APPENDIX G

Solutions to Problem Set 7

1. (Problem 6.4.2 in text)

Use the method of characteristics to show that the solution of

$$uu_x + u_y = 0 \qquad , \qquad u(x,0) = f(x) \tag{G.1}$$

is given implicitly by

$$u = f(x - uy) \tag{G.2}$$

and verify this result by direct differentiation. In what region is this solution valid?

The differential equations satisfied by the characteristic curves $(x(t), y(t), u(t))$ are

$$\begin{aligned} \frac{dx}{dt} &= u(t) \\ \frac{dy}{dt} &= 1 \\ \frac{du}{dt} &= 0 \quad . \end{aligned} \tag{G.3}$$

This system of ODEs is easily integrated to produce

$$\begin{aligned} u(t) &= C_1 \\ y(t) &= t + C_2 \\ x(t) &= C_1 t + C_3 \quad . \end{aligned} \tag{G.4}$$

Let $t = 0$ be the value of the parameter t when characteristic curves pass through the plane $y = 0$ and let $(x_o, 0, u_o)$ be the point where the characteristic through (x, y, u) passes through this plane. We then have

$$\begin{aligned} C_1 &= u_o \\ C_2 &= 0 \\ C_3 &= x_o \end{aligned} \tag{G.5}$$

Our initial conditions $u(x,0) = f(x)$, implies that

$$C_1 = u_o = f(x_o) \quad . \tag{G.6}$$

We can thus rewrite (G.4) as

$$\begin{aligned} x &= f(x_o)\, t + x_o \\ y &= t \\ u &= f(x_o) \quad . \end{aligned} \tag{G.7}$$

We can use the last two equations to replace $f(x_o)$ by u, x_o by $f^{-1}(u)$, and t by y in the first equation. This leads u

$$x = uy + f^{-1}(u) \quad .$$

or

$$u = f(x - uy) \quad . \tag{G.8}$$

It is apparent from (G.8) that

$$u(x,0) = f(x+0) = f(x) \quad .$$

On the other hand,

$$\begin{aligned}\frac{\partial u}{\partial x} &= f'\left(x-uy\right)\\ \frac{\partial u}{\partial y} &= -yf'\left(x-uy\right) \quad .\end{aligned}$$

It is thus clear that (G.8) is indeed the solution of (G.1) and (G.2).

$\square$

2. (Probelem 6.4.4 in text)

In a rotating fluid problem it is required that a function $v(r,\tau)$ be found in the region $0 < r < a$, $\tau > 0$, satisfying

$$\begin{aligned} v_\tau - \left(1 - \tfrac{v}{r}\right)(rv)_r &= 0 \\ v(r,0) &= 0 \\ v(a,\tau) &= a \quad . \end{aligned} \tag{G.9}$$

Use the method of characteristics to obtain the solution

$$\begin{aligned} v &= \tfrac{re^{2r} - \frac{a^2}{r}}{e^{2r}-1} \quad , \quad r \geq ae^{-r} \\ v &= 0 \quad , \quad r \leq ae^{-r} \quad . \end{aligned} \tag{G.10}$$

Sketch the two families of characteristics and discuss the nature of solutions near the point $r = a$ and $\tau = 0$.

Set

$$w = vr \quad .$$

Then PDE in (G.9) is equivalent to

$$\frac{1}{r}v_\tau - \left(1 - \frac{w}{r^2}\right) w_r = 0 \quad , \tag{G.11}$$

or

$$v_\tau + (\frac{w}{r} - r)w_r = 0$$

with boundary conditions

$$\begin{aligned} w = 0 \quad &, \quad \text{when } \tau = 0 \\ w = a^2 \quad &, \quad \text{when } r = a \quad . \end{aligned} \tag{G.12}$$

The differential equations satisfied by the characteristic curves , $(t) = (\tau(t), r(t), v(t))$ are

$$\begin{aligned} \tfrac{d\tau}{dt} &= 1 \\ \tfrac{dr}{dt} &= \tfrac{w}{r} - r \\ \tfrac{dw}{dt} &= 0 \quad . \end{aligned} \tag{G.13}$$

The last equation leads to

$$w(t) = c_1 \quad . \tag{G.14}$$

Notice that the differential equation for $r(t)$ takes a radically different form when $w(t) = c_1 = 0$. We shall handle this special case first.

