A Fine Partitioning of Cells

Atlas of Lie Groups Workshop

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1. The setting in Atlas

Let me begin by talking a bit about the organization of the set of irreducible admissible representations from the atlas point of view.

1.1. \( G \).

- \( G \) be a algebraic group defined over \( \mathbb{C} \),
- \( G^\vee \) its dual group,
- \( \tau \) an outer automorphism determining an inner class of real forms
- \( \delta \in \text{Aut}(G) \) a strong real form of \( G \) in the inner class of \( \tau \)
- \( G^\Gamma = G \times \Gamma = G \times \{1, \sigma\} \) with \( \sigma \) acting on \( G \) by \( \delta \)

1.2. Irreducible admissible representations. From the Atlas point of view, which is essentially a derivative of the Langlands point of view after several reductions/translations (Vogan-Zuckerman-Knapp-Adams-duCloux) the irreducible (admissible) representations (with integral infinitesimal character) are parameterized by triples \( (x, y, \lambda) \) where

- \( x \in G^\Gamma - G, x^2 \in Z(G) \), a representative of a strong real form of \( G \)
- \( y \in (G^\vee)^\Gamma - G^\vee, y^2 \in Z(G^\vee) \), a representative of a strong real form of \( G^\vee \)
- \( \lambda \in \mathfrak{d}t \cong \mathfrak{t}^* \)

Remark 1.1. In much of the atlas documentation you will also see \( x \) and \( y \) regarded as representing a pair \((\mathcal{O}, \mathcal{O}^\vee)\), were \( \mathcal{O} \) and \( \mathcal{O}^\vee \) are, respectively, a \( K \)-orbit in \( G/B \) and a \( K^\vee \)-orbit in \( G^\vee/B^\vee \). This is an equivalent parameterization.

1.3. Blocks of Representations. By restricting the parameters \( x \) and \( y \), respectively, to correspond to particular (equivalence classes of) strong real forms, the set \( \hat{G}_{adm}(\lambda) \) of irreducible admissible \((\mathfrak{g}, K)\)-modules with integral infinitesimal character \( \lambda \) can split into “blocks” of representations. From a purely atlas-centric point of view the notion of blocks is useful as it provides a minimal partitioning of \( \hat{G}_{adm}(\lambda) \) into subsets for which KL computations are self-contained. But it turns out that blocks of representations correspond to the collecting together of all irreducible admissible representations into groups connected by a non-trivial Ext.

Below is the beginning of the atlas session in which the KL polynomials for the “big block” of split E8 is computed; we remark that the choice of a weak real form at the user interface level is actually implemented as a choice of a strong real from within the software itself.

real: type
Lie type: E8
enter inner class(es): split
main: klwrite
(weak) real forms are:
0: e8
1.4. Cells of Representations. Within a block we can collect together those representations which belong to the same cell. Recall from Peter’s talk that a cell is an equivalence class of representations where

\[ X \sim Y \iff Y \text{ is a subquotient of } X \otimes F \text{ for some f.d.r. } F \text{ and } X \text{ is a subquotient of } Y \otimes F' \text{ for some f.d.r. } F' \]

and that the representations in the cell share the same associated variety. In particular, all the representations in a cell share the same Gelfand-Kirillov dimension.

The cell decomposition of a block is obtainable by the \texttt{wcells}, \texttt{wgraph}, or \texttt{extractgraph} commands of Atlas. Below is beginning of an atlas session in which the \texttt{wcells} command in run on the big block of F4.
1.5. **Tau-invariants.** The output of the `wcells` command contains a lot of information about the representations in a block. The elements of a cell are labeled by (strictly internal) indices from 0 to \((m - 1)\) where \(m\) is the number of elements in the cell; rather than pairs \((x, y)\).\(^1\) Following a cell index number \(i\) is a set \(\{t_{i,1}, \ldots, t_{i,\ell}\}\) which is the “tau-invariant” of the cell element. This is actually an invariant of the primitive ideal corresponding to the annihilator of the irreducible \((\mathfrak{g}, K)\)-module corresponding to \(i \sim (x, y, \lambda)\). The tau-invariant is a set of indices of simple roots that, roughly speaking, prescribes the “direction” in which the corresponding primitive ideal sits relative to the minimal primitive of ideal of infinitesimal character \(\lambda\).

