_\perp \) that is orthogonal to \( \gamma_\perp \) and is such that \( \gamma_\perp + \gamma_\perp \) is dominant integral. And there will be a unique root \( \gamma_\perp \) in \( W \cdot \gamma_\perp \) that is perpendicular to both \( \gamma_\perp \) and \( \gamma_\perp \) and such that \( \gamma_\perp + \gamma_\perp + \gamma_\perp \) is dominant integral. Continuing in this manner, we can construct a sequence \( \{ \gamma_1, \ldots, \gamma_n \} \) of \( \mathfrak{p} \)-roots. We can now set \( t_1 = \text{span} \{ \mathfrak{p}_i, \mathfrak{p}_{-i} \} \mid i = 1, \ldots, n \} \), and we’re back where we started from with \( \gamma_i = \gamma_i \).
2.3. BSZ and Sahi’s representations. Put \( r = d(n - 1) + e \) and let \( \nu \) be the positive character for \( L \) such that \( \nu^{2r} \) is the determinant of the adjoint action of \( L \) on \( n \). For \( t \in \mathbb{R} \), let \( (\Pi_t, I(t)) \) denote the (normalized) induced representation \( \text{Ind}_G^P(\nu_t) \).

**Theorem 2.3.** Assume \( \Sigma (\mathfrak{g}, t_1) \) is not of type \( D_2 \). Then \( I(t) \) has an unitarizable constituent \( \Pi_i \) of rank \( i - 1 < n \), if and only if \( t = d(n - i) + e + 1 \). Moreover, for this value of \( t \), this unitarizable constituent is actually a submodule, and its \( K \)-types are

\[
\{ \lambda = \sum a_j \gamma_j \mid a_i = a_{i+1} = \ldots = a_n = 0 \}
\]

and these \( K \)-types occur with multiplicity one.\(^1\)

**Notation 2.4.** Henceforth, \( G \) will consistently denote one of the groups listed in Table 1, \( n \) will denote its real rank (which happens to equal to the rank of the corresponding Jordan algebra), and \( \Pi_i, i \in \{1, \ldots, n\} \) will denote the irreducible unitarizable constituent of the corresponding induced representation as prescribed in the preceding theorem.

3. Asymptotics of the \( K \)-types of \( \Pi_i \)

Let \( X \) be the Harish-Chandra module of an irreducible admissible representation of a reductive Lie group \( G \). A filtration

(3.1)

\[ \{0\} \subset X_0 \subset X_1 \subset X_2 \subset \cdots \]

of \( X \) by subspaces \( X_i \) is said to be a good filtration \( ([V]) \) if

(i) \( \dim X_n < \infty \)

(ii) each \( X_n \) is \( K \)-invariant

(iii) \( g \cdot X_n \subseteq X_{n+1} \) for all \( n \in \mathbb{N} \).

Given a good filtration (3.1) of Harish-Chandra module \( X \), one has a corresponding graded object

\[ gr(X) = \bigoplus_{n=0}^{\infty} X_n/X_{n-1} \]

which is a finitely generated, graded \( (S(\mathfrak{g}), K) \)-module. As such, by a theorem of Hilbert and Serre, there exists a polynomial \( \varphi(n) \) such that

\[ \varphi_X(n) = \sum_{q \geq n} \dim_{\mathbb{C}}(X_q/X_{q-1}) \]

\(^1\)Here and henceforth we will use \( \gamma_i \) to denote either a restricted root in \( \Sigma \) or its (trivial) extension to \( \Delta (\mathfrak{g}, \mathfrak{t}) \).
Write

\[(3.2)\]
\[\varphi_X(n) = \frac{B_X}{D!} n^D + \text{lower order terms}\]

Then, the leading power \(D\) of \(\varphi_X\) is the Gelfand-Kirillov dimension of \(X\) and the leading coefficient of \(\varphi_X\) times \(D!\) is the Bernstein degree of \(X\).

3.1. Initial Formula for Leading Term of \(p_{\Pi_i}(k)\). Let \(\Pi_i\) be the Harish-Chandra module of the irreducible unitarizable constitute of \(Ind_G^F(\nu^i)\), occurring for \(t = d(n - i) + e + 1\). Its \(K\)-types\(^2\) are

\[
\Lambda = \left\{ \lambda = \sum_{j=1}^{i} a_j \gamma_j \mid a_i \in \mathbb{N} ; \quad a_1 \geq a_2 \geq \cdots \geq a_i \geq 0 \right\}
\]

if \(i < n\), \(i = n\) and \(\Sigma = C_n\), or

\[
\Lambda = \left\{ \lambda = \sum_{j=1}^{i} a_j \gamma_j \mid a_i \in \mathbb{N} ; \quad a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq |a_n| \geq 0 \right\}
\]

if \(i = n\) and \(\Sigma = D_n\).

We filter \(\Pi_i\) as follows

\(V_n = \{K\text{-types } \lambda = a_1 \gamma_1 + \cdots + a_i \gamma_i \text{ of } \Pi_i \text{ such that } a_1 \leq n\}\)

This is a good filtration of \(V\). (We note that each \(V_n\) is finite-dimensional and obviously stable under \(K\), and because the highest weight of \(s\) is \(\gamma_1\), the only \(K\)-types that can occur in \(g \cdot V_n\) are \(K\)-types \(\lambda = a_1 \gamma_1 + \cdots + a_i \gamma_i\) with \(a_1 \leq n + 1\).)

We will now compute the leading term of the Hilbert polynomial for \(\Pi_i\):

\[p_{\Pi_i}(k) = \dim(V_k) = \sum_{a_1=0}^{k} \sum_{a_2=0}^{a_1} \cdots \sum_{a_i=0}^{a_{i-1}} \dim F_{a_1 \gamma_1 + \cdots + a_i \gamma_i}\]

Here \(F_\lambda\) denotes the irreducible finite dimensional representation of \(K\) with highest weight \(\lambda\). If \(\Sigma = D_n\) and \(i = n\) the summation on the far right should actually run from \(a_n = -a_{n-1}\) to \(a_{n-1}\). Shortly, we’ll introduce a special factor to keep track of this special case.

According to the Weyl dimension formula, the dimension of \(F_\lambda\) is

\[\dim F_\lambda = \prod_{\alpha \in \Delta_K^+} \frac{(\lambda + \rho_K, \alpha)}{(\rho_K, \alpha)}\]

In order to use this formula to compute the dimensions of the \(K\)-types \(\lambda = a_1 \gamma_1 + \cdots + a_i \gamma_i\) appearing in \(\Pi_i\), we shall split the set \(\Delta_K^+\) into three disjoint sets:

\[\Delta_K^+ = \Delta_{\Sigma,i}^+ \cup \Delta_{\Sigma,i,c}^+ \cup \Delta_M^+\]

where

\[\Delta_{\Sigma,i}^+ = \{ \alpha \in \Delta_{\Sigma}^+ \mid (\alpha, \gamma_j) \neq 0 \text{ for some } j \in \{1, 2, \ldots, i\} \}\]

\[\Delta_{\Sigma,i,c}^+ = \{ \alpha \in \Delta_{\Sigma}^+ \mid (\alpha, \gamma_j) = 0 \text{ for all } j \in \{1, 2, \ldots, i\} \} = \Delta_{\Sigma}^+ - \Delta_{\Sigma,i}^+\]

\[\Delta_M^+ = \{ \alpha \in \Delta_K^+ \mid (\alpha, \gamma_i) = 0 \text{ for all } j \in \{1, \ldots, n\} \}\]

\(^2\)We specify \(K\)-types by their highest weight vectors.
We can now, accordingly, break up the product over the elements of $\Delta_K^+$ in the Weyl dimension formula

\[
\dim F_{a_1 \gamma_1 + \cdots + a_i \gamma_i} = \left( \prod_{\alpha \in \Delta_K^+} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right) \left( \prod_{\alpha \in \Delta_K^+ \cup \Delta_{K,c}} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right)
\]

\[
= \left( \prod_{\alpha \in \Delta_K^+} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right) \left( \prod_{\alpha \in \Delta_K^+ \cup \Delta_{K,c}} \frac{\langle \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right)
\]

\[
= \prod_{\alpha \in \Delta_K^+} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle}
\]

\[
= \left( \prod_{\alpha \in \Delta_K^+} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} \right)
\]

Thus, the Hilbert polynomial for $\Pi_i$ will be

\[
p_{\Pi_i}(k) = \sum_{a_i=0}^{k} \cdots \sum_{a_i=0}^{a_{i-1}} \dim F_{a_1 \gamma_1 + \cdots + a_i \gamma_i}
\]

\[
= \sum_{a_i=0}^{k} \cdots \sum_{a_i=0}^{a_{i-1}} \prod_{\alpha \in \Delta_{K_i}} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle}{\langle \rho_K, \alpha \rangle} + \text{lower order terms}
\]

\[
= 1 \frac{\langle \rho_K, \alpha \rangle}{\prod_{\alpha \in \Delta_{K_i}} \langle \rho_K, \alpha \rangle} \sum_{a_i=0}^{k} \cdots \sum_{a_i=0}^{a_{i-1}} \prod_{\alpha \in \Delta_{K_i}} \frac{\langle a_1 \gamma_1 + \cdots + a_i \gamma_i, \alpha \rangle}{\langle \rho_K, \alpha \rangle} + \text{lower order terms}
\]

