APPENDIX E

Solutions to Problem Set 5

1. (Problem 4.5.4 in text)

(a) Use a series expansion to find $\psi(r,\theta)$ satisfying

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} = -\frac{1}{r^2}\cos(\theta) \tag{E.1}$$

in the sector $1 < r < 3$, $0 < \theta < \frac{\pi}{2}$, where the boundary values on the sector are given by

$$\begin{array}{rcl} \psi(1,\theta) &=& 0 \\ \psi(3,\theta) &=& \frac{2}{3}\cos(\theta) \\ \psi(r,0) &=& \frac{r-1}{r} \\ \psi\left(r,\frac{\pi}{2}\right) &=& 0 \end{array} \tag{E.2}$$

We begin by representing $\psi(r,\theta)$ in terms of its Fourier series with respect to θ:

$$\psi(r,\theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(r)\sin(n\theta) \tag{E.3}$$

In view of the last boundary condition, ψ must vanish along the line $\theta = \frac{\pi}{2}$. Let us then immediately set all the coefficients $b_n(r) = 0$ (If we run into a consistency problem latter on we may have to go back and relax this condition. But so long as this assumption works for us we will not loose any solution; because if a solution exists it unique.) Thus, we take

$$\psi(r,\theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) \quad . \tag{E.4}$$

Plugging this expression into (E.1) we obtain

$$\frac{1}{2}\left(a_0'' + \frac{1}{r}a_0'\right) + \sum_{n=1}^{\infty}\left(a_n'' + \frac{1}{r}a_n' - n^2 a_n\right)\cos(n\theta) = -\frac{\cos}{r^2} \quad . \tag{E.5}$$

Multiplying both sides of (E.5) by $\frac{1}{\pi}\cos(m\theta)$, integrating between 0 and 2π, and employing the othogonality properties of the cosine functions we obtain

$$a_0'' + \frac{1}{r}a_0' = 0 \tag{E.6}$$

$$a_1'' + \frac{1}{r}a_1' - \frac{1}{r^2}a_1 = -\frac{1}{r^2} \tag{E.7}$$

$$a_n'' + \frac{1}{r}a_n' - \frac{n^2}{r^2}a_n = 0 \quad , \quad n = 2,3,4,5,\cdots \tag{E.8}$$

The general solution of (E.6) is

$$a_0(r) = A_0 + B_0 \ln|r| \quad . \tag{E.9}$$

The equations in (E.8) are homogeneous Euler type equations with general solutions of the form

$$a_n(r) = A_n r^{-n} + B_n r^n \quad . \tag{E.10}$$

Equation (E.7) is a non-homogeneous Euler-type equation. Its general solution can be determined by the Method of Undetermined Coefficients. The two linearly independent solutions of the corresponding homogeneous problem

$$a_1'' + \frac{1}{r}a_1' - \frac{1}{r^2}a_1$$

are

$$\begin{aligned} y_1(r) &= r^{-1} \\ y_2(r) &= r \end{aligned}$$

and

$$W[y_1, y_2] = r^{-1}(1) - \left(-r^{-2}\right) r = \frac{2}{r} \quad .$$

And so the general solution of (E.7) will be

$$\begin{aligned} a_1(r) &= r^{-1}\left(A_1 - \int \frac{(r)\left(-\frac{1}{r^2}\right)}{\frac{2}{r}} dr\right) + r\left(B_1 + \int \frac{\left(\frac{1}{r}\right)\left(-\frac{1}{r^2}\right)}{\frac{2}{r}} dr\right) \\ &= r^{-1}\left(A_1 + \tfrac{r}{2}\right) + r\left(B_1 + \tfrac{1}{2r}\right) \\ &= 1 + A_1 r^{-1} + B_1 r \quad . \end{aligned} \tag{E.11}$$

Replacing the coefficients $a_n(r)$, $n = 0, 1, \dots$, in (E.4) by the solutions (E.9), (E.10), and (E.12) we obtain

$$\psi(r,\theta) = \frac{1}{2}A_0 + \frac{1}{2}B_0 \ln|r| + \cos(\theta) + \sum_{n=1}^{\infty}\left(A_n r^{-n} + B_n r^n\right)\cos(n\theta) \quad . \tag{E.12}$$

