APPENDIX F

Solutions to Problem Set 6

1. (Problem 3.12.1 in text)

(a) Let $u(x,t)$ satisfy the equation

$$u_{tt} = c^2 u_{xx} \quad , \quad c = constant \quad ,$$

in some region of the (x,t) plane. Show that the quantity $(u_t - cu_x)$ is constant along each straight line defined by $x - ct = constant$, and that $(u_t + cu_x)$ is constant along each straight line of the form $x + ct = constant$. These straight lines are called *characteristics*; we will refer to typical members of the two families as C_+ and C_- curves, respectively; thus $(x - ct = constant)$ is a C_+ curve.

Set

$$\phi_+(x,t) = u_t\,(x,t) - cu_x\,(x,t) \quad . \tag{F.1}$$

Along a C_+ curve we have

$$x = k_1 + ct \tag{F.2}$$

and so along such a curve

$$\phi_+(x,t) = \phi_+(t) = u_t\,(k_1 + ct, t) - cu_x\,(k_1 + ct, t) \quad . \tag{F.3}$$

Differentiating ϕ_+ with respect to t we obtain

$$\begin{aligned} \frac{d\phi_+}{dt} &= cu_{tx} + u_{tt} - c^2 u_{xx} - cu_{tx} \\ &= u_{tt} - c^2 u_{xx} \\ &= 0 \end{aligned}$$

since u satisfies the wave equation. Therefore, ϕ_+ is constant along any curve of the form (F.2).

Similarly, if we set

$$\phi_-(x,t) = u_t\,(x,t) + cu_x\,(x,t) \quad . \tag{F.4}$$

Then along the curve

$$x = k_2 - ct \tag{F.5}$$

we have

$$\phi_-(x,t) = \phi_-(t) = u_t\,(k_2 - ct, t) + cu_x\,(k_2 - ct, t)$$

and so

$$\begin{aligned} \frac{d\phi_-}{dt} &= -cu_{tx} + u_{tt} - c^2 u_{xx} + cu_{tx} \\ &= u_{tt} - c^2 u_{xx} \\ &= 0 \end{aligned}$$

Thus, ϕ_- is constant along any curve of the form (F.5). $\square$

(b) Let $u(x,0)$ and $u_t(x,0)$ be prescribed for all values of x between $-\infty$ and $+\infty$, and let (x_o,t_o) be some point in the (x,t) plane, with $t_o > 0$. Draw the C_+ and C_- curves through (x_o,t_o) and let A and B denote, respectively, their intercepts with the x-axis. Use the properties of C_+ and C_- derived in part (a) to determine $u_t(x_o,t_o)$ in terms of initial data at points $(A,0)$ and $(B,0)$. Using a similar technique to obtain $u_t(x_o,\tau)$ with $0<\tau<t_o$, determine $u(x_o,t_o)$ by integration with respect to τ, and compare with Equation (3.7). Observe that this "method of characteristics" again shows that $u(x_o,t_o)$ depends only on that part of the initial data between points $(A,0)$ and $(B,0)$.

Let

$$k_\pm = x_o \mp ct_o \tag{F.6}$$

and set

$$c_\pm = \left\{(x,t)\in\mathbb{R}^2 \mid x \mp ct = k_\pm\right\} \tag{F.7}$$

From part (a) we know that

$$\begin{aligned}\phi_+ &= u_t(x,t) - cu_x(x,t)\\ \phi_- &= u_t(x,t) + cu_x(x,t)\end{aligned} \tag{F.8}$$

are, respectively, constant along the lines c_+ and c_-.

At the point $(A,0)$ where the line c_+ intersects the x-axis we have

$$\phi_+ = u_t(A,0) - cu_x(A,0) \tag{F.9}$$

and so the constant ϕ_+ is completely determined by the Cauchy data at the point $(A,0)$.

Similarly, at the point $(B,0)$ where the line c_- intersects the x-axis we have

$$\phi_- = u_t(B,0) + cu_x(B,0) \tag{F.10}$$

and so the constant ϕ_- is completely determined by the Cauchy data at the point $(B,0)$.

Using (F.9) and (F.10) we can rewrite equations (F.8) as

$$\begin{aligned}u_t(A,0) - cu_x(A,0) &= u_t(x_o,t_o) - cu_x(x_o,t_o)\\ u_t(B,0) + cu_x(B,0) &= u_t(x_o,t_o) + cu_x(x_o,t_o)\end{aligned} \tag{F.11}$$

Adding the second equation to the first and then dividing by 2 we obtain

$$u_t(x_o,t_o) = \frac{1}{2}\left(u_t(A,0) + u_t(B,0) - cu_x(A,0) + cu_x(B,0)\right) \quad . \tag{F.12}$$

We can be a even more explicit than this. For the value of A is precisely $k_+ = x_o - ct_o$, and the value of B is precisely $k_- = x_o + ct_o$. Thus,

$$\begin{aligned}u_t(x_o,t_o) &= \tfrac{1}{2}\left(u_t(x_o - ct_o,0) + u_t(x_o + ct_o,0)\right)\\ &\quad + \tfrac{c}{2}\left(-u_x(x_o - ct_o,0) + u_x(x_o + ct_o,0)\right)\end{aligned} \tag{F.13}$$