Now if $\Sigma = D_n$ and $i = n$ we actually have

\[
p_{\Pi_n}(k) = \sum_{a_n=0}^{k} \cdots \sum_{a_n=0}^{a_{n-1}} \dim F_{a_1 \gamma_1 + \cdots + a_n \gamma_n}
\]

\[
= \sum_{a_n=0}^{k} \cdots \sum_{a_n=0}^{a_{n-2}} \left( \sum_{a_{n-1}=0}^{a_{n-1}} \dim F_{a_1 \gamma_1 + \cdots + a_m \gamma_m + \sum_{a_n=0}^{a_n=\gamma_n+1}} \dim F_{a_1 \gamma_1 + \cdots + a_m \gamma_m} \right)
\]

\[
= \sum_{a_n=0}^{k} \cdots \sum_{a_n=0}^{a_{n-2}} \left( \sum_{a_{n-1}=0}^{a_{n-1}} \dim F_{a_1 \gamma_1 + \cdots + a_m \gamma_m} + \sum_{a_{n-1}=0}^{a_{n-1}+1} \dim F_{a_1 \gamma_1 + \cdots + a_m \gamma_m} \right)
\]

\[
= \sum_{a_n=0}^{k} \cdots \sum_{a_n=0}^{a_{n-2}} \left( \sum_{a_{n-1}=0}^{a_{n-1}} \dim F_{a_1 \gamma_1 + \cdots + a_m \gamma_m} \right) + \text{lower order terms.}
\]

\[
= \frac{2}{\prod_{\alpha \in \Delta_{K_i}} \langle \rho_K, \alpha \rangle} \sum_{a_i=0}^{k} \cdots \sum_{a_n=0}^{a_{n-1}} \prod_{\alpha \in \Delta_{K_i}} \frac{\langle a_1 \gamma_1 + \cdots + a_n \gamma_n, \alpha \rangle}{\langle \rho_K, \alpha \rangle} + \text{lower order terms}
\]

Introducing a factor

\[
\varepsilon_{G,i} = \begin{cases} 
2 & \text{if } i = n \text{ and } \Sigma \sim D_n \\
1 & \text{otherwise}
\end{cases}
\]
to accommodate the special case when $\Sigma = D_n$ and $i = n$, we thus have the following formula for the Hilbert polynomial of $\Pi_i$

$$
p_{\Pi_i}(k) = \frac{\varepsilon_{G,i}}{(\prod_{\alpha \in \Delta_{\Sigma}^+} \langle \rho_K, \alpha \rangle)} \sum_{a_1=0}^{k} \cdots \sum_{a_{i-1}}^{a_i-1} \left( \prod_{\alpha \in \Delta_{\Sigma}^+} \langle a_1\gamma_1 + \cdots + a_i\gamma_i, \alpha \rangle \right) + \text{lower order terms}
$$

Before preceding further we set

$$
D_{G,i} = \prod_{\alpha \in \Delta_{\Sigma}^+} \langle \rho_K, \alpha \rangle
$$

$$
N_{G,i}(k) = \sum_{a_1=0}^{k} \cdots \sum_{a_{i-1}}^{a_i-1} \left( \prod_{\alpha \in \Delta_{\Sigma}^+} \langle a_1\gamma_1 + \cdots + a_i\gamma_i, \alpha \rangle \right)
$$

so that

$$
p_{\Pi_i}(k) = \frac{\varepsilon_{G,i}}{D_{G,i}} N_{G,i}(k) + \text{lower order terms}
$$

Clearly, the Gelfand-Kirillov dimension is tied up in the asymptotic behavior of $N_{G,i}(k)$ as $k \to \infty$, while all three factors contribute to the Bernstein degree of $\Pi_i$. Indeed, if

$$
N_{G,i}(k) = ck^d + \text{lower order terms}
$$

then

$$
\text{GK dim (} \Pi_i \text{)} = 2d
$$

and so the Bernstein degree of $\Pi_i$ will be

$$
B_{\Pi_i} = \frac{\varepsilon_{G,i} cd!}{D_{G,i}}
$$

3.2. Computation of $N_{G,i}$ and the Gelfand-Kirillov dimension of $\Pi_i$. In this subsection we shall compute the constants $c$ and $d$ such that

$$
N_{G,i}(k) = \sum_{a_1=0}^{k} \cdots \sum_{a_{i-1}}^{a_i-1} \left( \prod_{\alpha \in \Delta_{\Sigma}^+} \langle a_1\gamma_1 + \cdots + a_i\gamma_i, \alpha \rangle \right) = ck^d + \text{lower order terms}
$$

Recall that we have the following description of the restricted system $\Sigma (\mathfrak{t}; t)$ in terms of the $\gamma_j$ :

- Short roots of the form $\gamma_i + \gamma_j$, which occur with a common multiplicity $d$.
- Long roots of the form $\gamma_i$, which occur with a common multiplicity $e$.

Writing

$$
t^* = t_1^* \oplus t_0^*
$$

and according split the roots in $\alpha \in \Delta_K = \Delta (\mathfrak{t}; t)$ into two components

$$
\alpha = (\alpha; \mu) , \quad \alpha \in t_1^* , \quad \mu \in t_0^*
$$

and extend the positive system $\Sigma^+$ to a positive system for $\Delta (\mathfrak{t}; \mathfrak{h})$.

**Formula 3.1.** If

$$
\prod_{\alpha \in \Delta_{\Sigma}^+} \langle a_1\gamma_1 + \cdots + a_i\gamma_i, \alpha \rangle = \left( \frac{1}{2} \right) \frac{d(2n-i-1)}{N_{a_1,\ldots,a_i}}
$$

where

$$
N_{a_1,\ldots,a_i} = \left( \prod_{j=1}^{i-1} \right. \prod_{j'+j+1}^{i} \left( (a_j)^2 - (a_j')^2 \right)^{d} \left( \prod_{j=1}^{i} (a_j)^{k(n-i)+e} \right)
$$
Proof. Let

\[ \Sigma^+_{i,1} = \left\{ \frac{\gamma_j + \gamma_{j'}}{2} \mid 1 \leq j < j' \leq i \right\} \]

\[ \Sigma^+_{i,2} = \left\{ \frac{\gamma_j + \gamma_{j'}}{2} \mid 1 \leq j \leq i < j' \leq n \right\} \]

\[ \Sigma^+_{i,3} = \left\{ \frac{\gamma_j - \gamma_{j'}}{2} \mid 1 \leq j < j' \leq i \right\} \]

\[ \Sigma^+_{i,4} = \left\{ \frac{\gamma_j - \gamma_{j'}}{2} \mid 1 \leq j \leq i < j' \leq n \right\} \]

\[ \Sigma^+_{i,5} = \{ \gamma_j \mid 1 \leq j < i \} \]

so that

\[ \Sigma^+_i = \bigcup_{k=1}^5 \Sigma^+_{i,k} \]

Then

\[
\prod_{\alpha \in \Delta^+_i} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \\
= \left( \prod_{\alpha \in \Sigma^+_{i,1}} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \right)^d \cdot \left( \prod_{\alpha \in \Sigma^+_{i,2}} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \right)^d \\
\cdot \left( \prod_{\alpha \in \Sigma^+_{i,3}} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \right)^e \\
\cdot \left( \prod_{\alpha \in \Sigma^+_{i,4}} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \right)^e \\
\cdot \left( \prod_{\alpha \in \Sigma^+_{i,5}} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \alpha \rangle \right)^e \\
= \left( \prod_{1 \leq j < j' \leq i} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \frac{\gamma_j - \gamma_{j'}}{2} \rangle \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \frac{\gamma_j - \gamma_{j'}}{2} \rangle \right)^d \\
\cdot \left( \prod_{1 \leq j \leq i < j' \leq n} \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \frac{\gamma_j + \gamma_{j'}}{2} \rangle \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \frac{\gamma_j - \gamma_{j'}}{2} \rangle \right)^d \\
\cdot \left( \prod_{j=1}^n \langle a_1 \gamma_1 + \cdots + a_i \gamma_i + \rho_K, \gamma_j \rangle \right)^e \\
= \left( \prod_{1 \leq j < j' \leq i} \frac{1}{4} (a_j^2 - a_{j'}^2) \right)^d \left( \prod_{1 \leq j \leq i} \frac{1}{4} (a_j^{(n-i)} - 2) \right)^d \left( \prod_{1 \leq j \leq i} a_j \right)^e \\
= \left( \frac{1}{2} \right)^{d(2n-i-1)} \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^i (a_j)^2 - (a_{j'})^2 \right)^d \left( \prod_{j=1}^i (a_j)^{2d(n-i)+e} \right) \\
= \left( \frac{1}{2} \right)^{d(2n-i-1)} N_{a_1, \ldots, a_i}
We now have

\[ N_{G,i}(k) = \left( \frac{1}{2} \right)^{di(2n-i-1)} \sum_{a_1=0}^{k} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} \]

but our task remains to compute the constants \( C_{G,i} \) and \( d \) such that

\[ N_{G,i}(k) = ck^d + \text{lower order terms} \]

The next step will be to establish an integral formula for the constant \( c \). Let

\[ I(k) \equiv \int_{a_1=0}^{k} \int_{a_2=0}^{a_1} \cdots \int_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} da_i \]