To fix the constants A_n, B_n, $n = 0, 1, 2, \dots$, we now impose the remaining boundary conditions. The first boundary condition

$$\psi(1,\theta) = 0$$

leads to

$$\frac{1}{2}A_0 + \cos(\theta) + \sum_{n=1}^{\infty}(A_n + B_n)\cos(\theta) = 0 \quad .$$

Applying the orthogonality properties of the cosine functions we thus obtain

$$\begin{aligned} A_0 &= 0 \\ 1 + A_1 + B_1 &= 0 \\ A_n + B_n &= 0 \end{aligned} \tag{E.13}$$

Utilizing these relations we thus obtain

$$\begin{aligned} \psi(r,\theta) &= \tfrac{1}{2}B_0 + \left(1 + A_1 r^{-1} - (1 + A_1) r\right)\cos(\theta) \\ &\quad + \textstyle\sum_{n=2}^{\infty} A_n \left(r^{-n} - r^n\right)\cos(n\theta) \end{aligned} \tag{E.14}$$

The second boundary condition

$$\psi(3,\theta) = \frac{2}{3}\cos(\theta)$$

now leads to

$$\begin{aligned} \frac{2}{3}\cos(\theta) &= \frac{1}{2}B_0 \ln|3| + \left(1 + A_1 3^{-1} - (1 + A_1) 3\right)\cos(\theta) \\ &\quad + \sum_{n=2}^{\infty} A_n \left(3^{-n} - 3^n\right)\cos(n\theta) \end{aligned}$$

which in view of the orthogonality properties of the cosine functions implies

$$\begin{aligned} B_0 &= 0 \\ 1 + \tfrac{1}{3}A_1 - 3A_1 - 3 &= \tfrac{2}{3} \\ A_n &= 0 \end{aligned} \tag{E.15}$$

The second equation in (E.15) is equivalent to

$$A_1 = -1 \quad ,$$

thus, (E.14) becomes

$$\psi(r,\theta) = \left(1 - r^{-1}\right)\cos(\theta) \quad . \tag{E.16}$$

This is the desired solution of (E.1) and (E.2). □

(b) Formulate the Dirichlet problem for the sector $0 < r < R$, $0 < \theta < \Theta$ (where R and Θ are given), involving Poisson's equation, and obtain a general series solution.

The general Dirichlet problem on the sector defined above would be the PDE/BVP

$$\psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} = f(r,\theta) \tag{E.17}$$

$$\begin{aligned} \psi(r,0) &= p(r) \\ \psi(r,\Theta) &= q(r) \\ \psi(R,\theta) &= t(\theta) \end{aligned} \tag{E.18}$$

As a Fourier expansion of the solution of this PDE/BVP we can use

$$\psi(r,\theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos(n\theta) + \sum_{n=1}^{\infty} b_n(r)\sin(n\theta) \quad . \tag{E.19}$$

since the functions $\cos(n\theta)$ and $\sin(n\theta)$ provide a complete set of functions on the interval $(0, 2\pi)$, and so also on the subinterval $(0,\Theta)$. However, this set of basis functions is not the most natural for our given boundary conditions. A better choice would be an expansion based on eigenfunctions of the form

$$\gamma_n(x) = \sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{E.20}$$

associated with to the Sturm-Louisville problem

$$y'' - \lambda^2 y = 0 \tag{E.21}$$

$$y(0) = 0 \tag{E.22}$$

$$y(\Theta) = 0 \tag{E.23}$$

Note that the ODE in (E.21) corresponds the ODE with respect to θ that arises when we apply Separation of Variables to Laplace's equation. However, unlike the boundary conditions for our general Dirichlet problem, the boundary conditions (E.22) and (E.23) are homogeneous (as they must be for Sturm-Louisville problems).