This equation is perfectly valid for any choice of x_o and t_o, and so we can write

$$\begin{aligned}u_t(x_o,t) &= \tfrac{1}{2}\left(u_t(x_o - ct,0) + u_t(x_o + ct,0)\right)\\ &\quad + \tfrac{c}{2}\left(-u_x(x_o - ct,0) + u_x(x_o + ct,0)\right)\end{aligned} \tag{F.14}$$

Integrating both sides with respect to t from 0 to t_o we obtain

$$\begin{aligned}u(x_o,t_o) - u(x_o,0) &= \tfrac{1}{2}\int_0^{t_o} u_t(x_o - ct,0)\,dt + \tfrac{1}{2}\int_0^{t_o} u_t(x_o + ct,0)\,dt\\ &\quad - \tfrac{c}{2}\int_0^{t_o} u_x(x_o - ct,0)\,dt + \tfrac{c}{2}\int_0^{t_o} u_x(x_o + ct,0)\,dt\end{aligned} \tag{F.15}$$

If we make a change of variables $\zeta = x_o - ct$ in the first and third integrals and a change of variables $\zeta = x_o + ct$ in the second and fourth integrals, the (F.15) becomes

$$
\begin{aligned}
u(x_o, t_o) - u(x_o, 0) &= -\frac{1}{2c}\int_{x_o}^{x_o - ct_o} u_t(\zeta, 0)\, d\zeta + \frac{1}{2c}\int_{x_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta \\
&\quad + \frac{1}{2}\int_{x_o}^{x_o - ct_o} u_x(\zeta, 0)\, d\zeta + \frac{1}{2}\int_{x_o}^{x_o + ct_o} u_x(\zeta, 0)\, d\zeta \\
&= \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta \\
&\quad + \frac{1}{2}\, u(\zeta, 0)\Big|_{x_o}^{x_o - ct_o} + \frac{1}{2}\, u(\zeta, 0)\Big|_{x_o}^{x_o + ct_o} \\
&= \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta \\
&\quad + \frac{1}{2}\left(u(x_o - ct_o) + u(x_o + ct_o)\right) + u(x_o, 0)
\end{aligned}
$$

or

$$
\boxed{u(x_o, t_o) = \frac{1}{2}\left(u(x_o - ct_o) + u(x_o + ct_o)\right) + \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta}
$$

which is precisely Equation (3.7). $\square$

2. (Problem 5.7.1 in text)

Let $u(x,y)$ satisfy

(F.16)
$$u_{xx} - 2u_{xy} + u_{yy} + 3u_x - u + 1 = 0 \quad .$$

The discriminant for this PDE is

(F.17)
$$(A_{xy})^2 - A_{xx}A_{yy} = (-1)^2 - (1)(1) = 0$$

and so this equation is parabolic.

Now under a general coordinate transformation

(F.18)
$$\begin{aligned} \zeta &= \tilde{\zeta}(x,y) \\ \eta &= \tilde{\eta}(x,y) \\ \\ x &= \tilde{x}(\zeta,\eta) \\ y &= \tilde{y}(\zeta,\eta) \end{aligned}$$

a second order linear PDE

(F.19)
$$A_{xx}u_{xx} + 2A_{xy}u_{xy} + A_{yy}u_{yy} + B_x u_x + B_y u_y + Cu + F = 0$$

becomes

(F.20)
$$\tilde{A}_{\zeta\zeta}U_{\zeta\zeta} + 2\tilde{A}_{\zeta\eta}U_{\zeta\eta} + \tilde{A}_{\eta\eta}U_{\eta\eta} + \tilde{B}_\zeta U_\zeta + \tilde{B}_\eta U_\eta + \tilde{C}U + \tilde{F} = 0$$

where

$$U(\zeta,\eta) = u(\tilde{x}(\zeta,\eta), \tilde{y}(\zeta,\eta))$$

and

(F.21)
$$\begin{aligned} \tilde{A}_{\zeta\zeta} &= A_{xx}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{xy}\tilde{\zeta}_x\tilde{\zeta}_y + A_{yy}\tilde{\zeta}_y\tilde{\zeta}_y \\ \tilde{A}_{\zeta\eta} &= A_{xx}\tilde{\zeta}_x\tilde{\eta}_x + A_{xy}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + A_{yy}\tilde{\zeta}_y\tilde{\eta}_y \\ \tilde{A}_{\eta\eta} &= A_{xx}\tilde{\eta}_x\tilde{\eta}_x + 2A_{xy}\tilde{\eta}_x\tilde{\eta}_y + A_{yy}\tilde{\eta}_y\tilde{\eta}_y \\ \tilde{B}_\zeta &= A_{xx}\tilde{\zeta}_{xx} + 2A_{xy}\tilde{\zeta}_{xy} + A_{yy}\tilde{\zeta}_{yy} + B_x\tilde{\zeta}_x + B_y\tilde{\zeta}_y \\ \tilde{B}_\eta &= A_{xx}\tilde{\eta}_{xx} + 2A_{xy}\tilde{\eta}_{xy} + A_{yy}\tilde{\eta}_{yy} + B_x\tilde{\eta}_x + B_y\tilde{\eta}_y \\ \tilde{C}(\zeta,\eta) &= C(\tilde{x}(\zeta,\eta), \tilde{y}(\zeta,\eta)) \\ \tilde{F}(\zeta,\eta) &= F(\tilde{x}(\zeta,\eta), \tilde{y}(\zeta,\eta)) \quad . \end{aligned}$$