A little more precisely. Let \(\text{Prim}_\rho\) be the set of primitive ideals of infinitesimal character \(\rho\) endowed with the natural partial ordering by inclusion.

\[
I \leq I' \iff I \subseteq I'
\]

Then there is a unique maximal primitive ideal within \(\text{Prim}_\rho\) (the annihilator of the trivial representation) and a unique minimal primitive ideal \(I_0\) which is the annihilator of the irreducible Verma module \(M_\rho\) of highest weight \(-2\rho\). The other primitive ideals of infinitesimal character \(\rho\) can be thought of as sitting on the vertices of a Hasse diagram associated with the above partial ordering. Since the minimal primitive ideal is contained in every primitive ideal, every primitive ideal in the \(\text{Prim}_\rho\) is connected to \(I_0\) by certain sequences of inclusions \(I \supset I' \supset \cdots \supset I^{(k)} \supset I_0\), which can be visualized as certain directed paths through a Hasse diagram. It turns out that the penultimate primitive ideals in such a sequence are always primitive ideals of the form \(\text{Ann}(M_{-s\rho}/M_{-\rho})\) where \(M_{-s\rho}\) is the Verma module of highest weight \(-s\rho - \rho\), \(s\) being a reflection by a simple root in \(W\). For a given primitive ideal \(I\) let \(\tau(I)\) denote the set of simple roots \(s\) for which \(\text{Ann}(M_{-s\rho}/M_{-\rho})\) sits between \(I\) and \(I_0\) in the Hasse diagram of \(\text{Prim}_\rho\). In other words, the tau invariant \(\tau(I)\) is the set of next-to-last-stops on the paths from \(I\) to \(I_0\).

I should perhaps remark that the sets \(\{t_{i,1}, \ldots, t_{i,\ell}\}\) that occur in the output of `wcells` are actually the indices of the simple roots that lie in the descent set of the representation indexed by \(i\). The identification with tau-invariants comes via

**Theorem 1.2.** If \(X\) is an irreducible \((\mathfrak{g}, K)\)-module, then the tau-invariant of \(\text{Ann}(X)\) is equal to the descent set of \(X\).

Note that in the example above, some cell elements share the same tau-invariant. This, however, does not mean that they share same annihilator; it simply means that their annihilators lie in the same direction(s) from \(I_0\). On the other hand, since \(\tau(X) \equiv \tau(\text{Ann}(X))\),

**Fact 1.3.** If \(X, Y\) are two irreducible \((\mathfrak{g}, K)\)-modules such that \(\text{Ann}(X) = \text{Ann}(Y)\), then \(\tau(X) = \tau(Y)\)

1.6. **Edges.** The output of the `wcells` command contains one last bit of cell element data; the `edges` and multiplicities of the Wgraph “star”\(^2\) originating from a cell element. If \((j, m)\) is listed in the edge/multiplicity data of a cell element \(i\) then the representation \(\pi_j\) corresponding to (cell index) \(j\) occurs in the HC module \(\pi_i \otimes \mathfrak{g}\) with multiplicity \(m\) (\(\pi_i\) being the irreducible HC module corresponding to cell index \(i\)).

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\(^1\)The exact correspondence between the internal cell indices \(i\) and the pairs \((x, y)\) of real forms can be deduced from the output of the `extractgraph` and `block` commands.

\(^2\)Here I mean the Wgraph of the cell which is the restriction of the Wgraph of the block to the cell.
1.7. **Primitive Ideals.** In the above we allude to the possibility of grouping together cell elements which share the same primitive ideal; however, Atlas will not do that for us. Yet, ....