We will eventually apply the method of Lecture 5 to handle the inhomogeneous boundary conditions of the PDE; but first let us concentrate on solving the following simpler problem:

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = F(x,y) \tag{E.24}$$

$$\phi(r,0) = 0 \tag{E.25}$$

$$\phi(r,\Theta) = 0 \tag{E.26}$$

$$\phi(R,\theta) = T(\theta) \tag{E.27}$$

We begin by writing

$$\phi(r,\theta) = \sum_{n=1}^{\infty} a_n(r)\sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{E.28}$$

and

$$f(r,\theta) = \sum_{n=1}^{\infty} f_n(r)\sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{E.29}$$

where coefficients $f_n(r)$ are determined by

$$f_n(r) = \frac{2}{\Theta}\int_0^{\Theta} f(r,\theta)\sin\left(\frac{n\pi\theta}{\Theta}\right)d\theta \tag{E.30}$$

and the coefficients $a_n(r)$ are to be determined by the requirement that $\phi(r,\theta)$ satisfies (E.21) - (E.23).

Plugging (E.28) and (E.29), respectively, into the left and right hands sides of (E.21) we obtain

$$\sum_{n=1}^{\infty}\left[a_n'' + \frac{1}{r}a_n' - \frac{n^2\pi^2}{r^2\Theta^2}a_n\right]\sin\left(\frac{n\pi\theta}{\Theta}\right) = \sum_{n=1}^{\infty} f_n(r)\sin\left(\frac{n\pi\theta}{\Theta}\right)$$

or

$$\sum_{n=1}^{\infty}\left[a_n'' + \frac{1}{r}a_n' - \frac{n^2\pi^2}{r^2\Theta^2}a_n - f_n(r)\right]\sin\left(\frac{n\pi\theta}{\Theta}\right) = 0.$$

From this we can conclude that the coefficients function $a_n(r)$ obeys the following differential equation:

$$a_n'' + \frac{1}{r}a_n' - \frac{n^2\pi^2}{r^2\Theta^2}a_n = f_n(r). \tag{E.31}$$

Now the homogeneous equation corresponding to (E.31) is the Euler-type equation

$$r^2R'' + rR' + \left(\frac{n\pi}{\Theta}\right)^2 R = 0.$$

This equation has the following pair of linearly independent solutions

$$\begin{aligned} R_1(r) &= r^{-\frac{n\pi}{\Theta}} \\ R_2(r) &= r^{\frac{n\pi}{\Theta}} \end{aligned}$$

Therefore the Method of Variation of Parameters yields

$$a_n(r) = r^{-\frac{n\pi}{\Theta}}\left[A_n - \frac{\Theta}{2n\pi}\int_R^r s^{\frac{n\pi}{\Theta}}f_n(s)ds\right] + r^{\frac{n\pi}{\Theta}}\left[B_n + \frac{\Theta}{2n\pi}\int_R^r s^{-\frac{n\pi}{\Theta}}f_n(s)ds\right]. \tag{E.32}$$

In order to ensure that the coefficient functions are well-behaved as $r \to 0$, we must take

$$A_n = \frac{\Theta}{2n\pi}\int_R^0 s^{\frac{n\pi}{\Theta}}f_n(s)ds \tag{E.33}$$

We can also impose (E.23) which after expanding $T(\theta)$ in terms of the eigenfunctions $\sin\left(\frac{n\pi\theta}{\Theta}\right)$ leads to

$$\sum_{n=1}^{\infty} a_n(R)\sin\left(\frac{n\pi\theta}{\Theta}\right) = \sum_{n=1}^{\infty} t_n\sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{E.34}$$

where, of course, the coefficients t_n are given by

$$t_n = \frac{2}{\Theta}\int_0^{\Theta} T(\theta)\sin\left(\frac{n\pi\theta}{\Theta}\right)d\theta. \tag{E.35}$$

From (E.34) we can conclude that

$$a_n(R) = t_n. \tag{E.36}$$

Evaluating (E.32) at $r = R$ and using (E.36) and (E.33) we obtain

$$t_n = R^{-\frac{n\pi}{\Theta}}A_n + R^{\frac{n\pi}{\Theta}}B_n$$

or

$$B_n = R^{-\frac{n\pi}{\Theta}}t_n - R^{-\frac{2n\pi}{\Theta}}A_n \tag{E.37}$$

We now have formulas for all the ingredients of the expansion (E.28).