Thus, under a general coordinate transformation (F.16) takes the form (F.20) with

(F.22)
$$\begin{aligned} \tilde{A}_{\zeta\zeta} &= \tilde{\zeta}_x\tilde{\zeta}_x - 2\tilde{\zeta}_x\tilde{\zeta}_y + \tilde{\zeta}_y\tilde{\zeta}_y \\ \tilde{A}_{\zeta\eta} &= \tilde{\zeta}_x\tilde{\eta}_x - \left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + \zeta_y\tilde{\eta}_y \\ \tilde{A}_{\eta\eta} &= \tilde{\eta}_x\tilde{\eta}_x - 2\tilde{\eta}_x\tilde{\eta}_y + \tilde{\eta}_y\tilde{\eta}_y \\ \tilde{B}_\zeta &= \tilde{\zeta}_{xx} - 2\tilde{\zeta}_{xy} + \tilde{\zeta}_{yy} + 3\tilde{\zeta}_x \\ \tilde{B}_\eta &= \tilde{\eta}_{xx} - 2\tilde{\eta}_{xy} + \tilde{\eta}_{yy} + 3\tilde{\eta}_x \\ \tilde{C}(\zeta,\eta) &= -1 \\ \tilde{F}(\zeta,\eta) &= 1 \end{aligned}$$

In order to put (F.16) in standard form we must find a coordinate transformation for which $0 = \tilde{A}_{\zeta\zeta} = \tilde{A}_{\zeta\eta}$.

Let us represent the level curves of $\tilde{\zeta}$ as the graph of a function f of x. Then the condition

(F.23)
$$\tilde{\zeta}(x, f(x)) = const$$

leads to the relation

(F.24)
$$\tilde{\zeta}_x = -f'\tilde{\zeta}_y$$

Using (F.24) to eliminate the expressions $\tilde{\zeta}_x$ in

$$0 = \tilde{A}_{\zeta\zeta} = \tilde{\zeta}_x \tilde{\zeta}_x - 2\tilde{\zeta}_x \tilde{\zeta}_y + \tilde{\zeta}_y \tilde{\zeta}_y$$

we obtain

$$0 = \left((f')^2 + 2f' + 1 \right) \tilde{\zeta}_y \tilde{\eta}_y$$

or

$$(f')^2 + 2f' + 1 = 0 \quad . \tag{F.25}$$

Solving (F.25) for f' we find

$$f' = -1$$

or

$$f(x) = -x + c \quad .$$

Thus, we should choose the coordinate ζ so that the level curves $\tilde{\zeta}(x, y) = const$ coincide with the lines

$$y = -x + c \quad .$$

Therefore, we set

$$\tilde{\zeta}(x, y) = x + y \quad . \tag{F.26}$$

Since the original PDE is parabolic, there is only one family of characteristics, and so we will not be able to find another coordinate η such that $\tilde{A}_{\eta\eta} = 0$. Therefore, we shall not even bother looking for a better choice for a second coordinate and we'll simply set $\eta = y$.

Let us now write down the original PDE in terms of our new coordinates

$$\begin{aligned} \zeta(x, y) &= x + y \\ \eta(x, y) &= y \end{aligned}$$

We have

$$\begin{aligned}
\tilde{A}_{\zeta\zeta} &= A_{xx}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{xy}\tilde{\zeta}_x\tilde{\zeta}_y + A_{yy}\tilde{\zeta}_y\tilde{\zeta}_y = 1 - 2 + 1 = 0 \\
\tilde{A}_{\zeta\eta} &= A_{xx}\tilde{\zeta}_x\tilde{\eta}_x + A_{xy}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + A_{yy}\tilde{\zeta}_y\tilde{\eta}_y = 0 - (1 + 0) + 1 = 0 \\
\tilde{A}_{\eta\eta} &= A_{xx}\tilde{\eta}_x\tilde{\eta}_x + 2A_{xy}\tilde{\eta}_x\tilde{\eta}_y + A_{yy}\tilde{\eta}_y\tilde{\eta}_y = 0 + 0 + 1 = 1 \\
\tilde{B}_{\zeta} &= A_{xx}\tilde{\zeta}_{xx} + 2A_{xy}\tilde{\zeta}_{xy} + A_{yy}\tilde{\zeta}_{yy} + B_x\tilde{\zeta}_x + B_y\tilde{\zeta}_y = 0 + 0 + 0 + 3 + 0 = 3 \\
\tilde{B}_{\eta} &= A_{xx}\tilde{\eta}_{xx} + 2A_{xy}\tilde{\eta}_{xy} + A_{yy}\tilde{\eta}_{yy} + B_x\tilde{\eta}_x + B_y\tilde{\eta}_y = 0 + 0 + 0 + 0 + 0 = 0 \\
\tilde{C}(\zeta, \eta) &= -1 \\
\tilde{F}(\zeta, \eta) &= 1
\end{aligned}$$

Thus, (F.20) becomes

$$U_{\eta\eta} + 3U_{\zeta} - U + 1 = 0$$

or

$$U_{\eta\eta} + 3U_{\zeta} + U_{\eta} - U + 1 = 0 \quad . \tag{F.27}$$

Equation (F.16) is now in standard form.