Case 1. $c_1 = 0$.

If $w(t) = c_1 = 0$ we have from (5)

$$\frac{dr}{dt} = -r$$

or

$$\frac{dr}{r} = -dt$$

or

$$\log|r| = -t + c_1$$

or

$$r(t) = r_o e^{-t} \tag{G.15}$$

The differential equation for $\tau(t)$

$$\frac{d\tau}{dt} = 1 \tag{G.16}$$

easily integrates to

$$\tau(t) = t + \tau_o. \tag{G.17}$$

Fixing $t = 0$ to correspond to the point along the characteristic curve where $\tau = 0$, we get the following one parameter family of characteristic curves passing through the line $\tau = w = 0$;

$$\begin{aligned} \tau(t) &= t \\ r(t) &= r_o e^{-t} \\ w(t) &= 0 \end{aligned} \tag{G.18}$$

Note that $r(\tau)$ never quite vanishes. This implies that

$$0 = w(t) = v(t)r(t) \qquad \Rightarrow \qquad v(t) = 0 \quad . \tag{G.19}$$

We conclude from this that if (τ, r) is any point in the (τ, r)-plane lying on a curve of the form

$$\begin{aligned} \tau(t) &= t \\ r(t) &= r_o e^{-t} \qquad , \qquad 0 \leq r_o \leq a \end{aligned} \tag{G.20}$$

then

$$v(\tau, r) = 0 \quad . \tag{G.21}$$

Case 2. $c_1 \neq 0$.

We will now proceed with a construction of a general solution of the differential equations of the characteristics:

$$\begin{aligned} \tfrac{d\tau}{dt} &= 1 \\ \tfrac{dr}{dt} &= \tfrac{w}{r} - r \\ \tfrac{dw}{dt} &= 0 \end{aligned} \tag{G.22}$$

Again the general solution to the first and last equations is

$$\begin{aligned} \tau(t) &= t + \tau_o \\ v(t) &= w_o \quad . \end{aligned} \tag{G.23}$$

Inserting the last result into the differential equation for $r(t)$ yields

$$\frac{dr}{dt} = \frac{w_o}{r} - r \tag{G.24}$$

or

$$\frac{rdr}{w_o - r^2} = dt \quad .$$

The latter equation is easily integrated to produce

$$-\frac{1}{2} \ln \left| w_o - r^2 \right| = t + C$$

or

$$w_o - r^2 = Ae^{-2t} \quad . \tag{G.25}$$

or

$$r(t) = \sqrt{w_o - Ae^{-2t}} \quad . \tag{G.26}$$

Our characterisitics are thus curves of the form

$$\begin{aligned} \tau(t) &= t + \tau_o \\ r(t) &= \sqrt{w_o - Ae^{-2t}} \\ w(t) &= w_o \quad . \end{aligned} \tag{G.27}$$

Recalling that

$$w(t) =^{def} r(t)v(t)$$

we have

$$\tau(t) = t + \tau_o$$
$$r(t) = \sqrt{w_o - Ae^{-2t}} \tag{G.28}$$
$$v(t) = \frac{w_o}{\sqrt{w_o - Ae^{-2t}}} \quad .$$

The natural thing to do next would be to impose the boundary condition

$$r = v = a \quad , \quad \forall\, \tau > 0 \quad , \tag{G.29}$$

at $t = \tau$ identify the constants w_o and A. However, note that the Cauchy data (G.29) lies on the line

$$\begin{aligned} \tau(t) &= t \\ r(t) &= a \\ v(t) &= a \end{aligned} \tag{G.30}$$

which corresponds to the characteristic curve with $\tau_o = 0$, $w_o = a^2$, and $A = 0$. Since (G.29) is a characteristic curve, we can not use it as Cauchy data. By the uniqueness property of characteristics no other characteristic will pass through the line (G.29).