In summary, the solution to

$$\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} = F(x,y) \tag{E.38}$$
$$\phi(r,0) = 0 \tag{E.39}$$
$$\phi(r,\Theta) = 0 \tag{E.40}$$
$$\phi(R,\theta) = T(\theta) \tag{E.41}$$

is given by

$$\phi(r,\theta) = \sum_{n=1}^{\infty} a_n(r) \sin\left(\frac{n\pi\theta}{\Theta}\right) \tag{E.42}$$

where

$$a_n(r) = r^{-\frac{n\pi}{\Theta}}\left[A_n - \frac{\Theta}{2n\pi}\int_R^r s^{\frac{n\pi}{\Theta}} f_n(s)ds\right] + r^{\frac{n\pi}{\Theta}}\left[B_n + \frac{\Theta}{2n\pi}\int_R^r s^{-\frac{n\pi}{\Theta}} f_n(s)ds\right] \tag{E.43}$$
$$A_n = \frac{\Theta}{2n\pi}\int_R^0 s^{\frac{n\pi}{\Theta}} f_n(s)ds \tag{E.44}$$
$$B_n = R^{-\frac{n\pi}{\Theta}} t_n - R^{-\frac{2n\pi}{\Theta}} A_n \tag{E.45}$$
$$t_n = \frac{2}{\Theta}\int_0^{\Theta} T(\theta)\sin\left(\frac{n\pi\theta}{\Theta}\right) d\theta \tag{E.46}$$

We will now (finally) address the solution of Laplace's equation with the inhomogeneous boundary conditions (E.18). So as we did in Lecture 5, we relate solutions of

$$\begin{array}{rcl} \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} & = & f(r,\theta) \\ \psi(r,0) & = & p(r) \\ \psi(r,\Theta) & = & q(r) \\ \psi(R,\theta) & = & t(\theta) \end{array} \tag{E.47}$$

to solutions of

$$\begin{array}{rcl} \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} & = & F(r,\theta) \\ \phi(r,0) & = & 0 \\ \phi(r,\Theta) & = & 0 \\ \phi(R,\theta) & = & T(\theta) \end{array} \tag{E.48}$$

If we set

$$\phi(r,\theta) = \psi(r,\theta) - \cos\left(\frac{\pi\theta}{2\Theta}\right)p(r) - \sin\left(\frac{\pi\theta}{2\Theta}\right)q(r)$$

then we obviously have

$$\begin{aligned} \phi(r,0) &= \psi(r,0) - p(r) = 0 \\ \phi(r,\Theta) &= \psi(r,\Theta) - q(r) = 0 \end{aligned}$$

But we also have

$$\begin{aligned} \phi(R,\theta) &= \psi(r,\theta) - \cos\left(\frac{\pi\theta}{2\Theta}\right)p(R) - \sin\left(\frac{\pi\theta}{2\Theta}\right)q(R) \\ &= t(\theta)\psi - \cos\left(\frac{\pi\theta}{2\Theta}\right)p(R) - \sin\left(\frac{\pi\theta}{2\Theta}\right)q(R) \end{aligned}$$

and

$$\begin{aligned}
\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} &= \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta} \\
&\quad - \left(p(r)'' + \frac{1}{r}p(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 p(r)\right)\cos\left(\frac{\pi\theta}{2\Theta}\right) \\
&\quad - \left(q(r)'' + \frac{1}{r}q(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 q(r)\right)\sin\left(\frac{\pi\theta}{2\Theta}\right) \\
&= f(r,\theta) - \left(p(r)'' + \frac{1}{r}p(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 p(r)\right)\cos\left(\frac{\pi\theta}{2\Theta}\right) \\
&\quad - \left(q(r)'' + \frac{1}{r}q(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 q(r)\right)\sin\left(\frac{\pi\theta}{2\Theta}\right)
\end{aligned}$$