(a)

Let us now try to construct the power series solution corresponding to the following Cauchy data:

$$\begin{aligned} u(x, 0) &= 2 \\ u_y(x, 0) &= 0 \end{aligned} \tag{F.28}$$

Solving the PDE (F.16) for u_{yy} we have

$$u_{yy} = 2u_{xy} - u_{xx} - 3u_x + u - 1 \quad . \tag{F.29}$$

Using the data (F.28) we can explicitly evaluate the right hand side of (F.29) along the x-axis;

$$\begin{aligned} v_{yy}(x,0) &= \left.(2u_{xy} - u_{xx} - 3u_x + u - 1)\right|_{y=0} \\ &= \left.\left(2\tfrac{\partial}{\partial x}u_y - \tfrac{\partial^2}{\partial x^2}u - 3\tfrac{\partial}{\partial x}u + u - 1\right)\right|_{y=0} \\ &= 0 - 0 - 0 + 2 - 1 \\ &= 1 \end{aligned} \tag{F.30}$$

If we can now differentiate (F.29) with respect to y and evaluate the result along the x-axis

$$\begin{aligned} u_{yyy}(x,0) &= \left.(2u_{xyy} - u_{xxy} - 3u_{xy} + u_y)\right|_{y=0} \\ &= \left.\left(2\frac{\partial}{\partial x}u_{yy} - \frac{\partial^2}{\partial x^2}u_y - 3\frac{\partial}{\partial x}u_y + u_y\right)\right|_{y=0} \\ &= 0 \end{aligned}$$

Similarly, we can compute

$$\begin{aligned} u_{yyyy} &= \left.(2u_{xyyy} - u_{xxyy} - 3u_{xyy} + u_{yy})\right|_{y=0} \\ &= 1 \end{aligned}$$

The following pattern emerges

$$\left.\frac{\partial^n u}{\partial y^n}\right|_{y=0} = \begin{cases} 1 & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases}$$

We thus arrive at the following power series solution to (F.16) and (F.28).

$$\begin{aligned} u(x,y) &= \sum_{n=0}^{\infty} \frac{y^n}{n!}\left.\left(\frac{\partial^n u}{\partial y^n}\right)\right|_{(x,0)} \\ &= u(x,0) + yu_y(x,0) + \frac{1}{2}y^2 u_{yy}(x,0) + \frac{1}{6}y^3 u_{yyy}(x,0) + \\ &= 2 + \frac{1}{2}y^2 + \frac{1}{4!}y^4 + \frac{1}{6!}y^6 + \cdots \\ &= 1 + \cosh(y) \end{aligned}$$

(b)

Since the line $x + y = 0$ is a characteristic for (F.16), the Cauchy problem corresponding to the data $u = 2$, $\frac{\partial u}{\partial n} = 0$ along the line $x + y$ is not well posed.

□

3. (Problem 5.7.3(b) in text)

Consider the PDE

(F.31)
$$\phi_{xx} + y\phi_{yy} - x\phi_y + y = 0 \quad .$$

The discriminant of this PDE is

(F.32)
$$(A_{xy})^2 - A_{xx}A_{yy} = 0 - (1)(y) = -y \quad .$$

We thus see that (F.31) is hyperbolic in the region where $y < 0$, parabolic in the region where $y = 0$, and elliptic in the region where $y > 0$.

Case 1: $y < 0$ (Hyperbolic Region)

We seek a coordinate transformation

(F.33)
$$\begin{aligned} \zeta &= \tilde{\zeta}(x,y) \\ \eta &= \tilde{\eta}(x,y) \end{aligned}$$

such that (F.31) takes the form

$$\Phi_{\zeta\eta} + B_\zeta \Phi_\zeta + B_\eta \Phi_\eta + \tilde{y}\,(\zeta,\eta) = 0 \quad .$$

Under an arbitrary coordinate transformation the coefficients of the second order terms become

(F.34)
$$\begin{aligned} A_{\zeta\eta} &= A_{xx}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{xy}\tilde{\zeta}_x\tilde{\zeta}_y + A_{yy}\tilde{\zeta}_y\tilde{\zeta}_y \\ &= \left(\tilde{\zeta}_x\tilde{\eta}_x\right) + y\left(\tilde{\zeta}_y\tilde{\eta}_y\right) \\ A_{\zeta\zeta} &= A_{xx}\tilde{\zeta}_x\tilde{\eta}_x + A_{xy}\tilde{\zeta}_x\tilde{\eta}_y + A_{xy}\tilde{\zeta}_y\tilde{\eta}_x + A_{yy}\tilde{\zeta}_y\tilde{\eta}_y \\ &= \left(\tilde{\zeta}_x\right)^2 + y\left(\tilde{\zeta}_y\right)^2 \\ A_{\eta\eta} &= A_{xx}\tilde{\eta}_x\tilde{\eta}_x + 2A_{xy}\tilde{\eta}_x\tilde{\eta}_y + A_{yy}\tilde{\eta}_y\tilde{\eta}_y \\ &= (\tilde{\eta}_x)^2 + y\,(\tilde{\eta}_y)^2 \end{aligned}$$

Let us first try to choose $\tilde{\eta}(x,y)$ so that $A_{\eta\eta}$ vanishes identically.