Using a Riemann sum argument we can bound the multiple summation between two integrals:

\[ I(k) \leq \sum_{a_1=0}^{k} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} \leq I(k+1) \]

for all \( k \). Now observe that the integrand \( N_{a_1 \ldots a_i} \) is a homogeneous polynomial of total degree \( 2ndi - di^2 - id + ie \). It is easy to see that at each stage of evaluation the integrand increases its total degree but remains homogeneous. Thus, upon evaluation \( I(k) \) will result in a monomial of the form \( ck^{2ndi-di^2-id+ie} \) and \( I(k+1) \) will evaluate to \( c(k+1)^{2ndi-di^2-id+ie+i} \), with the same constant \( c \). We thus have

\[ c^{2ndi-di^2-id+ie} \leq \sum_{a_1=0}^{k} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} \leq c(k+1)^{2ndi-di^2-id+ie+i} \]

Dividing everything by \( k^{2ndi-di^2-id+ie} \), we have

\[ c \leq \frac{1}{k^{2ndi-di^2-id+ie}} \sum_{a_1=0}^{k} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} \leq c + \text{terms of order } (1/k) \]

Now letting \( k \to \infty \), we can conclude for sufficiently large \( k \) we must have

\[ \sum_{a_1=0}^{k} \cdots \sum_{a_i=0}^{a_i-1} N_{a_1 \ldots a_i} \sim ck^{2ndi-di^2-id+ie} + \text{lower order terms} \]

In fact, since \( I(k) \) is a monomial in \( k \), the constant \( c \) can be obtained by simply evaluating \( I(k) \) at \( k = 1 \).

\[ c = I(1) = \int_{0}^{1} da_1 \int_{0}^{a_1} da_2 \cdots \int_{0}^{a_i-1} N_{a_1 \ldots a_i} da_i = \left( \frac{1}{2} \right)^{di(2n-i-1)} \int_{0}^{1} da_1 \int_{0}^{a_1} da_2 \cdots \int_{0}^{a_i-1} \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \left( (a_j)^2 - (a_{j'})^2 \right)^d \right) \left( \prod_{j=1}^{i} (a_j)^{2d(n-i)+e} \right) da_i \]

Note that in the preceding calculation we have shown that the dimension of the \( K \)-types of \( \Pi_i \) grow like \( k^{di(2n-i)+i(e-d)} \). Thus,

**Proposition 3.2.** The Gelfand-Kirillov dimension of \( \Pi_i \) is

\[ D = 2di(2n-i) + 2i(e-d) \]

We also the following formula for the leading term of \( N_{G,i}(k) \)

\[ N_{G,i}(k) = C_{G,i}k^{di(2n-i)+i(e-d)} + \text{lower order terms} \]

with the constant \( C_{G,i} \) determined by the integral formula

\[ C_{G,i} \equiv \frac{1}{2^{di(2n-i-1)}} \int_{0}^{1} da_1 \int_{0}^{a_1} da_2 \cdots \int_{0}^{a_i-1} \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \left( (a_j)^2 - (a_{j'})^2 \right)^d \right) \left( \prod_{j=1}^{i} (a_j)^{2d(n-i)+e} \right) da_i \]
We’ll now proceed with the evaluation of the integral in the formula for $C_{G,i}$. Set

$$I_o(i,d,p) = \int_0^1 da_1 \int_0^{a_1} da_2 \cdots \int_0^{a_{i-1}} da_i \left( \prod_{j=1}^{i-1} \prod_{j' = j+1}^i (a_j)^2 - (a_{j'})^2 \right)^d \left( \prod_{j=1}^i (a_j)^p \right)$$

After a change of variables, $x_j = (a_j)^2$, $j=1,\ldots,i$ we have

$$I(i,d,p) = \frac{1}{2^i} \int_0^1 dx_1 \cdots \int_0^{x_{i-1}} dx_i \left( \prod_{j=1}^{i-1} \prod_{j' = j+1}^i (x_j - x_{j'})^2 \right) \left( \prod_{j=1}^i (x_j)^{\frac{p+1}{2}} \right)$$

To evaluate this integral we can employ Selberg’s integral formula (as stated by I.G. Macdonald, Some conjectures for root systems, SIAM J. Math. Anal., 13 (1982), 988-1007).

**Lemma 3.3** (Selberg, Macdonald). Let

$$J_i(a,b;c) = \int_{C_i} \prod_{j=1}^i \left( x_j^a (1 - x_j)^b \right) |D(x)|^{2c} dx_1 \cdots dx_i$$

where $C_i = \{(x_1,\ldots,x_i) \in \mathbb{R}^i \mid 0 \leq x_j \leq 1, \ j = 1,\ldots,i\}$,

$$D = \prod_{1 \leq j < j' \leq i} (x_j - x_{j'})$$

and $a, b, c$ are complex numbers satisfying

- $\text{Re}(a) > -1$
- $\text{Re}(b) > -1$
- $\text{Re}(c) > -\min\left(\frac{1}{n}, \frac{\text{Re}(a+1)}{n-1}, \frac{\text{Re}(b+1)}{n-1}\right)$

Then

$$J_i(a,b;c) = \prod_{j=1}^i \frac{\Gamma(jc+1) \Gamma(a+(k-1)c+1) \Gamma(b+(j-1)c+1)}{\Gamma(c+1) \Gamma(a+b+2+(i+j-2)c)}$$

We can readily massage the Selberg integral formula to handle our particular integral. First of all we note that the integrand of $(\ast)$ coincides with that of the Selberg integral when we take $a = (p-1)/2$, $b = 0$, and $c = d/2$ and restrict the domain of integration to \( \{ x \in \mathbb{R}^i \mid 0 \leq x_1 \leq x_{i-1} \leq \cdots \leq x_2 \leq x_1 \leq 1 \} \). We note also that our parameters $p$ and $d$ will always be non-negative real numbers.

Next we partition the domain of integration for Selberg’s integral, the unit cube in $\mathbb{R}^i$, into a set of $n!$ disjoint wedge-like regions:

$$D_\sigma = \{ x \in \mathbb{R}^i \mid 0 \leq x_{\sigma(i)} \leq x_{\sigma(i-1)} \leq \cdots \leq x_{\sigma(2)} \leq x_{\sigma(1)} \leq 1 \} \quad , \quad \sigma \in S_i$$

$$C_n = \bigcup_{\sigma \in S_i} D_\sigma$$

Note how we have used the permutation group $S_i$ to parameterize sets $D_\sigma$. We can thus write

$$J_i(p,0;\frac{d}{2}) = \sum_{\sigma \in S_i} \int_{D_\sigma} \left( \prod_{j=1}^i (x_j)^p \right) |D(x)|^d dx_1 \cdots dx_i$$

But, since the integrand is manifestly invariant with respect to interchanges of variables, we can, by a change of variables, write each of the integrals in the sum as an integral over

$$D_1 = \{ x \in \mathbb{R}^i \mid 0 \leq x_1 \leq x_{i-1} \leq \cdots \leq x_2 \leq x_1 \leq 1 \}$$
We thus obtain
\[
J_i \left( p, 0; \frac{d}{2} \right) = \sum_{\sigma \in S_i} \int_{D_1} \left( \prod_{j=1}^{i} (x_{\sigma(j)})^p \right) |D(\sigma(x))|^d \, dx_{\sigma(1)} \cdots dx_{\sigma(i)}
\]
But now we note that the integrand is invariant with respect to permutations of variables, and so
\[
J_i \left( p, 0; \frac{d}{2} \right) = \sum_{\sigma \in S_i} \int_{\mathcal{D}_1} \left( \prod_{j=1}^{i} (x_{\sigma(j)})^p \right) |D(\sigma(x))|^d \, dx_{\sigma(1)} \cdots dx_{\sigma(i)}
\]
\[
= \sum_{\sigma \in S_i} \int_{0}^{1} dx_{1} \cdots \int_{0}^{x_{-1}} dx_{i} \left( \prod_{j=1}^{i} (x_{j})^p \right) |D(x)|^d
\]
And so we can conclude that
\[
I (i, d, p) = \frac{1}{2^i i!} J_i \left( p, 0; \frac{d}{2} \right)
\]
Putting this all together we have,

**Proposition 3.4.**

(i) The Gelfand-Kirillov dimension \( D \) of a representation \( \Pi_i \) is given by
\[
\text{GK dim} \left( \Pi_i \right) = 2i (2n - 1) + 2i (e - d + 1)
\]
(ii) The leading coefficient of \( \text{NG}_i (k) \) is
\[
C_{G,i} \equiv \frac{1}{i! 2^{i(2n - d(i + 1) + 1)}} J_i \left( \frac{2d(n - i) + e - 1}{2}, 0; \frac{d}{2} \right)
\]
\[
= \frac{1}{i! 2^{i(2n - i - 1) + 1}} \prod_{j=1}^{i} \frac{\Gamma \left( \frac{jd}{2} + 1 \right) \Gamma \left( d(n - i) + \frac{e + 1 + (j - 1)d}{2} \right) \Gamma \left( \frac{d(j - 1)}{2} + 1 \right)}{\Gamma \left( \frac{d}{2} + 1 \right) \Gamma \left( d(n - i) + \frac{e + 1 + (i + j - 2)d}{2} \right)}
\]

3.3. Evaluating \( D_{G,i} \). Recall
\[
D_{G,i} \equiv \prod_{\alpha \in \Delta^{\pm}_{\Sigma,i}} \langle \rho_K, \alpha \rangle
\]
where
\[
\Delta^{\pm}_{\Sigma,i} = \{ \alpha \in \Delta^\pm (\mathfrak{t}; t_1 + t_0) \mid \langle \alpha, \gamma_j \rangle \neq 0 \text{ for some } j \in \{1, \ldots, i\} \}
\]
Once one has expressions for the \( \gamma_i \) in terms of the basis vectors \( \{e_i\} \) of the Euclidean space in which the roots of \( \mathfrak{t} \) are usually expressed, the \( \Delta^{\pm}_{\Sigma,i} \) are evaluated fairly easily.