So here's the problem. The Cauchy data above the line $\tau = 0$, $0 < r < a$ give rises to characteristics that fill only the region of the (r, τ)-plane lying below the curve $r = ae^{-\tau}$, while the initial data along the line $r = a$, $\tau > 0$ is itself a characteristic and so intersects no other characteristics. Thus, we seem to have no means of determining the characteristic curves of our solution whose projections in the (r, τ)-plane lie the curve $r = ae^{-\tau}$ and below the line $r = a$.

Let's proceed by ignoring the problem. Actually, what we will do is construct a general solution of the PDE in (1) from the characteristic curves passing over the line $r = a$ in the (r, τ)-plane and then take impose appropriate boundary conditions on this general solution.

From (G.28) we see that along any characteristic curve for (1) we have

$$\begin{aligned} r^2 &= w_o - Ae^{-2\tau} \\ v &= \frac{w_o}{r} \end{aligned} \tag{G.31}$$

with w_o and A fixed constants (on any particular characteristic). Suppose $P = (r_1, \tau_1)$ is a point in the (r, τ) plane in the region above the curve $r = ae^{-\tau}$ and the line $r = a$. We want to figure out under what circumstances the characteristic through P will cross over the line $r = a$. Suppose it crosses this line when $\tau = \tau_o$. Then

$$a^2 = w_o - Ae^{-2\tau_o}.$$

or

$$w_o = a^2 - Ae^{-2\tau_o} \quad . \tag{G.32}$$

But we also have

$$r_1^2 = w_o - Ae^{-2\tau_1}$$

or

$$w_o = r_1^2 - Ae^{-2\tau_1} \tag{G.33}$$

We can now use (G.32) and (G.33) to eliminate the constants w_o and A. One finds

$$\begin{aligned} A &= \frac{a^2 - r_1^2}{e^{-2\tau_1} - e^{-2\tau_o}} \\ w_o &= \frac{a^2 e^{-2\tau_1} - r_1^2 e^{-2\tau_o}}{e^{-2\tau_1} - e^{-2\tau_o}} \end{aligned} \tag{G.34}$$

Thus,

$$v_1 = \frac{w_o}{r_1} = \frac{\frac{1}{r_1} a^2 e^{-2\tau_1} - r_1 e^{-2\tau_o}}{e^{-2\tau_1} - e^{-2\tau_o}} \quad . \tag{G.35}$$

Now note have yet to impose any boundary condition on v we have only shifted the ambiguity in the value of v along the line $r = a$ to an unspecified number τ_o, the value of τ when the characteristic through (r_1, τ_1, v_1) crosses through the plane $r = a$. As such we can regard τ_1 as a constant depending only on initial conditions. Indeed, we can regard the relation (G.35) as a relation that holds at arbitrary points $r_1 = r$, $\tau_1 = \tau$, $v_1 = v$. We thus obtain

$$v(r,\tau) = \frac{\frac{1}{r}a^2 e^{-2\tau} - re^{-2\tau_o}}{e^{-2\tau} - e^{-2\tau_o}} \quad . \tag{G.36}$$

It is a simple, albeit tedius, task to confirm that (G.36) is indeed a solution of the PDE in (G.9).

We now (finally) ask the question. Can we match the solution (G.36) to the boundary conditions in (G.9). The answer is yes; we simply take $\tau_o = 0$ to obtain

$$\begin{aligned} v(r,\tau) &= \frac{\frac{1}{r}a^2 e^{-2\tau} - r}{e^{-2\tau} - 1} \\ &= \frac{re^{2\tau} - \frac{a^2}{r}}{e^{2\tau} - 1} \quad . \end{aligned} \tag{G.37}$$

Note that

$$\begin{aligned} \lim_{r \to ae^{-\tau}} v(r,\tau) &= 0 \\ \lim_{r \to a} v(r,\tau) &= a \end{aligned} \tag{G.38}$$

and this solution is the appropriate one matching the solution in region below the curve $r = ae^{-\tau}$ and the data along the line $r = a$.