If the level curve $\tilde{\eta}(x,y) = const$ of the new coordinate η corresponds to the graph $x = f(y)$, then we have

(F.35)
$$\tilde{\eta}_y = -f'(y)\tilde{\eta}_x$$

and so (F.34) becomes

(F.36)
$$A_{\eta\eta} = \left(1 + y\,(f')^2\right)(\tilde{\eta}_x)^2 \quad .$$

Setting $A_{\eta\eta} = 0$, then yields

$$f'(y) = \pm\frac{1}{\sqrt{-y}}$$

so

(F.37)
$$x = f(y) = 2\sqrt{-y} + C \quad .$$

Corresponding to these two independent solutions for $f(y)$, we adopt the following coordinates

(F.38)
$$\begin{aligned} \zeta &= x + 2\sqrt{-y} \\ \eta &= x - 2\sqrt{-y} \quad . \end{aligned}$$

Then

$$\begin{aligned} \tilde{\zeta}_x = \tilde{\eta}_x &= 1 \\ \tilde{\zeta}_y = -\tilde{\eta}_y &= -\frac{1}{\sqrt{-y}} \\ \tilde{\zeta}_{xx} = \tilde{\eta}_{xx} &= 0 \\ \tilde{\zeta}_{yy} = -\tilde{\eta}_{yy} &= \frac{1}{2}\,[-y]^{-3/2} \end{aligned}$$

We now compute the coefficients of the PDE relative to the coordinates (F.38).

$$
\begin{aligned}
A_{\zeta\zeta} &= \left(\tilde{\zeta}_x\right)^2 + y\left(\tilde{\zeta}_y\right)^2 \\
&= (1)^2 + y\left(\frac{1}{-y}\right) \\
&= 0 \\
A_{\zeta\eta} &= \left(\tilde{\zeta}_x \tilde{\eta}_x\right) + y\left(\tilde{\zeta}_y \tilde{\eta}_y\right) \\
&= (1) + y(\frac{1}{y}) \\
&= 2 \\
A_{\eta\eta} &= (\tilde{\eta}_x)^2 - y\,(\tilde{\eta}_y)^2 \\
&= (1)^2 - y\left(\tfrac{1}{y}\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
B_\zeta &= \tilde{\zeta}_{xx} + y\tilde{\zeta}_{yy} - x\zeta_y \\
&= 0 + y\left(-\frac{1}{2}\left[-y\right]^{-3/2}\right) + \frac{x}{\sqrt{-y}} \\
&= \frac{1+2x}{2\sqrt{-y}} \\
&= \frac{2(1+\zeta+\eta)}{\zeta-\eta} \\
B_\eta &= \tilde{\eta}_{xx} + y\tilde{\eta}_{yy} - x\eta_y \\
&= 0 + y\left(\frac{1}{2}\left[-y\right]^{-3/2}\right) - \frac{x}{\sqrt{-y}} \\
&= \\
&= \frac{-2(1+\zeta+\eta)}{\zeta-\eta}
\end{aligned}
$$

Thus, (F.31) is equivalent to

$$
2\Phi_{\zeta\eta} + \frac{2(1+\zeta+\eta)}{\zeta-\eta}\Phi_\zeta - \frac{2(1+\zeta+\eta)}{\zeta-\eta}\Phi_\eta - \frac{(\zeta-\eta)^2}{16} = 0
$$

or

$$
\Phi_{\zeta\eta} + \frac{(1+\zeta+\eta)}{\zeta-\eta}\Phi_\zeta - \frac{(1+\zeta+\eta)}{\zeta-\eta}\Phi_\eta - \frac{(\zeta-\eta)^2}{32} = 0 \quad . \tag{F.39}
$$

Case 2: $y > 0$ (Elliptic Region)

We now try to find a coordinate transformation such that

$$
\begin{aligned}
0 &= A_{\zeta\eta} = \left(\tilde{\zeta}_x \tilde{\eta}_x\right) + y\left(\tilde{\zeta}_y \tilde{\eta}_y\right) \\
1 &= A_{\zeta\zeta} = \left(\tilde{\zeta}_x\right)^2 + y\left(\tilde{\zeta}_y\right)^2 \\
1 &= A_{\eta\eta} = (\tilde{\eta}_x)^2 + y\,(\tilde{\eta}_y)^2
\end{aligned}
\tag{F.40}
$$

The simplest way to satisfy the first two equations is to take

$$
\begin{aligned}
\tilde{\eta}(x,y) &= f(y) \\
\tilde{\zeta}(x,y) &= x \quad .
\end{aligned}
$$

Let us represent level curve of $\tilde{\eta}$. The third equation then implies

$$
1 = y\,(f')^2
$$

or

$$f' = \pm \frac{1}{\sqrt{y}} \quad .$$

We can thus take

$$\begin{aligned} \tilde{\zeta}(x,y) &= x \\ \tilde{\eta}(x,y) &= 2\sqrt{y} \quad . \end{aligned}$$