Below we present a summary of the results of these calculations; we’ll confine the explicit calculations of the denominators to an appendix.

4. SUMMARY: GELFAND-KIRILLOV DIMENSIONS AND BERNSTEIN DEGREES

**Theorem 4.1.** Let \( G \) be one of the Lie groups listed in Table 1. Let \( n \) denote the real rank of \( G \), and let \( d \) and \( e \) denote, respectively, the common multiplicity of the short and long roots in the restricted root system \( \Delta (\mathfrak{t}; t_1) \). Let \( \Pi_i, i = 1, \ldots, n \) be the \( (\mathfrak{g}, K) \)-module of the unitarizable constituent of one of Sahi’s representations. Then

- The Gelfand-Kirillov of \( \Pi_i \) is
\[
\text{GK dim} \left( \Pi_i \right) = 2i (2n - 1) + 2i (e - d + 1)
\]
• The Bernstein degree of $\Pi_i$ is

$$B_{\Pi_i} = \frac{\varepsilon_{\Sigma}(i) \cdot (d(2n - 1) + i(e - d + 1))!C_{G,i}}{\left(\prod_{\alpha \in \Delta_{\Sigma_i}} \langle \rho_K, \alpha \rangle\right)}$$

where

$$\varepsilon_{\Sigma}(i) = \begin{cases} 2 & \text{if } i = n \text{ and } \Sigma \sim D_n \\ 1 & \text{otherwise} \end{cases}$$

$$C_{G,i} = \prod_{j=1}^{i} \frac{\Gamma\left(\frac{i(i+1)}{2} + 1\right) \Gamma\left(d(n-i) + \frac{e+1+(j-1)d}{2}\right) \Gamma\left(d(n-i) + \frac{e+3+i+j-2d}{2}\right)}{\Gamma\left(\frac{i}{2} + 1\right) \Gamma\left(d(n-i) + \frac{e+3+i+j-2d}{2}\right)}$$

and the remaining data is contained in the following table:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\Sigma_n$</th>
<th>$d$</th>
<th>$e$</th>
<th>$D_{G,i}$ = $\prod_{\alpha \in \Delta_{\Sigma_i}} \langle \rho_K, \alpha \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL_{2n}$</td>
<td>$SO_{2n}$</td>
<td>$D_n$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$SO_{2n,2n}$</td>
<td>$SO_{2n} \times SO_{2n}$</td>
<td>$D_n$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$E_{7}(7)$</td>
<td>$SU_8$</td>
<td>$D_3$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$SO_{p,q}$</td>
<td>$SO_p \times SO_q$</td>
<td>$D_2$</td>
<td>2</td>
<td>$\frac{p+q-4}{2}$</td>
</tr>
<tr>
<td>$Sp(n, C)$</td>
<td>$Sp_n$</td>
<td>$C_n$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$SL(2n, C)$</td>
<td>$SU_{2n}$</td>
<td>$C_n$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$SO(4n, C)$</td>
<td>$SO_{4n}$</td>
<td>$C_n$</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$E_{7}(C)$</td>
<td>$E_7$</td>
<td>$C_3$</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>$SO(p, C)$</td>
<td>$SO_p$</td>
<td>$C_2$</td>
<td>2</td>
<td>$p - 4$</td>
</tr>
<tr>
<td>$Sp(n, n)$</td>
<td>$Sp_n \times Sp_n$</td>
<td>$C_n$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$SL(2n, \mathbb{H})$</td>
<td>$Sp_{2n}$</td>
<td>$C_n$</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>$SO(p, 1)$</td>
<td>$SO(p)$</td>
<td>$C_1$</td>
<td>0</td>
<td>$p - 1$</td>
</tr>
</tbody>
</table>

Appendix A. Explicit Calculations of the Denominators $D_{G,i}$

A.1. Useful Formulas. Here’s a table from Bourbaki

<table>
<thead>
<tr>
<th>$A_n$</th>
<th>$\Delta^+ = {e_i \pm e_j \mid 1 \leq i &lt; j \leq n + 1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$\frac{1}{2}(ne_1 + (n - 2)e_2 + \cdots + (2 - n)e_n - ne_{n+1})$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$\Delta^+ = {e_i \mid 1 \leq i &lt; n} \cup {e_i \pm e_j \mid 1 \leq i &lt; j \leq n}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\frac{1}{2}((2n - 1)e_1 + (2n - 3)e_2 + \cdots + 3e_{n-1} + e_n)$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\Delta^+ = {2e_i \mid 1 \leq i &lt; n} \cup {e_i \pm e_j \mid 1 \leq i &lt; j \leq n}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$ne_1 + (n - 1)e_2 + \cdots + 2e_{n-1} + e_n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\Delta^+ = {e_i \pm e_j \mid 1 \leq i &lt; j \leq n}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$(n - 1)e_1 + (n - 2)e_2 + \cdots + e_{n-1}$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\Delta^+ = {\pm e_i \pm e_j \mid 1 \leq i &lt; j \leq 7} \cup {e_8 - e_7} \cup \left{\frac{1}{2}(e_8 - e_7) + \sum_{i=1}^{6} (e_i \pm e_i) \mid \sum_{i=1}^{6} v(i)\right}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$e_2 + 2e_3 + 3e_4 + 4e_5 + 5e_6 - \frac{17}{2}e_7 + \frac{17}{2}e_8$</td>
</tr>
</tbody>
</table>

Formula A.1.

\[
\prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} (N - j - j') \left(j' - j\right) = \prod_{j=1}^{i-1} \left(\frac{(N - 2j + 1)!}{(N - i - j - 1)!} (i - j)\right)
\]
Proof.

\[ \text{LHS} = \prod_{j=1}^{i-1} \prod_{j' = j+1}^{i} (N - j - j') (j' - j) \]

\[ = \prod_{j=1}^{i-1} (N - 2j - 1) (1) (N - 2j - 2) (2) \cdots (N - i - j + 1) (i - j - 1) (N - i - j) (i - j) \]

\[ = (N - 3) (N - 4) (2) \cdots (N - i) (i - 2) (N - i - 1) (i - 1) \]

\[ \cdot (N - 5) (N - 6) (2) \cdots (N - i - 1) (i - 2) (N - i - 2) (i - 2) \]

\[ = \left( \frac{(N - 3)!}{(N - i - 2)! (i - 1)!} \right) \left( \frac{(N - 5)!}{(N - i - 3)! (i - 2)!} \right) \cdots \left( \frac{(N - 2i + 3)!}{(N - 2i + 1)! (2)!} \right) \left( \frac{(N - 2i + 1)!}{(N - 2i)! (1)!} \right) \]

\[ = \prod_{j=1}^{i-1} \left( \frac{(N - 2j + 1)!}{(N - i - j - 1)! (i - j)!} \right) = \text{RHS} \]

Formula A.2.

\[ \prod_{j=1}^{i} \prod_{j' = i+1}^{n} (N - j - j') (j' - j) = \prod_{j=1}^{i} \left( \frac{(N - i - j - 1)! (n - j)!}{(N - n - j - 1)! (i - j)!} \right) \]

Proof.

\[ \text{LHS} = \prod_{j=1}^{i} \prod_{j' = i+1}^{n} (N - j - j') (j' - j) \]

\[ = \prod_{j=1}^{i} (N - i - j - 1) (i - j + 1) (N - i - j - 2) (i - j + 2) \cdots (N - n - j + 1) (n - 1 - j) (N - n - j) (n - j) \]

\[ = (N - i - 2) (i) (N - i - 3) (i + 1) \cdots (N - n) (n - 2) (N - n - 1) (n - 1) \]

\[ \cdot (N - i - 3) (i - 1) (N - 2 - i - 2) (i + 2 - 2) \cdots (N - n - 1) (n - 3) (N - n - 2) (n - 2) \]

\[ \vdots \]

\[ \cdot (N - 2i - 2) (2) (N - 2i - 3) (3) \cdots (N - n - i + 2) (n - i) (N - n - i + 1) (n - i + 1) \]

\[ \cdot (N - 2i - 1) (1) (N - 2i - 2) (2) \cdots (N - n - i + 1) (n - i - 1) (N - n - i) (n - i) \]

\[ = \left( \frac{(N - i - 2)! (n - 1)!}{(N - n - 2)! (i - 1)!} \right) \left( \frac{(N - i - 3)! (n - i + 1)!}{(N - n - 3)! (i - 2)!} \right) \cdots \left( \frac{(2n - 2i)! (n - i + 1)!}{(n - i)!} \right) \]

\[ \cdot \left( \frac{(2n - 2i - 1)! (n - i)!}{(n - i - 1)! 0!} \right) \]

\[ = \prod_{j=1}^{i} \left( \frac{(N - i - j - 1)! (n - j)!}{(N - n - j - 1)! (i - j)!} \right) = \text{RHS} \]