□

3. (Problem 9.3.1 in text)

4. (Problem 9.3.9 in text)

A two-dimensional Green's function $G(x, y; \zeta, \eta)$ for the Laplacian operator on a region $D \subset \mathbb{R}^2$ may be defined by

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y; \zeta, \eta) = \delta(x - \zeta)\delta(y - \eta) \tag{G.39}$$

$$G(x, y; \zeta, \eta)|_{\partial D} = 0 \quad . \tag{G.40}$$

Show that in the case of the square region

$$D = \{(x, y) \mid 0 < x < L \quad , \quad 0 < y < L\} \quad , \tag{G.41}$$

$G(x, y; \zeta, \eta)$ is given by

$$\begin{aligned} G(x, y; \zeta, \eta) \;\; &= \;\; \textstyle\sum_{m=1}^{\infty} \frac{\sin\left(\frac{m\pi\zeta}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}{m\pi \sinh(m\pi)} \\ &\times \left[\cosh\left[\frac{m\pi(L-y-\eta)}{L}\right] - \cosh\left[\frac{m\pi(L-|y-\eta|)}{L}\right]\right] \quad . \end{aligned} \tag{G.42}$$

We will solve this problem using series expansions. Recall that when we looked for solutions of

$$\nabla^2\phi(x, y) = 0 \tag{G.43}$$

using separation of variables, we were lead to solutions of the form

$$\phi(x, y) = \left(Ae^{i\lambda x} + Be^{-i\lambda x}\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right) \tag{G.44}$$

or equivalently

$$\phi(x, y) = (c_1 \sin(\lambda x) + c_2 \cos(\lambda x))\,(d_1 \sinh(\lambda x) + d_2 \cosh(\lambda x)) \tag{G.45}$$

Noting the boundary conditions (G.40), we might thus try an expansion of the form

$$G(x, y; \zeta, \eta) = \sum_{m=1}^{\infty} a_m(y, \zeta, \eta) \sin\left(\frac{m\pi x}{L}\right) \quad . \tag{G.46}$$

However, before we start, I should point out that

$$\delta(x - \zeta) = \sum_{m=1}^{\infty} \frac{2}{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{m\pi\zeta}{L}\right) \quad . \tag{G.47}$$

The justification for this formula is as follows. Suppose $f(x)$ is a piecewise continuous function on $(0, L)$ and let

$$f(x) = \sum_{m=1} \alpha_m \sin\left(\frac{m\pi x}{L}\right) \tag{G.48}$$

be its Fourier expansion. Then using the identities

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2}\delta_{m,n} \tag{G.49}$$

is easy to show that

$$f(\zeta) = \sum_{m=1} \alpha_m \sin\left(\frac{m\pi\zeta}{L}\right) = \int_0^L f(x)\delta(x - \zeta)dx \quad . \tag{G.50}$$

If we now insert the expansions (G.46) and (G.47) into (G.39) we get

$$\sum_{m=0}^{\infty}\left(\left(-\frac{m^2\pi^2}{L^2}a_m + \frac{\partial^2 a_m}{\partial y^2}\right)\sin\left(\frac{m\pi x}{L}\right)\right) = \sum_{m=1}^{\infty}\frac{2}{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{m\pi\zeta}{L}\right)\delta(y - \eta)$$

Setting the total coefficient of $\sin\left(\frac{m\pi x}{L}\right)$ equal to zero, we get

$$-\frac{m^2\pi^2}{L^2}a_m + \frac{\partial^2 a_m}{\partial y^2} = \frac{2}{L}\sin\left(\frac{m\pi\zeta}{L}\right)\delta(y-\eta) \tag{G.51}$$

The coefficients a_m should thus correspond to

$$a_m = \frac{2}{L}\sin\left(\frac{m\pi\zeta}{L}\right)g(y,\eta) \tag{G.52}$$

where $g(y,\eta)$ is the Green's function satisfying

$$\begin{aligned} \tfrac{d^2}{dy^2}g(y,\eta) - \tfrac{m^2\pi^2}{L^2}g(y,\eta) &= \delta(y-\eta) \\ g(0,\eta) &= 0 \\ g(L,\eta) &= 0 \quad . \end{aligned} \tag{G.53}$$