We then find

$$\begin{aligned} A_{\zeta\zeta} &= \left(\tilde{\zeta}_x\right)^2 + y\left(\tilde{\zeta}_y\right)^2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} A_{\zeta\eta} &= \left(\tilde{\zeta}_x \tilde{\eta}_x\right) + y\left(\tilde{\zeta}_y \tilde{\eta}_y\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} A_{\eta\eta} &= \left(\tilde{\eta}_x\right)^2 + y\left(\tilde{\eta}_y\right)^2 \\ &= y\left(\tfrac{1}{y}\right) \\ &= 1 \end{aligned}$$

$$\begin{aligned} B_{\zeta} &= \tilde{\zeta}_{xx} + y\tilde{\zeta}_{yy} - x\zeta_y \\ &= 0 \end{aligned}$$

$$\begin{aligned} B_{\eta} &= \tilde{\eta}_{xx} + y\tilde{\eta}_{yy} - x\eta_y \\ &= y\left(\frac{1}{2}[y]^{-3/2}\right) + \frac{x}{\sqrt{y}} \\ &= \frac{1}{2\sqrt{y}} + \frac{x}{\sqrt{y}} \\ &= \frac{1+2\zeta}{\eta} \end{aligned}$$

Thus, (F.31) is equivalent to

$$\Phi_{\zeta\zeta} + \Phi_{\eta\eta} + \frac{1+2\zeta}{\eta}\Phi_{\eta} + \frac{\eta^2}{4} = 0 \quad .$$

Alternatively, one could use the formulas developed in Lecture 16. To make contact with the formulas there, we take

$$\begin{aligned} x_1 &= y \\ x_2 &= x \\ y_1 &= \eta \\ y_2 &= \zeta \end{aligned}$$

and set

$$
\begin{aligned}
f' &= \frac{A_{12} + \sqrt{A_{11}A_{22} - (A_{12})^2}}{A_{11}} \\
&= \frac{1}{\sqrt{y}} \\
g' &= \frac{A_{12} - \sqrt{A_{11}A_{22} - (A_{12})^2}}{A_{11}} \\
&= -\frac{1}{\sqrt{y}}
\end{aligned}
$$

and so we have

$$
\begin{aligned}
x = f(y) &= 2\sqrt{y} + const \\
x = g(y) &= -2\sqrt{y} + const
\end{aligned}
$$

We might thus take

$$
\begin{aligned}
\tilde{\zeta}(x,y) &= x + 2\sqrt{y} \\
\tilde{\eta}(x,y) &= x - 2\sqrt{y}
\end{aligned}
$$

One then finds

$$
\begin{aligned}
A_{\zeta\zeta} &= \left(\tilde{\zeta}_x\right)^2 + y\left(\tilde{\zeta}_y\right)^2 \\
&= 1 + 1 \\
&= 2 \\
A_{\zeta\eta} &= \left(\tilde{\zeta}_x \tilde{\eta}_x\right) + y\left(\tilde{\zeta}_y \tilde{\eta}_y\right) \\
&= 1 - 1 \\
&= 0 \\
A_{\eta\eta} &= \left(\tilde{\eta}_x\right)^2 + y\left(\tilde{\eta}_y\right)^2 \\
&= 1 + 1 \\
&= 2 \\
B_\zeta &= \tilde{\zeta}_{xx} + y\tilde{\zeta}_{yy} - x\zeta_y \\
&= 0 + y\left(-\frac{1}{2}[y]^{-3/2}\right) - \frac{x}{\sqrt{y}} \\
&= \frac{-1 + 2x}{2\sqrt{y}} \\
&= \frac{-2(1 + \zeta + \eta)}{\zeta - \eta} \\
B_\eta &= \tilde{\eta}_{xx} + y\tilde{\eta}_{yy} - x\eta_y \\
&= 0 + y\left(\frac{1}{2}[y]^{-3/2}\right) + \frac{x}{\sqrt{-y}} \\
&= \\
&= \frac{2(1 + \zeta + \eta)}{\zeta - \eta}
\end{aligned}
$$

Thus, (F.31) is equivalent to

$$2\Phi_{\zeta\eta} - \frac{2(1+\zeta+\eta)}{\zeta-\eta}\Phi_\zeta + \frac{2(1+\zeta+\eta)}{\zeta-\eta}\Phi_\eta + \frac{(\zeta-\eta)^2}{16} = 0$$

or

$$\Phi_{\zeta\eta} - \frac{(1+\zeta+\eta)}{\zeta-\eta}\Phi_\zeta + \frac{(1+\zeta+\eta)}{\zeta-\eta}\Phi_\eta + \frac{(\zeta-\eta)^2}{32} = 0 \quad .$$

$\square$

4. (Problem 5.7.3(d) in text)

Consider the PDE

$$\phi_{xy} + y\phi_{yy} + \sin(x+y) = 0 \quad . \tag{F.41}$$

Comparing this equation with the general form of a second order linear equation

$$\sum_{i,j=1}^{2} A_{ij}\phi_{x_i x_j} + \sum_{i=1}^{2} B_i \phi_{x_i} + C\phi + F = 0 \tag{F.42}$$

we see that we must take

$$\begin{aligned} A_{11} &= 0 \\ A_{12} &= \tfrac{1}{2} \\ A_{22} &= y \\ B_1 &= 0 \\ B_2 &= 0 \\ C &= 0 \\ F &= \sin(x+y) \quad . \end{aligned} \tag{F.43}$$