Therefore, the solution $\psi(r,\theta)$ of (E.47) can be constructed by setting

$$\psi(r,\theta) = \phi(r,\theta) + \cos\left(\frac{\pi\theta}{2\Theta}\right)p(r) + \sin\left(\frac{\pi\theta}{2\Theta}\right)q(r)$$

where $\phi(r,\theta)$ is the solution of (E.48) with

$$\begin{aligned}
F(r,\theta) &= f(r,\theta) - \left(p(r)'' + \frac{1}{r}p(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 p(r)\right)\cos\left(\frac{\pi\theta}{2\Theta}\right) \\
&\quad - \left(q(r)'' + \frac{1}{r}q(r)' - \frac{1}{r^2}\left(\frac{\pi}{2\Theta}\right)^2 q(r)\right)\sin\left(\frac{\pi\theta}{2\Theta}\right) \\
T(\theta) &= t(\theta)\psi - \cos\left(\frac{\pi\theta}{2\Theta}\right)p(R) - \sin\left(\frac{\pi\theta}{2\Theta}\right)q(R)
\end{aligned}$$

The solution of (E.48) can be calculated via the formulas (E.42) - (E.46).

□

2. (Problem 4.8.4 in text)

Discuss the properties of the mapping function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{E.49}$$

What happens to circles and straight-line rays in the z plane? Can this mapping function, together with the result of Problem 4.5.6(c), be used to solve the exterior Dirichlet problem for a region outside an ellipse? If so, provide a recipe.

A circle (with center $z = 0$) in the z plane can be represented as a curve of the form

$$z(t) = \rho e^{it} \quad .$$

The image of such a circle under the mapping (E.49) is just

$$\begin{aligned}
w(t) &= \tfrac{1}{2}\left(\rho e^{it} + \tfrac{1}{\rho e^{it}}\right) \\
&= \tfrac{1}{2\rho}\left(\rho^2 e^{it} + e^{-it}\right) \\
&= \tfrac{1}{2\rho}\left(\left(\rho^2+1\right)\cos(t) + i\left(\rho^2-1\right)\sin(t)\right) \\
&= \left(\tfrac{\rho^2+1}{2\rho}\right)\cos(t) + i\left(\tfrac{\rho^2-1}{2\rho}\right)\sin(t)
\end{aligned} \tag{E.50}$$

This is the equation of an ellipse in the (u,v)-plane with major and minor axes given by

$$
\begin{aligned}
r_{max} &= \tfrac{1}{2}\left(\rho + \tfrac{1}{\rho}\right) \\
r_{min} &= \tfrac{1}{2}\left|\rho - \tfrac{1}{\rho}\right|
\end{aligned}
\tag{E.51}
$$

Note that in the limit $\rho \to 1$ the image of the circle of radius ρ under the map ω collapses to the line segment $[-1,+1]$ on the real line.

A straight line ray in the z plane can be represented by a curve of the form

$$z(t) = te^{i\theta_o}.$$

The image of such a ray under the mapping (E.49) is

$$w(t) = \left(\frac{t^2+1}{2t}\right)\cos(\theta_o) + i\left(\frac{t^2-1}{2t}\right)\sin(\theta_o) \tag{E.52}$$

As $t \to +\infty$, this curve asymptotically approaches the straight line

$$w_\infty(t) = \frac{1}{2}\left(\cos(\theta_o) + i\sin(\theta_o)\right)t \quad , \tag{E.53}$$

which is just a straight line with slope θ_o with respect to the real axis.

As $t \to 1$, the curve (E.52) tends to the point

$$w_1 = \cos(\theta_o) \tag{E.54}$$

As $t \to 0$, the curve (E.52) approaches the curve

$$w_0(t) = \frac{1}{2}\left(\cos(\theta_o) - i\sin(\theta_o)\right)\frac{1}{t} \tag{E.55}$$

which is a straight line heading to infinity in a direction perpendicular to the line (E.53).

Let us look again at the elliptical images of circles under the map w. From (E.51) we see that if $\rho > 1$ then $r_{max}(\rho)$ and $r_{min}(\rho)$ are monotonically increasing. This implies that the exterior of a circle of radius $\rho > 1$ is mapped 1:1 onto the exterior of an ellipse with semi-major axis $\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$ and semi-minor axix $\frac{1}{2}\left(\rho - \frac{1}{\rho}\right)$.