Formula A.3.

\[ \left( \prod_{j=1}^{i-1} \prod_{j' = j+1}^{i} (N - j - j') (j' - j) \right) \left( \prod_{j=1}^{i} \prod_{j' = i+1}^{n} (N - j - j') (j' - j) \right) = \prod_{j=1}^{i} \left( \frac{(N - 2j + 1)!}{(N - n - j - 1)! (n - j)!} \right) \]
Proof. Using Formulas A and B we have

\[ LHS = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \]

\[ = \left( \prod_{j=1}^{i-1} \left( \frac{(N - 2j + 1)!}{(N - i - j - 1)!} (i - j)! \right) \right) \left( \prod_{j=1}^{i} \left( \frac{(N - 2j - 1)!}{(N - n - j - 1)!} (n - j)! \right) \right) \]

\[ = \left( \prod_{j=1}^{i-1} \frac{(N - 2j + 1)!}{(N - n - j - 1)!} (n - j)! \right) \left( \prod_{j=1}^{i} \frac{(N - 2j - 1)!}{(N - n - i - 1)!} (n - j)! \right) \]

\[ = \prod_{j=1}^{i} \frac{(N - 2j + 1)!}{(N - n - j - 1)!} (n - j)! = RHS \]

Formula A.4. (Another version of the preceding formula)

\[ (D) \quad \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) = \prod_{j=1}^{i} \frac{(N - 2j + 1)!}{(N - n - j - 1)!} (n - j)! \]

Proof. In view of the preceding formula suffices to show that

\[ \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \]

\[ RHS = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \left( \prod_{j'=i+1}^{n} (N - i - j') (j' - i) \right) \]

\[ = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) \left( \prod_{j'=i+1}^{n} (N - i - j') (j' - i) \right) \]

\[ = \prod_{j=1}^{i} \frac{(N - 2j + 1)!}{(N - n - j - 1)!} (n - j)! = LHS \]

The Calculations, Case by Case

A.2. \( G = SL_{2n} \), \( K = SO_{2n} \), \( \Sigma = D_n \)

\[ \gamma_1 = [2, 0, \ldots, 0] = 2e_1 \]
\[ \gamma_2 = [-2, 2, 0, \ldots, 0] = 2e_2 \]
\[ \vdots \]
\[ \gamma_{n-2} = [0, \ldots, 0, -2, 2, 0, 0] = 2e_{n-2} \]
\[ \gamma_{n-1} = [0, \ldots, 0, -2, 2, 2] = 2e_{n-1} \]
\[ \gamma_n = [0, \ldots, 0, -2, 2] = 2e_n \]

\[ a_1\gamma_1 + \cdots + a_n\gamma_n = [2a_1 - 2a_2, 2a_2 - 2a_3, \ldots, 2a_{n-2} - 2a_{n-1}, 2a_{n-1} - 2a_n, 2a_n] \]
\[ \rho_K = (n - 1) e_1 + (n - 2) e_2 + \cdots + e_{n-1} \]
Thus, \( \gamma \) \\

So

\[ \langle \rho_K, e_j \rangle = n - j \]

In this case we have \( \gamma_j = 2e_j \), and so

\[ \Delta^+_{\Sigma, i} = \{ e_j \pm e_{j'} \mid 1 \leq j < j' \leq i \} \cup \{ e_j \pm e_{j'} \mid 1 \leq j \leq i, \ i + 1 < j' \leq n \} \]

Hence,

\[
\prod_{\alpha \in \Delta^+_{\Sigma, i}} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^i \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right) \left( \prod_{j=1}^i \prod_{j'=i+1}^n \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right)
\]

\[= \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^i (n-j+n-j') (j'-j) \right) \left( \prod_{j=1}^i \prod_{j'=i+1}^n (n-j+n-j') (j'-j) \right) \]

\[= \left( \prod_{j=1}^i (2n-2j+1)! (n-j)! \right) \left( (2n-2j+1)! (n-j) \right) \]

\[= \prod_{j=1}^i ((2n-2j+1)! (n-j)) \]

\[= \frac{(n-1)!}{(n-i-1)!} \prod_{j=1}^i (2n-2j+1)! \]

where in the final step we applied Formula C with \( N = 2n \).

Thus,

\[
\prod_{\alpha \in \Delta^+_{\Sigma, i}} \langle \rho_K, \alpha \rangle = \frac{(n-1)!}{(n-i-1)!} \prod_{j=1}^i (2n-2j+1)! 
\]

A.3. \( G = SO_{2n,2n} \times K = SO_{2n} \times SO_{2n} \times D_n \)

\[
\begin{align*}
\gamma_1 &= [1,0,\ldots,0;1,0,\ldots,0] = e_1 + \tilde{e}_1 \\
\gamma_2 &= [-1,1,0,\ldots,0;1,0,\ldots,0] = e_2 + \tilde{e}_2 \\
\gamma_3 &= [0,-1,1,0,\ldots,0;0,-1,1,0,\ldots,0] = e_3 + \tilde{e}_3 \\
&\vdots \\
\gamma_{n-2} &= [0,\ldots,0,-1,1,0,\ldots,0,-1,1,0,0] = e_{n-2} + \tilde{e}_{n-2} \\
\gamma_{n-1} &= [0,\ldots,0,-1,1,1,0,\ldots,0,-1,1,1] = e_{n-1} + \tilde{e}_{n-1} \\
\gamma_n &= [0,\ldots,0,-1,1,0,\ldots,0,-1,1,0,\ldots,0] = e_n + \tilde{e}_n \\
a_1\gamma_1 + \cdots + a_n\gamma_n &= [a_1 - a_2,\ldots,a_{n-1} - a_n,a_{n-1} + a_n,a_1 - a_2,\ldots,a_{n-1} - a_n,a_{n-1} + a_n] \\
\rho_K &= (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1} + (n-1)\tilde{e}_1 + (n-2)\tilde{e}_2 + \cdots + \tilde{e}_{n-1} \\
\Rightarrow \quad \langle \rho_K, e_j \rangle = \langle \rho_K, \tilde{e}_j \rangle = n - j
\end{align*}
\]

In this case we have

\[ \Delta^+_{\Sigma, i} = \{ e_j \pm e_{j'} \mid 1 \leq j < j' \leq i \} \cup \{ e_j \pm e_{j'} | 1 \leq j \leq i, \ i + 1 < j' \leq n \} \]

\[\cup \{ \tilde{e}_j \pm \tilde{e}_{j'} \mid 1 \leq j < j' \leq i \} \cup \{ \tilde{e}_j \pm \tilde{e}_{j'} | 1 \leq j \leq i, \ i + 1 < j' \leq n \} \]
\[ \prod_{\alpha \in \Delta_{\Sigma,i}} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \langle e_j + e_{j'}, \rho_K \rangle \langle e_j - e_{j'}, \rho_K \rangle \right) \left( \prod_{j=1, j' = j+1}^{i} \langle e_j + e_{j'}, \rho_K \rangle \langle e_j - e_{j'}, \rho_K \rangle \right) \cdot \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \langle e_j + \hat{e}_{j'}, \rho_K \rangle \langle e_j - \hat{e}_{j'}, \rho_K \rangle \right) \left( \prod_{j=1, j' = j+1}^{i} \langle e_j + \hat{e}_{j'}, \rho_K \rangle \langle e_j - \hat{e}_{j'}, \rho_K \rangle \right) \]
\[ = \left( \prod_{j=1}^{i} \frac{(2n - j + 1)!}{(n - j - 1)! (n - j)!} \right)^2 \left( \prod_{j=1}^{n} \frac{(2n - 2j + 1)!}{(n - j)!} \right)^2 \]
\[ = \left( \frac{(n - 1)!}{(n - i - 1)!} \prod_{j=1}^{i} (2n - 2j + 1)! \right)^2 \]

A.4. \( G = E_7 \) \( \Sigma = D_3 \)

\( \gamma_1 = [0, 0, 0, 1, 0, 0, 0] = \frac{1}{2} (e_1 + e_2 + e_3 + e_4 - e_5 - e_6 - e_7 - e_8) \)
\( \gamma_2 = [0, 1, 0, -1, 0, 0, 0] = \frac{1}{2} (e_1 + e_2 - e_3 - e_4 + e_5 + e_6 - e_7 - e_8) \)
\( \gamma_3 = [0, 1, 0, 0, 0, -1, 0] = \frac{1}{2} (e_1 + e_2 - e_3 - e_4 - e_5 + e_6 + e_7 + e_8) \)
\( a_1 \gamma_1 + a_2 \gamma_2 + a_3 \gamma_3 = [0, a_2 + a_3, 0, a_1 - a_2, 0, a_2 - a_3, 0] \)
\( \rho_K = \frac{1}{2} (7e_1 + 5e_2 + 3e_3 + e_4 - e_5 - 3e_6 - 5e_7 - 7e_8) \)

We carried out the calculation of \( \prod_{\alpha \in \Delta_{\Sigma,i}} \langle \rho_K, \alpha \rangle \) for \( i = 1, 2, 3 \) using Maple, and obtained the following results.

\[ \prod_{\alpha \in \Delta_{\Sigma,1}} \langle \rho_K, \alpha \rangle = 2^{12}3^{5}5^{3}7 \]
\[ \prod_{\alpha \in \Delta_{\Sigma,2}} \langle \rho_K, \alpha \rangle = 2^{16}3^{7}5^{3}7 \]
\[ \prod_{\alpha \in \Delta_{\Sigma,3}} \langle \rho_K, \alpha \rangle = 2^{16}3^{7}5^{3}7 \]