The discriminant of (F.41) is thus

$$(A_{12})^2 - A_{11}A_{12} = \left(\frac{1}{2}\right)^2 - (0)(y) = \frac{1}{4} > 0 \tag{F.44}$$

and so the PDE (F.41) is hyperbolic. This means there must exist a coordinate transformation that puts (F.41) in the form

$$A'_{12}\Phi_{y_1y_2} + \sum_{i=1}^{2} B'_i \Phi_{y_i} + F' = 0 \quad . \tag{F.45}$$

Now for a general nonsingular coordinate transformation

$$\begin{aligned} \zeta &= \tilde{\zeta}(x,y) \\ \eta &= \tilde{\eta}(x,y) \\ x &= \tilde{x}(\zeta,\eta) \\ y &= \tilde{y}(\zeta,\eta) \end{aligned}$$

we have

$$\begin{aligned} A'_{\zeta\zeta} &= A_{11}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{12}\tilde{\zeta}_x\tilde{\zeta}_y + A_{22}\tilde{\zeta}_y\tilde{\zeta}_y \\ &= \tilde{\zeta}_x\tilde{\zeta}_y + y\tilde{\zeta}_y\tilde{\zeta}_y \\[1ex] A'_{\zeta\eta} &= A_{11}\tilde{\zeta}_x\tilde{\eta}_x + A_{12}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + A_{22}\tilde{\zeta}_y\tilde{\eta}_y \\ &= \tfrac{1}{2}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + y\tilde{\zeta}_y\tilde{\eta}_y \\[1ex] A'_{\eta\eta} &= A_{11}\tilde{\eta}_x\tilde{\eta}_x + 2A_{12}\tilde{\eta}_x\tilde{\eta}_y + A_{22}\tilde{\eta}_y\tilde{\eta}_y \\ &= \tilde{\eta}_x\tilde{\eta}_y + y\tilde{\eta}_y\tilde{\eta}_y \\[1ex] B'_{\zeta} &= A_{11}\tilde{\zeta}_{xx} + 2A_{12}\tilde{\zeta}_{xy} + A_{22}\tilde{\zeta}_{yy} + B_1\tilde{\zeta}_x + B_2\tilde{\zeta}_y \\ &= \tilde{\zeta}_{xy} + y\tilde{\zeta}_{yy} \\[1ex] B'_{\eta} &= A_{11}\tilde{\eta}_{xx} + 2A_{12}\tilde{\eta}_{xy} + A_{22}\tilde{\eta}_{yy} + B_1\tilde{\eta}_x + B_2\tilde{\eta}_y \\ &= \tilde{\eta}_{xy} + y\tilde{\eta}_{yy} \end{aligned} \tag{F.46}$$

Of course since $A_{11} = 0$, already we can simply take

$$\tilde{\zeta}(x,y) = x \quad ; \tag{F.47}$$

but note that (F.46) tells us that this is consistent.

Let's try to eliminate $A'_{\eta\eta}$. Let $f(x)$ be a function whose graph corresponds the level curve $\tilde{\eta}(x,y) = const$. We then have

$$\tilde{\eta}\,(x, f(x)) = const \tag{F.48}$$

which when differentiated with respect to x yields

$$\tilde{\eta}_x = -f'\,\tilde{\eta}_y \quad . \tag{F.49}$$

Plugging this into the expression (F.46) for $A'_{\eta\eta}$ and setting the resulting expression equal to zero yields

$$0 = -f'\,\tilde{\eta}_y\tilde{\eta}_y + y\tilde{\eta}_y\tilde{\eta}_y = \left(-f' + f\right)\tilde{\eta}_y\tilde{\eta}_y \tag{F.50}$$

or

$$f'(x) - f(x) = 0 \quad . \tag{F.51}$$

The general solution to (F.51) is

$$f(x) = Ce^x. \tag{F.52}$$

Setting $y = f(x)$ and $\eta = C$, we obtain a suitable expression for η

$$\eta = ye^{-x} \quad . \tag{F.53}$$

Inverting (F.47) and (F.53) we obtain

$$\begin{array}{rcl} x & = & \zeta \\ y & = & \eta e^{\zeta} \quad . \end{array} \tag{F.54}$$

Inserting (F.47), (F.53) and (F.54) into (F.46) we find that all the coefficients A'_{ij} and B'_i vanish except

$$\begin{array}{rcl} A'_{\zeta\eta} & = & \frac{1}{2}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + y\tilde{\zeta}_y\tilde{\eta}_y \\ & = & \frac{1}{2}\left((1)\left(e^{-\zeta}\right) + (0)\,(-\eta)\right) + \eta e^{\zeta}(0)\left(e^{-\zeta}\right) \\ & = & \frac{1}{2}e^{-\zeta} \end{array} \tag{F.55}$$

and

$$\begin{array}{rcl} B'_\eta & = & \tilde{\eta}_{xy} + y\tilde{\eta}_{yy} \\ & = & -e^{-\zeta} \end{array}$$

Thus, (F.41) is equivalent to

$$\begin{array}{rcl} 0 & = & 2A'_{\zeta\eta}\Phi_{\zeta\eta} + B'_\eta\Phi_\eta + F\left(\tilde{x}(\zeta,\eta), \tilde{y}(\zeta,\eta)\right) \\ & = & e^{-\zeta}\Phi_{\zeta\eta} - e^{-\zeta}\Phi_\eta + \sin\left(\zeta + \eta e^{\zeta}\right) \end{array} \tag{F.56}$$

or

$$\Phi_{\zeta\eta} - \Phi_\eta = -e^{\zeta}\sin\left(\zeta + \eta e^{\zeta}\right) \quad . \tag{F.57}$$

To solve this equation we first integrate (to find anti-derivatives of) both sides of (F.57) with respect to η. We get

$$\Phi_\zeta - \Phi = \cos\left(\zeta + \eta e^{\zeta}\right) + H(\zeta) \tag{F.58}$$

Now we regard η as being fixed so that (F.58) can be viewed as an first order linear ODE for Φ_ζ; however, in solving this ODE we must regard any constant of integration that is introduced as being potentially dependent on η.