In the preceding lecture, we developed a general formula for constructing Green's functions for Sturm-Liouville problems with homogeneous boundary conditions:

$$\begin{aligned} \left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]G(x,\zeta) &= \delta(x-\zeta) \\ G(a,\zeta) &= 0 \\ G(b,\zeta) &= 0 \end{aligned}$$

$$\Rightarrow \qquad G(x,\zeta) = \begin{cases} \frac{u_1(x)u_2(\zeta)}{W[u_1,u_2](\zeta)p(\zeta)} & , \quad a < x < \zeta \\ \frac{u_1(\zeta)u_2(x)}{W[u_1,u_2](\zeta)p(\zeta)} & , \quad \zeta < x < b \end{cases} \tag{G.54}$$

where $u_1(x)$ and $u_2(x)$ satisfy respectively,

$$\begin{aligned} \left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]u_1(x) &= 0 \\ u_1(a) &= 0 \end{aligned}$$

and

$$\begin{aligned} \left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right]u_2(x) &= 0 \\ u_2(b) &= 0 \quad , \end{aligned}$$

and $W[u_1,u_2](x)$ is the Wronskian of $u_1(x)$ and $u_2(x)$.

In the case at hand we may take

$$\begin{aligned} u_1(y) &= \sinh\left(\tfrac{m\pi y}{L}\right) \\ u_2(y) &= \sinh\left(\tfrac{m\pi}{L}(L-y)\right) \end{aligned} \tag{G.55}$$

as solutions to the homogeneous equations corresponding to (14), respectively satisfying $u_1(0) = 0$ and $u_2(L) = 0$. Note that the $p(x) = 1$ in (G.52) and that

$$\begin{aligned} W[u_1,u_2](x) &= \left(\sinh\left(\tfrac{m\pi y}{L}\right)\right)\left(-\tfrac{m\pi}{L}\cosh\left(\tfrac{m\pi}{L}\right)(L-y)\right) \\ &\quad - \left(\tfrac{m\pi}{L}\cosh\left(\tfrac{m\pi y}{L}\right)\right)\left(\sinh\left(\tfrac{m\pi}{L}(L-y)\right)\right) \\ &= \tfrac{-m\pi}{L}\left(\sinh\left(m\pi\right)\right) \end{aligned} \tag{G.56}$$

In the last step we used the identity

$$\sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta) = \sinh(\alpha+\beta) \quad .$$

Thus,

$$g(y,\eta) = \begin{cases} \frac{L\sinh\left(\frac{m\pi}{L}y\right)\sinh\left(\frac{m\pi}{L}(L-\eta)\right)}{-m\pi\sinh(m\pi)} & , \quad 0 < y < \eta \\ \frac{L\sinh\left(\frac{m\pi}{L}(L-y)\right)\sinh\left(\frac{m\pi}{L}\eta\right)}{-m\pi\sinh(m\pi)} & , \quad \eta < y < L \quad . \end{cases} \tag{G.57}$$

Using the identity

$$\sinh(\alpha)\sinh(\beta) = \frac{1}{2}\left[\cosh(\alpha+\beta) - \cosh(\alpha-\beta)\right] \quad ,$$

we get

$$g(y,\eta) = \begin{cases} \frac{L\left(\cosh\left(\frac{m\pi}{L}(L-y-\eta)\right)-\cosh\left(\frac{m\pi}{L}(L+y-\eta)\right)\right)}{2m\pi\sinh(m\pi)} & , \quad 0<y<\eta \\ \frac{L\left(\cosh\left(\frac{m\pi}{L}(L-\eta-y)\right)-\cosh\left(\frac{m\pi}{L}(L+\eta-y)\right)\right)}{2m\pi\sinh(m\pi)} & , \quad \eta<y<L \end{cases}$$

or

$$g(y,\eta) = \frac{L\left(\cosh\left(\frac{m\pi}{L}\left(L-(y+\eta)\right)\right) - \cosh\left(\frac{m\pi}{L}\left(L-|y-\eta|\right)\right)\right)}{2m\pi\sinh(m\pi)} \quad . \tag{G.58}$$

Finally, inserting (G.58) into (G.52), we get

$$a_m = \frac{\sin\left(\frac{m\pi\zeta}{L}\right)\left(\cosh\left(\frac{m\pi}{L}\left(L-(y+\eta)\right)\right) - \cosh\left(\frac{m\pi}{L}\left(L-|y-\eta|\right)\right)\right)}{m\pi\sinh(m\pi)}$$

and plugging this expression for a_m into (G.46) yields (G.42). □