2. **A fine partitioning of cells**

What one can obtain from immediately from the output of the `wcells` command is a partitioning of a cell into subcells with the same tau-invariant. As remarked above, this is compatible with the partitioning of a cell via primitive ideals, but it is much coarser. However, besides having the same tau-invariant, representations with the same primitive ideal also have the property that their collections of tau-invariants of their edge vertices are the same. By this I mean the following. Let $\tau_0(i) = \{t_{i,1}, \ldots, t_{i,m}\}$ denote the tau invariant of vertex $i$ of a cell. Let $e(i) = \{e_{i,1}, \ldots, e_{i,k}\}$ be the set of edge vertices for vertex $i$, and let

$$\tau_1(i) = \{\tau_0(e_{i,1}), \ldots, \tau_0(e_{i,k})\}$$

be the corresponding set (without multiplicity) of tau-invariants of the edge vertices of vertex $i$. If two vertices $i, j$ share the same primitive ideal then

$$\tau_1(i) = \tau_1(j)$$

This equivalence relation still fails to complete separate the representations in a cell into subgroups with a common primitive ideal; however, it is compatible with the partitioning by primitive ideals and it is compatible with the partitioning than by tau-invariants.

So what to do next? Simply repeat. That is, set

$$\tau_2(i) = \{\tau_1(e_{i,1}), \ldots, \tau_1(e_{i,k})\}$$

and group together vertices with the same $\tau_0$, the same $\tau_1$ and the same $\tau_2$. If you continue this process until the sub-partitioning process stabilizes (as it must since the cells are finite), and do this for, say, all real forms of all exceptional groups, then the following empirical fact emerges:

**Fact 2.1.** The number of elements in the infinite order (stable) partitioning of a cell is always the dimension of a special representation of the Weyl group of $G$.

What makes this so striking is the following.

**Fact 2.2.** Attached (by other means [Lu]) to each cell $C$ is a unique special representation $\sigma_C$ of $W$, and

$$\# \{\text{Ann}(X) \mid X \text{ an irreducible HC module in } C\} = \dim \sigma_C$$

That is, the number of distinct primitive ideals arising from a given cell is equal to the dimension of the special representation of $W$ attached to that cell.

**Conjecture 2.3.** The infinite ordering partitioning of a cell corresponds to a partitioning of the cell by primitive ideals.

Patrick Polo asked the following question. Why should it be that representations with the same primitive ideal share the same $\tau_1$ invariant (as well as the higher derived $\tau$-invariants, $\tau_2, \ldots$)? David offered the following explanation.

Fix a irreducible $(\mathfrak{g}, K)$-module $X$ of infinitesimal character $\rho$, and consider $I = \text{Ann}(X)$. $X \otimes \mathfrak{g}$ is a $(\mathfrak{g}, K)$-module that is also a faithful module for $U(\mathfrak{g})/I \otimes U(\mathfrak{g})/\text{Ann}(\mathfrak{g})$. You can diagonally embed $U(\mathfrak{g})$

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})/I \otimes U(\mathfrak{g})/\text{Ann}(\mathfrak{g})$$

Then

$$\Delta^{-1}(U(\mathfrak{g})/I \otimes U(\mathfrak{g})/\text{Ann}(\mathfrak{g})) = \text{Ann}_{U(\mathfrak{g})}(X \otimes \mathfrak{g})$$
depends only on $I$. Now $X \otimes g$ will decompose as

$$X \otimes g = \text{several copies of } X + \bigoplus Y_i \text{ (irreducible } (g,K)\text{-module in same cell as } X; \text{ in fact, Wgraph neighbors of } X) + (g,K)\text{-modules of lower GK-dim (bigger annihilators)}$$

Therefore, the set of minimal primes in $\text{Ann}(X \otimes g)$ will consist of $\{I, \text{Ann}(Y_i)\}$. Since $\text{Ann}(X \otimes g)$ depends only on the primitive ideal $I$ containing $X$, the set of minimal primes in $\text{Ann}(X \otimes g)$, will also depend only on $I$ and so if we have two $(g,K)$-modules $X, X'$ in the same cell and consider the sets of primitive ideals corresponding to the Wgraph neighbors of $X, X'$ we must have

$$\{\text{Ann}(Y_1), \ldots, \text{Ann}(Y_k)\} = \left\{\text{Ann}(Y'_1), \ldots, \text{Ann}(Y'_\ell)\right\}$$

This implies

$$\tau_1(X) = \tau_1(X')$$

and, in fact, infers the equality of the higher derived tau-invariants as well.

References


[M] W. McGovern, *Primitive Ideals*, Monty’s article in the private atlas Wiki http:\\wiki.math.umd.edu\atlas\Primitive_ideals