A.5. \( G = SO_{p,q} \) \( \Sigma = D_2 \)

\( \gamma_1 = [1, 0, \ldots, 0; 1, 0, \ldots, 0] = e_1 + \hat{e}_1 \)
\( \gamma_2 = [1, 0, \ldots, 0; -1, 0, \ldots, 0] = e_1 - \hat{e}_1 \)
\( a_1 \gamma_1 + a_2 \gamma_2 = [a_1 + a_2, 0, \ldots, 0; a_1 - a_2, 0, \ldots, 0] \)

There are four case here, corresponding to \( p \) and \( q \) being even or odd. However, since

\[ \Delta^+_{\Sigma,i} = \left( \Delta^+_{\Sigma,i} \cap \Delta_{\text{so}(p)} \right) \cup \left( \Delta^+_{\Sigma,i} \cap \Delta_{\text{so}(q)} \right) \]
\[ = \left( \Delta^+_{\Sigma,i} \right)_p \cup \left( \Delta^+_{\Sigma,i} \right)_q \]
we'll have

\[
\prod_{\alpha \in \Delta_{\Sigma,i}^+} \langle \rho_K, \alpha \rangle = \left( \prod_{\alpha \in (\Delta_{\Sigma,i}^+)_p} \langle \rho_K, \alpha \rangle \right) \left( \prod_{\alpha \in (\Delta_{\Sigma,i}^+)_q} \langle \rho_K, \alpha \rangle \right)
\]

and it will suffice to develop formulas for

\[
\prod_{\alpha \in (\Delta_{\Sigma,i}^+)_2k} \langle \rho_K, \alpha \rangle
\]

and

\[
\prod_{\alpha \in (\Delta_{\Sigma,i}^+)_2k+1} \langle \rho_K, \alpha \rangle
\]

Another convenience for the \(SO(p, q)\) case is that

\[
\Delta_{\Sigma,1}^+ = \Delta_{\Sigma,2}^+
\]

A.5.1. Even Case. \((\Delta_{\Sigma,i}^+)^{2k}\).

\[
\rho_K = (k - 1)e_1 + (k - 2)e_2 + \cdots + e_{k-1}
\]

\[
\Rightarrow \quad \langle \rho_K, e_j \rangle = k - j
\]

\[
(\Delta_{\Sigma,i}^+)^{2k} = \{e_1 + e_j \mid 2 \leq j \leq k\} \cup \{e_1 - e_j \mid 2 \leq j \leq k\}
\]

And so

\[
\prod_{\alpha \in (\Delta_{\Sigma,i}^+)^{2k}} \langle \rho_K, \alpha \rangle = \prod_{j=2}^{k} \langle \rho_K, e_1 + e_j \rangle \langle \rho_K, e_1 - e_j \rangle
\]

\[
= \prod_{j=2}^{k} (2k - j - 1) (j - 1)
\]

\[
= (k - 1) (2k - 3)!
\]

\[
= \frac{1}{2} (2k - 2)!
\]

A.5.2. Odd Case. \((\Delta_{\Sigma,i}^+)^{2k+1}\).

\[
\rho_K = \frac{1}{2} (2k - 1) e_1 + \frac{1}{2} (2k - 3) e_2 + \cdots + \frac{1}{2} e_{k-1}
\]

\[
\Rightarrow \quad \langle \rho_K, e_j \rangle = \frac{1}{2} (2k - 2j + 1)
\]

\[
(\Delta_{\Sigma,i}^+)^{2k+1} = \{e_1 + e_j \mid 2 \leq j \leq k\} \cup \{e_1 - e_j \mid 2 \leq j \leq k\} \cup \{e_1\}
\]
And so

\[
\prod_{\alpha \in (\Delta_{\Sigma,i})_{2k+1}} \langle \rho_K, \alpha \rangle = \langle \rho_K, e_1 \rangle \prod_{j=2}^{k} \langle \rho_K, e_j + e_j \rangle \langle \rho_K, e_1 - e_j \rangle
\]

\[
= \frac{1}{2} (2k - 1) \prod_{j=2}^{k} (2k - j) (j - 1)
\]

\[
= \frac{1}{2} (2k - 1) (2k - 2)!
\]

\[
= \frac{1}{2} (2k - 1)!
\]

\[
= \frac{1}{2} (2k + 1 - 2)!
\]

And so we see that whether \( p \) is even or odd

\[
\prod_{\alpha \in (\Delta_{\Sigma,i})_p} \langle \rho_K, \alpha \rangle = \frac{1}{2} (p - 2)!
\]

Hence, for \( SO(p,q) \)

\[
\prod_{\alpha \in \Delta_{\Sigma,i}} \langle \rho_K, \alpha \rangle = \frac{1}{4} (p - 2)! (q - 2)!
\]

A.6. \( G = Sp(n, \mathbb{C}) \). \( K = Sp_n \) \( \Sigma = C_n \)

\[\gamma_1 = [2, 0, \ldots, 0] = 2e_1\]
\[\gamma_2 = [-2, 2, 0, \ldots, 0] = 2e_2\]
\[\vdots\]
\[\gamma_n = [0, \ldots, 0, -2, 2] = 2e_n\]

\[a_1\gamma_1 + \cdots + a_n\gamma_n = [2a_1 - 2a_2, \ldots, 2a_{n-1} - 2a_n, 2a_n]\]

\[\rho_K = ne_1 + (n - 1)e_2 + \cdots + 2e_{n-1} + e_n\]

\[\Rightarrow \langle \rho_K, e_j \rangle = (n - j + 1)\]

In this case we have \( \gamma_j = 2e_j \), and so

\[\Delta_{\Sigma,i}^+ = \{ e_j \pm e_j' | 1 \leq j < j' \leq i \} \cup \{ e_j \pm e_j' | 1 \leq j \leq i, i + 1 < j' \leq n \} \cup \{ 2e_j | 1 \leq j \leq i \}\]
\[
\prod_{\alpha \in \Delta_{2,i}^+} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{i} \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=i+1}^{n} \langle \rho_K, e_j + e_{j'} \rangle \langle \rho_K, e_j - e_{j'} \rangle \right) \\
\cdot \left( \prod_{j=1}^{i} \langle \rho_K, 2e_j \rangle \right) \\
= \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{i} (2n - j - j' + 2) (j' - j) \right) \left( \prod_{j=1}^{i} \prod_{j'=i+1}^{n} (2n - j - j' + 2) (j' - j) \right) \\
\cdot \left( \prod_{j=1}^{i} (2n - 2j + 2) \right) \\
= \left( \frac{i!}{(2n + 2 - n - j - 1)! (n-j)!} \right) \frac{2^i n!}{(n-i)!} \\
= \frac{2^i n! (n-i)!}{(n-i)!} \left( \prod_{j=1}^{i} (2n - 2j + 3)! \right) \frac{(n-i)!}{n!} \\
= 2^i \left( \prod_{j=1}^{i} (2n - 2j + 3)! \right)
\]

A.7. \( G = SL(2n, \mathbb{C}) \). \( K = SU_{2n} \) \( \Sigma = C_n \)

\[
\gamma_1 = [1, 0, \ldots, 0, 1] = e_1 - e_{2n} \\
\gamma_2 = [-1, 1, 0, \ldots, 0, 1, -1] = e_2 - e_{2n-1} \\
\vdots \\
\gamma_n = [0, \ldots, -1, 2, -1, \ldots, 0] = e_n - e_{n+1}
\]

\[
a_1 \gamma_1 + \cdots + a_n \gamma_n = [a_1 - a_2, \ldots, a_{n-1} - a_n, 2a_n, a_{n-1} - a_n, \ldots, a_1 - a_2] \]

\[
\rho_K = \frac{1}{2} (2n - 1) e_1 + (2n - 3) e_2 + \cdots - (2n - 3) e_{2n-1} - (2n - 1) e_{2n} \\
\Rightarrow \langle \rho_K, e_j \rangle = \frac{1}{2} (2n - 2j + 1)
\]

\[
\Delta_{2,i}^+ = \{ e_j - e_{j'} \mid 1 \leq j \leq i ; \ j < j' \leq 2n \} \\
\quad \cup \{ e_j - e_{j'} \mid i + 1 \leq j \leq 2n - i ; \ 2n - i < j' < n \} \\
\quad \cup \{ e_j - e_{j'} \mid 2n - i < j < j' \leq 2n \} 
\]
\[
\prod_{\alpha \in \Delta_{\Sigma,i}^+} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{2n} \langle \rho_K, e_j - e_{j'} \rangle \right) \left( \prod_{j=i+1}^{2n-i} \prod_{j'=2n-i+1}^{2n} \langle \rho_K, e_j - e_{j'} \rangle \right) \left( \prod_{j=2n-i+1}^{2n-1} \prod_{j'=j+1}^{2n} \langle \rho_K, e_j - e_{j'} \rangle \right)
\]

\[
= \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{2n} \frac{1}{2} (j' - j) \right) \left( \prod_{j=i+1}^{2n-i} \prod_{j'=2n-i+1}^{2n} \frac{1}{2} (j' - j) \right) \left( \prod_{j=2n-i+1}^{2n-1} \prod_{j'=j+1}^{2n} \frac{1}{2} (j' - j) \right)
\]

\[
= \left( \prod_{j=1}^{i} \left( \frac{1}{2} \right)^{2n-j} (2n-j)! \right) \left( \prod_{j=i+1}^{2n-i} \left( \frac{1}{2} \right)^{i(2n-2i)} (2n-j)! \right) \left( \prod_{j=2n-i+1}^{2n} \left( \frac{1}{2} \right)^{i(2n-2i)}/2 (2n-j)! \right)
\]

\[
= \left( \frac{1}{2} \right)^{i(4n-2i-1)/2} \left( \prod_{j=1}^{2n-2i} \frac{1}{(2n-2i-j)!} \right) \left( \prod_{j=1}^{2n-2i} \frac{1}{(2n-2i-j)!} \right)
\]