The general solution of an ODE of the form

$$y' + p(x)y = g(x) \tag{F.59}$$

is

$$f(x) = \frac{1}{\mu(x)}\left[\int_0^x \mu(s)g(s)ds + C\right] \tag{F.60}$$

where

$$\mu(x) = e^{\int^x p(t)\,dt} \quad . \tag{F.61}$$

In our case,

$$p(\zeta) = -1 \quad , \quad g(\zeta) = \cos\left(\zeta + \eta e^{\zeta}\right) + H(\zeta) \quad , \tag{F.62}$$

so

$$\mu(\zeta) = e^{-\zeta} \tag{F.63}$$

and

$$\begin{aligned} \Phi(\zeta, \eta) &= \frac{1}{e^{-\zeta}} \left[\int_0^{\zeta} e^{-\alpha} \left(\cos\left(\alpha + \eta e^{\alpha}\right) + H(\alpha) \right) d\alpha + C(\eta) \right] \\ &= e^{\zeta} \int_0^{\zeta} e^{-\alpha} \cos\left(\alpha + \eta e^{\alpha}\right) d\alpha + e^{\zeta} F(\eta) + G(\zeta) \end{aligned} \tag{F.64}$$

where

$$\begin{aligned} F(\eta) &= C(\eta) \\ G(\zeta) &= e^{\zeta} \int_0^{\zeta} e^{-\alpha} H(\alpha) d\alpha \end{aligned} \tag{F.65}$$

can both be regarded as arbitrary functions. We thus write the general solution of (F.58) as

$$\Phi(\zeta, \eta) = e^{\zeta} \int_0^{\zeta} e^{-\alpha} \cos\left(\alpha + \eta e^{\alpha}\right) d\alpha + e^{\zeta} F(\eta) + G(\zeta) \quad . \tag{F.66}$$

The general solution to (F.41) is now obtained by changing back to the original variables x and y using (F.47) and (F.53). One thus obtains

$$\begin{aligned} \phi(x, y) &= \Phi(x, ye^{-x}) \\ &= e^{\zeta} \int_0^x e^{-\alpha} \cos\left(\alpha + ye^{-x} e^{\alpha}\right) d\alpha + e^x F(ye^{-x}) + G(x) \\ &= e^{\zeta} \int_0^x e^{-\alpha} \cos\left(\alpha + ye^{\alpha - x}\right) d\alpha + e^x F(ye^{-x}) + G(x) \quad . \end{aligned} \tag{F.67}$$

□

5. (Problem 5.7.6 in text)

Let a function $u(\zeta, \eta)$ satisfy the equation

$$u_{\zeta\eta} + \alpha u_\zeta + \beta u_\eta + \gamma u + \delta = 0 \tag{F.68}$$

where $\alpha, \beta, \gamma, \delta$ are functions of ζ and η. In the (ζ, η)-plane, the characteristics are lines parallel to the coordinate axes. We have shown that the Cauchy data on a characteristic is not adequate for computation of the solution elsewhere; show, moreover, that Cauchy data cannot be arbitrarily prescribed on a characteristic in any event because of a compatibility condition imposed by the PDE itself. What does this result imply about Cauchy data on the (x, y) plane characteristic of Eq. (5.1)

$$A\phi_{xx} + 2B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi + G = 0$$

in the hyperbolic case? In the parabolic case? Is there any similar restriction in the elliptic case?

Suppose we try to impose the following condition on a solution of (F.68)

$$\begin{aligned} u(\zeta, \eta_o) &= f(\zeta) \\ u_\eta(\zeta, \eta_o) &= g(\zeta) \end{aligned}$$

Then along the characteristic $\eta = \eta_o$ we have

$$\begin{aligned} u_\zeta(\zeta, \eta_o) &= f'(\zeta) \\ u_{\eta\zeta}(\zeta, \eta_o) &= g'(\zeta) \end{aligned}$$

Evaluating (F.68) along the characteristic $\eta = \eta_o$ thus yields

$$0 = g' + \alpha f' + \beta g + \gamma f + \delta \tag{F.69}$$

Thus, the function g and f can **not** be completely arbitrary (they must satisfy (F.69) or else the PDE will not be satisfied along the curve $\eta = \eta_o$).

In the hyperbolic case, where there are two families of characteristics, this result implies that there will be two families of curves for which the Cauchy problem is ill-posed.

In the parabolic case, where there is a single family of characteristics, this result implies that there will be a 1-parameter family of curves for which the Cauchy problem is ill-posed.

In the parabolic case, where there are no characteristics, the Cauchy problem will always be well-posed.