When $0 < \rho < 1$ r_{max} and r_{min} are both monotonically decreasing. Thus, the interior of a circle or radius $\rho < 1$ is mapped onto the exterior of an ellipse with semi-major axis $\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$ and semi-minor axis $\frac{1}{2}\left(\frac{1}{\rho} - \rho\right)$.

Suppose now we seek a solution of $u_{xx} + u_{yy} = 0$ in region exterior exterior to the ellipse

$$\frac{4x^2}{\left(R + \frac{1}{R}\right)^2} + \frac{4y^2}{\left(\frac{1}{R} - R\right)^2} = 1 \quad , \quad 0 < R < 1 \quad ,$$

satisfying $u = f(\theta)$ on the boundary of the ellipse. We can find such a solution by first solving the following Dirichlet problem and then applying the map ω: find a solution ϕ of Laplace's equation in the region interior to the circle of radius R satisfying

$$\psi(R,\theta) = f(-\theta) \quad .$$

(The minus sign is needed since the $arg(w(z)) = -arg(z)$ when $|z| < 1$. See Eq. (E.50).)

To obtain a appropriate solution of the Dirichlet problem on the circle we can simply employ the Poisson integral formula

$$\psi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(-\theta)\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\alpha - \theta)}d\theta \quad . \tag{E.56}$$

We now pull this solution back to the original ellipse using the inverse mapping

$$z = w - \sqrt{w^2 - 1} \quad .$$

(Note: the map w is actually 2:1, this is the inverse that maps 1:1 from the region outside the ellipse onto the region inside the corresponding circle.) Thus, the desired solution will be

$$\psi\left(w - \sqrt{w^2 - 1}\right)$$

with $\psi(z)$ determined by (E.56). □

3. (Problem 4.8.5 in text)

With $w = u + iv$, $z = x + iy$, let $w = f(z)$ be analytic in a region of the z plane. Show that the (x, y)-plane curves $u = const$, $v = const$ intersect orthogonally. Show also that the directional derivatives of u in any direction is equal to the directional derivative of v in the orthogonal direction, and use this fact to explain how a Neumann problem for a finite region R can be transformed into a Dirichlet problem for a conjugate function.

First let us recall some simple facts from Analytic Geometry.

Two parameterized curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^2$ and $\beta : J \subset \mathbb{R} \to \mathbb{R}^2$ will intersect orthogonally at a point $(x, y) \in \mathbb{R}^2$ if there exist $s_o \in I$, $t_o \in J$ such that $\alpha(s_o) = \beta(t_o)$ and

$$0 = \left.\frac{d\alpha}{ds}\right|_{s_o} \cdot \left.\frac{d\beta}{dt}\right|_{t_o} = \frac{d\alpha_x}{dt}(t_o)\,\frac{d\beta_x}{ds}(s_o) + \frac{d\alpha_y}{dt}(t_o)\,\frac{d\beta_y}{ds}(s_o)\,. \tag{E.57}$$

If we represent the curves α and β locally as the graphs of two functions f and g of x

$$\begin{array}{rcl} \alpha(t) & = & (t, f(t)) \quad , \\ \beta(s) & = & (s, g(t)) \quad , \end{array} \tag{E.58}$$

then the condition for intersection when $t = s = 0$ is

$$0 = (1, f'(0)) \cdot (1, g'(0)) = 1 + f'(0)g'(0)$$

or

$$f'(0) = -\frac{1}{g'(0)} \quad . \tag{E.59}$$

Now let $w(z) = u(x, y) + iv(x, y)$ be an analytic function. The Cauchy-Riemann conditions then require

$$\begin{array}{rcl} \frac{\partial u}{\partial x} & = & \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & = & -\frac{\partial u}{\partial y} \end{array} \tag{E.60}$$

Let $f(x)$ be a function whose graph corresponds, in a neighborhood of a point $(x_o, y_0) = (x_o, f(x_o))$, to a curve $u(x, y) = c_1$. We then have the following identity

$$u(x, f(x)) = c_1 \quad . \tag{E.61}$$

Differentiating this equation with respect to x yields

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} f' = 0 \quad . \tag{E.62}$$