A.8. \( G = SO(4n, \mathbb{C}) \), \( K = SO_{4n} \), \( \Sigma = C_n \)

\[
\gamma_1 = [0, 1, 0, \ldots, 0] = e_1 + e_2
\]

\[
\gamma_2 = [0, -1, 0, 1, 0, \ldots, 0] = e_3 + e_4
\]

\[
\gamma_3 = [0, 0, 0, -1, 0, 1, 0, \ldots, 0] = e_5 + e_6
\]

\[ \vdots \]

\[
\gamma_{n-1} = [0, \ldots, 0, -1, 0, 1, 0, \ldots, 0] = e_{2n-3} + e_{2n-2}
\]

\[
\gamma_n = [0, \ldots, 0, 0, -1, 0, 2] = e_{2n-1} + e_{2n}
\]

\[
a_1 \gamma_1 + \cdots + a_n \gamma_n = [0, a_1 - a_2, 0, a_2 - a_3, \ldots, 0, a_{n-1} - a_n, 0, 2a_n]
\]

\[
\rho_K = (2n-1) e_1 + (2n-2) e_2 + \cdots + e_{2n-1}
\]

\[
= \langle \rho_K, e_j \rangle = 2n - j
\]

\[
\Delta_{\Sigma,i}^+ = \{ e_j + e_{j'} | 1 \leq j \leq 2i, \ j < j' \leq 2n \}
\]

\[
\cup \{ e_{2j-1} - e_{j'} | 1 \leq j \leq i, \ 2j < j' \leq 2n \}
\]

\[
\cup \{ e_{2j} - e_{j'} | 1 \leq j \leq i \ }, \ 2j < j' \leq 2n \}
\[
\prod_{\alpha \in \Delta^\perp_{E,1}} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} \langle \rho_K, e_j + e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{2n} \langle \rho_K, e_{2j-1} - e_{j'} \rangle \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{2n} \langle \rho_K, e_{2j} - e_{j'} \rangle \right)
\]

\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j') \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{2n} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{2n} (j' - 2j) \right)
\]

\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j') \right) \left( \prod_{j=1}^{i} \prod_{j'=2j}^{2n} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{2n} (j' - 2j) \right)
\]

\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j') (j - j') \right)
\]

Adopting Formula (D) to the case at hand \((i \to 2i, n \to 2n, N \to 2n)\)

\[
\left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j') (j' - j) \right) = \frac{2i}{j=1} \frac{(4n - 2j + 1)!}{(2n - j - 1)! (2n - j)!} = \frac{2i}{j=1} \frac{(4n - 2j + 1)!}{(2n - j - 1)! (2n - j)!}
\]

we conclude

\[
\prod_{\alpha \in \Delta^\perp_{E,1}} \langle \rho_K, \alpha \rangle = \prod_{j=1}^{2i} \frac{(4n - 2j + 1)!}{(2n - j - 1)! (2n - j)!}
\]

A.9. \(G = E_7 (\mathbb{C}), \quad K = E_7 \quad \Sigma = C_3\)

\[
\gamma_1 = [1, 0, 0, 0, 0, 0, 0] = -e_7 + e_8 \\
\gamma_2 = [-1, 0, 0, 0, 1, 0] = e_5 + e_6 \\
\gamma_3 = [0, 0, 0, 0, 0, -1, 2] = -e_5 + e_6 \\
\]

\[
a_1 \gamma_1 + a_2 \gamma_2 + a_3 \gamma_3 = [a_1 - a_2, 0, 0, 0, 2a_2 - a_3, 2a_3]
\]

We carried out the calculation of \(\prod_{\alpha \in \Delta^\perp_{E,1}} \langle \rho_K, \alpha \rangle\) for \(i = 1, 2, 3\) using Maple, and obtained the following results.

\[
\prod_{\alpha \in \Delta^\perp_{E,1}} \langle \rho_K, \alpha \rangle = 2^{29}3^{13}5^{7}7^{4}11^{3}13^{2}17 \\
\prod_{\alpha \in \Delta^\perp_{E,2}} \langle \rho_K, \alpha \rangle = 2^{42}3^{19}5^{6}7^{6}11^{3}13^{2}17 \\
\prod_{\alpha \in \Delta^\perp_{E,3}} \langle \rho_K, \alpha \rangle = 2^{42}3^{19}5^{6}7^{6}11^{3}13^{2}17
\]

A.10. \(G = SO (\rho, \mathbb{C}), \quad K = SO_p \quad \Sigma = C_2\)

\[
\gamma_1 = [0, 1, 0, \ldots, 0] = e_1 + e_2 \\
\gamma_2 = [0, -1, 0, 1, 0, \ldots, 0] = e_3 + e_4 \\
\]

\[
a_1 \gamma_1 + a_2 \gamma_2 = [a_1 - a_2, 0, a_2, 0, a_2, 0, \ldots, 0]
\]
\[ p = 2k. \]

\[
\rho_K = (k - 1) e_1 + (k - 2) e_2 + \cdots + e_{k-1}
\]
\[ \Rightarrow \langle \rho_K, e_j \rangle = k - j \]

\[ \Delta^+_{\Sigma,i} = \{ e_j + e_{j'} \mid 1 \leq j \leq 2i, \ j < j' \leq k \} \]
\[ \cup \{ e_{2j-1} - e_{j'} \mid 1 \leq j \leq i, \ 2j < j' \leq k \} \]
\[ \cup \{ e_{2j} - e_{j'} \mid 1 \leq j \leq i, \ 2j < j' \leq k \} \]

\[
\prod_{\alpha \in \Delta^+_{\Sigma,i}} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} \langle \rho_K, e_j + e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j-1} - e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j} - e_{j'} \rangle \right)
\]
\[ = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j') \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j) \right)
\]
\[ = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j') \right) \left( \prod_{j=1}^{2i} (2k - j - j') \right)
\]
\[ = \prod_{j=1}^{2i} \left( \frac{(2k - 2j + 1)!}{(k - j - 1)!} \right) (k - j)!
\]
\[ = \prod_{j=1}^{2i} (2k - 2j + 1)! (k - j)
\]
\[ = \left( \frac{1}{2} \right) \prod_{j=1}^{2i} (2k - 2j + 1)! (2k - 2j)
\]

\[ p = 2k + 1. \]

\[
\rho_K = (k - \frac{1}{2}) e_1 + (k - \frac{3}{2}) e_2 + \cdots + \frac{1}{2} e_k
\]
\[ \Rightarrow \langle \rho_K, e_j \rangle = \frac{1}{2} (2k - 2j + 1) \]

\[ \Delta^+_{\Sigma,i} = \{ e_j + e_{j'} \mid 1 \leq j \leq 2i, \ j < j' \leq k \} \]
\[ \cup \{ e_{2j-1} - e_{j'} \mid 1 \leq j \leq i, \ 2j < j' \leq k \} \]
\[ \cup \{ e_{2j} - e_{j'} \mid 1 \leq j \leq i, \ 2j < j' \leq k \} \]
\[ \cup \{ e_j \mid 1 \leq j \leq 2i \} \]
\[
\prod_{\alpha \in \Delta_{\Sigma,i}^+} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} \langle \rho_K, e_j + e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j-1} - e_{j'} \rangle \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{k} \langle \rho_K, e_{2j} - e_{j'} \rangle \right)
\]
\[
\times \left( \prod_{j=1}^{2i} \langle \rho_K, e_j \rangle \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j' + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{k} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{k} (j' - 2j) \right)
\]
\[
\times \left( \prod_{j=1}^{2i} \frac{1}{2} (2k - 2j + 1) \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - j - j' + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=2j}^{k} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j}^{k} (j' - 2j) \right)
\]
\[
\times \left( \prod_{j=1}^{2i} \frac{1}{2} (2k - 2j + 1) \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - 2j + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{k} (2k - j - j' + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{k} (j' - j) \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - 2j + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{k} (2k - j - j' + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{k} (j' - j) \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - 2j + 1) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{k} \frac{(2k - 2j + 2)!}{(k-j)!} \right) \left( \prod_{j=1}^{i-1} \prod_{j'=j+1}^{k} (j' - j) \right)
\]
\[
= \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{k} (2k - 2j + 2)! (2k - 2j + 1) \right)
\]

Thus, we see again we can consolidate the formulas for \( p = 2k \) and \( p = 2k + 1 \).

\[
\prod_{\alpha \in \Delta_{\Sigma,i}^+} \langle \rho_K, \alpha \rangle = \frac{2i}{2} \left( p - 2j \right) (p - 2j + 1)!
\]

A.11. \( G = Sp(n,n) \). \( K = Sp_n \times Sp_n \) \( \Sigma = C_n \)

\[
\begin{align*}
\gamma_1 &= [1,0,\ldots,0;1,0,\ldots,0] = e_1 + \tilde{e}_1 \\
\gamma_2 &= [-1,1,0,\ldots,0;1,0,\ldots,0] = e_2 + \tilde{e}_2 \\
&\vdots \\
\gamma_n &= [0,\ldots,0,-1,1;0,\ldots,0,-1,1] = e_n + \tilde{e}_n \\
a_1 \gamma_1 + \cdots + a_n \gamma_n &= [a_1 - a_2, \ldots, a_{n-1} - a_n, a_n; a_1 - a_2, \ldots, a_{n-1} - a_n] \\
\rho_K &= ne_1 + (n-1)e_2 + \cdots + 2e_n - e_1 + ne_1 + (n-1)e_2 + \cdots + 2e_n - e_n \\
\Rightarrow \langle \rho_K, e_j \rangle &= n - j + 1 \\
\langle \rho_K, \tilde{e}_j \rangle &= n - j + 1
\end{align*}
\]

\( \Delta_{\Sigma,i}^+ = \{ e_j + e_{j'} \mid 1 \leq j \leq i, \ j < j' \leq n \} \cup \{ e_j - e_{j'} \mid 1 \leq j \leq i, \ j < j' \leq n \} \cup \{ 2e_j \mid 1 \leq j \leq i \} \cup \{ 2\tilde{e}_j \mid 1 \leq j \leq i \} \)

\( \Delta_{\Sigma,i}^- = \{ e_j + e_{j'} \mid 1 \leq j \leq i, \ j < j' \leq n \} \cup \{ e_j - e_{j'} \mid 1 \leq j \leq i, \ j < j' \leq n \} \cup \{ 2e_j \mid 1 \leq j \leq i \} \cup \{ 2\tilde{e}_j \mid 1 \leq j \leq i \} \)
\[
\prod_{\alpha \in \Delta_{x,i}^{2}} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} \langle \rho_K, e_j + e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} \langle \rho_K, e_j - e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \langle \rho_K, 2e_j \rangle \right)
\]
\[
\cdot \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} \langle \rho_K, \bar{e}_j + \bar{e}_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} \langle \rho_K, \bar{e}_j - \bar{e}_{j'} \rangle \right) \left( \prod_{j=1}^{i} \langle \rho_K, 2\bar{e}_j \rangle \right)
\]
\[
= \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} (2n-j-j'+2)(j'-j) \right)^2 \left( \prod_{j=1}^{i} 2(n-j+1) \right)^2
\]
\[
= 2^{2i} \left( \frac{(n)!}{(n-i)!} \right)^2 \left( \prod_{j=1}^{i} \frac{(2n-2j+3)!}{(n-j+1)!} \right)^2 \left( \prod_{j=1}^{i} 2(n-j+1) \right)^2
\]
\[
= 2^{2i} \left( \frac{(n)!}{(n-i)!} \right)^2 \left( \prod_{j=1}^{i} \frac{(2n-2j+3)!}{n-j+1} \right)^2 \left( \prod_{j=1}^{i} (2n-2j+3)! \right)^2
\]
\[
= 2^{2i} \left( \prod_{j=1}^{i} (2n-2j+3)! \right)^2
\]

A.12. \( G = SL(2n, \mathbb{H}) \), \( K = Sp_{2n} \), \( \Sigma = C_n \)

\[
\gamma_1 = [0, 1, 0, \ldots, 0] = e_1 + e_2
\]
\[
\gamma_2 = [0, -1, 0, 1, 0, \ldots, 0] = e_3 + e_4
\]
\[
\gamma_3 = [0, 0, 0, -1, 0, 1, 0, \ldots, 0] = e_5 + e_6
\]
\[\vdots\]
\[
\gamma_n = [0, \ldots, 0, -1, 0, 1] = e_{2n-1} + e_{2n}
\]

\[
a_1 \gamma_1 + \cdots + a_n \gamma_n = [0, a_1 - a_2, 0, a_2 - a_3, \ldots, 0, a_{n-1} - a_n, 0, a_n]
\]

\[
\rho_K = 2ne_1 + (2n-1) e_2 + \cdots + 2e_{2n-1} + e_{2n}
\]
\[
\Rightarrow \quad \langle \rho_K, e_j \rangle = 2n - j + 1
\]

\[
\Delta_{x,i}^{2} = \{ e_j + e_{j'} \mid 1 \leq j \leq 2i , \; j < j' \leq 2n \}
\]
\[
\cup \{ e_{2j-1} - e_{j'} \mid 1 \leq j \leq i , \; 2j < j' \leq 2n \}
\]
\[
\cup \{ e_{2j} - e_{j'} \mid 1 \leq j \leq i , \; 2j < j' \leq 2n \}
\]
\[
\cup \{ 2e_j \mid 1 \leq j \leq 2i \}
\]
Case A.

\[ \prod_{\alpha \in \Delta^+_\Sigma} \langle \rho_K, \alpha \rangle = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} \langle \rho_K, e_j + e_{j'} \rangle \right) \left( \prod_{j=1}^{i} \prod_{j'=2j+1}^{2n} \langle \rho_K, e_{2j-1} - e_{j'} \rangle \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{2n} \langle \rho_K, e_{2j} - e_{j'} \rangle \right) \]

\[ \cdot \left( \prod_{j=1}^{2i} \langle \rho_K, 2e_j \rangle \right) \]

\[ = \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j' + 2) \right) \left( \prod_{j=1}^{i} \prod_{j'=2j}^{2n} (j' - 2j + 1) \right) \left( \prod_{j=1}^{i-1} \prod_{j'=2j+1}^{2n} (j' - 2j) \right) \]

\[ \cdot \left( \prod_{j=1}^{2i} 2(2n - j + 1) \right) \]

\[ = 2^{2i} \frac{(2n)!}{(2n - 2i)!} \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j' + 2) \right) \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{2n} (j' - j) \right) \]

\[ = 2^{2i} \frac{(2n)!}{(2n - 2i)!} \left( \prod_{j=1}^{2i} \prod_{j'=j+1}^{2n} (4n - j - j' + 2)(j - j') \right) \]

\[ = 2^{2i} \frac{(2n)!}{(2n - 2i)!} \left( \prod_{j=1}^{2i} \frac{(4n - 2j + 3)!}{(2n - j + 1)!} (n - j)! \right) \]

where in the last step we employed Formula (D).

\[ \left( \prod_{j=1}^{i} \prod_{j'=j+1}^{n} (N - j - j') (j' - j) \right) = \prod_{j=1}^{i} \left( \frac{(N - 2j + 1)!}{(N - n - j - 1)!} (n - j)! \right) \]

---

A.13. \( G = SO(p, 1) \), \( K = SO(p) \), \( \Sigma = C_1 \)

\[ \gamma_1 = [1, 0, \ldots, 0] = e_1 \]

\[ a_1 \gamma_1 = [a_1, 0, \ldots, 0] \]

A.13.1. Case A. \( p = 2k \).

\[ \rho_K = (k - 1)e_1 + (k - 2)e_2 + \cdots + e_{k-1} \]

\[ \Rightarrow \langle \rho_K, e_j \rangle = k - j \]

\[ \Delta^+_{\Sigma,1} = \{e_1 + e_j \mid j = 2, \ldots, k\} \cup \{e_1 - e_j \mid j = 2, \ldots, k\} \]
\[
\prod_{\alpha \in \Delta^+_{\Sigma,1}} \langle \rho_K, \alpha \rangle = \prod_{j=2}^{k} (2k - j - 1) (j - 1)
\]

\[
= (2k - 3) (1) (2k - 4) (2) \cdots (k) (k - 2) (k - 1) (k - 1)
\]

\[
= (k - 1) (2k - 3)!
\]

\[
= \frac{1}{2} (2k - 2) (2k - 3)!
\]

\[
= \frac{1}{2} (2k - 2)!
\]

**A.13.2. Case B.** \( p = 2k + 1 \).

\[
\rho_K = \frac{1}{2} ((2k - 1) e_1 + (2k - 3) e_2 + \cdots + 3e_{k-1} + e_k)
\]

\[
\Delta^+_{\Sigma,1} = \{e_1\} \cup \{e_1 + e_j \mid 1 < j \leq k\} \cup \{e_1 - e_j \mid 1 < j \leq k\}
\]

\[
\prod_{\alpha \in \Delta^+_{\Sigma,1}} \langle \rho_K, \alpha \rangle = \langle \rho_K, e_1 \rangle \prod_{j=2}^{k} \langle \rho_K, e_1 + e_j \rangle \langle \rho_K, e_1 - e_j \rangle
\]

\[
= \frac{1}{2} (2k - 1) \prod_{j=2}^{k} (2k - j) (j - 1)
\]

\[
= \frac{2k - 1}{2} (2k - 2) (1) (2k - 3) (2) \cdots (k) (k - 1)
\]

\[
= \frac{1}{2} (2k - 1)!
\]

\[
= \frac{1}{2} ((2k + 1) - 2)!
\]

Thus, whether \( p \) is even or odd we have

\[
\prod_{\alpha \in \Delta^+_{\Sigma,1}} \langle \rho_K, \alpha \rangle = \frac{1}{2} (p - 2)!
\]