Similarly, let $g(x)$ be a function whose graph corresponds to a curve $v(x, y) = c_2$ in a neighborhood of the point $(x_o, y_o) = (x_o, g(x_o))$. We then have

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} g' = 0 \quad . \tag{E.63}$$

Using (E.60) we can rewrite (E.63) as

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} g' \quad . \tag{E.64}$$

Using (E.64) to eliminate $\frac{\partial u}{\partial y}$ in (E.62) we obtain

$$\left(1 + f'g'\right) \frac{\partial u}{\partial x} = 0 \tag{E.65}$$

which so long as $\frac{\partial u}{\partial x} \neq 0$ implies that the two curves $u(x,y) = c_1$ and $v(x,y) = c_2$ intersect orthogonally.

The directional derivative of a function $f(x,y)$ with respect to the direction $\mathbf{e} = (e_x, e_y)$ is given by

$$\nabla_{\mathbf{e}} f = \nabla f \cdot \mathbf{e} = e_x \frac{\partial f}{\partial x} + e_y \frac{\partial f}{\partial y} \quad .$$

Let $\mathbf{e}^{\perp} = (-e_y, e_x)$, so that $\mathbf{e} \cdot \mathbf{e}^{\perp} = 0$. Then if $w(x,y) = u(x,y) + iv(x,y)$we have

$$\begin{aligned} \nabla_{\mathbf{e}} u &= e_x \frac{\partial u}{\partial x} + e_y \frac{\partial u}{\partial y} \\ &= e_x \frac{\partial v}{\partial y} - e_y \frac{\partial v}{\partial x} \\ &= \nabla_{\mathbf{e}^{\perp}} v \end{aligned}$$

(In the second step we simply employed the Cauchy-Riemann conditions).

Now suppose $C = \{z = \sigma(t) \mid t \in I \subset \mathbb{R}\}$ is a closed curve and consider the following Neumann problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ u_{\mathbf{n}}\big|_C &= 0 \end{aligned}$$

where $u_{\mathbf{n}}$ is the directional derivative of u along C in the direction normal to C.

Assume the parameterization $\sigma(t)$ of C is such that $\left|\frac{d\sigma}{dt}\right| = 1$ is (we can always find such a parameterization). The tangent vector to the curve C at the point $\sigma(t)$ will be

$$\frac{d\sigma}{dt}(t) = \left(\frac{d\sigma_x}{dt}(t), \frac{d\sigma_y}{dt}(t)\right)$$

and so

$$\mathbf{n}(t) = \left(\frac{d\sigma_y}{dt}(t), -\frac{d\sigma_x}{dt}(t)\right)$$

will be a unit vector that always be orthogonal to $\frac{d\sigma}{dt}$. The Neumann boundary condition thus takes the form

$$0 = u_{\mathbf{n}}\big|_C = \nabla\phi \cdot u\big|_C = \frac{\partial u}{\partial x}\frac{d\sigma_y}{dt} - \frac{\partial u}{\partial x}\frac{d\sigma_x}{dt} \quad . \tag{E.66}$$

We know that the real and imaginary part of an analytic function will always satisfy Laplace's equation. The converse is also true, if u satisfies Laplace's equation in a region R then u can be regarded as the real (or imaginary) part of an analytic function on R. Let v be a harmonic conjugate of u; i.e., v is a function such that $w(x,y) = u(x,y) + iv(x,y)$ is analytic. Then the Cauchy-Riemann conditions imply

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

Applying these relations to the far right hand side of (E.66) we obtain

$$0 = \frac{\partial v}{\partial y}\frac{d\sigma_y}{dt} + \frac{\partial v}{\partial x}\frac{d\sigma_x}{dt} = \frac{d}{dt}\left(v\left(\sigma(t)\right)\right) \tag{E.67}$$

But now (E.67) implies

$$v(x,y)\left.\right|_C = const. \tag{E.68}$$

Also since $v(x,y)$ is the imaginary part of an analytic function

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad . \tag{E.69}$$

Thus, the harmonic conjugate of a solution of a Neumann problem on C is the solution of a Dirichlet problem on C. $\square$