APPENDIX D

Solutions to Problem Set 4

1. (Problem 3.4.3 in text)

(a) Consider an infinite-interval problem, $-\infty < x < +\infty$, for which

$$
\begin{aligned}
u(x,0) &= \begin{cases} h(x) & , \quad \text{for } x > 0 \\ -h(-x) & , \quad \text{for } x < 0 \end{cases} \\
u_t(x,0) &= 0
\end{aligned}
\tag{D.1}
$$

Show that the solution of

$$u_{tt} - c^2 u_{xx} = 0$$

satisfying these initial conditions also solves the following semi-infinite problem: find $u(x,t)$ satisfying $u_{tt} - c^2 u_{xx} = 0$, $x \in (0, +\infty)$, with initial conditions $u(x,0) = h(x)$, $u_t(x,0) = 0$, and with fixed end condition $u(0,t) = 0$. [Here $h(x)$ is any given function, with $h(0) = 0$]. Sketch the solution for the case where $h(x) = \frac{1}{2} - \left|x - \frac{3}{2}\right|$ for $1 < x < 2$, $h(x) = 0$ elsewhere.

(b) Use a similar idea to explain how you could use

$$u(x,t) = \frac{1}{2}\left[u(x+ct,0) + u(x-ct,0)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(\tau,0)\,d\tau \tag{D.2}$$

to solve any finite interval problem in which $u(0,t) = u(l,t) = 0$ for all t, with $u(x,0) = h(x)$ and $u_t(x,0) = 0$ for $0 < x < l$. [We take $h(0) = h(l) = 0$.]

(c) Reconsider parts (a) and (b) for situations in which $u_t(x,0)$ is prescribed, with $u(x,0) = 0$. Sketch the solution for a simple case.

(a) Equation (D.2) gives the unique solution of the wave equation in the region $-\infty < x < +\infty$, $0 < t < +\infty$, in terms of its Cauchy data along the x-axis. If we use Eq. (D.1) to extend a Cauchy problem on the positive x-axis to the entire x-axis, then by restricting Eq. (D.2) to the region $(0,+\infty)\times(0,+\infty)$ we obtain a solution of the wave equation satisfying the boundary conditions $u(x,0) = h(x)$, $u_t(x,0) = 0$, for all $x \in (0, +\infty)$. We only have to check that if the boundary condition $u(0,t) = 0$ is also satisfied. From (D.2) we have

$$
\begin{aligned}
u(x,0) &= \tfrac{1}{2}\left[u(x,0) + u(x,0)\right] + \tfrac{1}{2c}\textstyle\int_x^x u_t(\tau,0)\,d\tau \\
&= \tfrac{1}{2}h(x) \\
&= 0
\end{aligned}
\tag{D.3}
$$

$\square$

(b) Our problem now is to define an extension $H(x)$ of the function $h(x)$ defined on $(0,l)$ to the entire x-axis so that Eq. (8.22) can be used to write down the solution of the wave equation in the region $(0,l)\times(0,+\infty)$ satisfying

$$
\begin{aligned}
u(x,0) &= h(x) \\
u_t(x,0) &= 0 \\
u(0,t) &= 0 \\
u(l,t) &= 0
\end{aligned}
\tag{D.4}
$$

The validity of the first two boundary conditions will be automatic (since the restriction of our extension must give us exactly what we started with).

Setting $u(x,0) = H(x)$, $u_t(x,0) = 0$ we obtain from Eq. (8.22)

$$u(x,t) = \frac{1}{2}\left[H(x+ct) + H(x-ct)\right]$$

The third boundary condition in (D.4) thus leads to

$$0 = u(0,t) = \frac{1}{2}\left[H(ct) + H(-ct)\right] \quad .$$

This will be satisfied automatically if we extend $h(x)$ is such a way that the new function $H(x)$ is odd with respect to reflections about $x = 0$.

The last boundary condition thus leads to

$$0 = u(l,t) = \frac{1}{2}\left[H(l+ct) + H(l-ct)\right] \quad .$$

This will be satisfied automatically if we extend $h(x)$ is such a way that the new function $H(x)$ is odd with respect to reflections about $x = l$.

We thus define $H(x)$ as follows:

$$g(x) = \begin{cases} h(2l+x) & , \quad -2l < x < -l \\ -h(-x) & , \quad -l < x < 0 \\ h(x) & , \quad 0 < x < l \\ -h(x-l) & , \quad l < x < 2l \end{cases}$$

$$H(x) = g\left(x - 4nl\right) \quad , \quad 4nl - 2l < x < 4nl + 2l \quad , \quad n \in \mathbb{Z} \quad .$$

□

(c) If instead we had boundary conditions of the form

$$\begin{aligned} &(D.5) & u(x,0) &= 0 \quad , \quad 0 < x < l \\ &(D.6) & u_t(x,0) &= p(x) \quad , \quad 0 < x < l \\ &(D.7) & u(0,t) &= 0 \\ &(D.8) & u(l,t) &= 0 \end{aligned}$$

we would seek to extend the definition of $p(x)$ to the entire x-axis so that the last two boundary conditions are satisfied automatically. We would thus need to define $P(x)$ such that

$$(D.9) \qquad 0 = u(0,t) = \frac{1}{2c}\int_{-ct}^{ct} P(\tau)d\tau$$

$$(D.10) \qquad 0 = u(l,t) = \frac{1}{2c}\int_{l-ct}^{l+ct} P(\tau)d\tau$$

automatically. To accomplish this we can simply extend $p(x)$ in such a way that it is periodic with period $4l$ and antisymmetric with respect to reflections about $x = 0$ and $x = l$. □

2. (Problem 3.4.4 in text)

Consider the "whip-cracking" problem:

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= 0 \\ \phi(x,0) &= 0 \\ \phi_t(x,0) &= 0 \\ \phi(0,t) &= \gamma(t) \\ \phi(0,0) &= 0 \quad , \end{aligned} \tag{D.11}$$

in the region $x > 0$, $t > 0$.

We know from the discussion in Lecture 9 that

$$\phi(x,t) = \alpha(x+ct) + \beta(x-ct) \tag{D.12}$$

is the general solution to the wave equation

$$\phi_{tt} - c^2\phi_{xx} = 0 \quad . \tag{D.13}$$

The boundary conditions in (D.11) imply

$$\begin{aligned} \alpha(x) + \beta(x) &= 0 \\ c\alpha'(x) - c\beta'(x) &= 0 \\ \alpha(ct) - \beta(-ct) &= \gamma(t) \quad . \end{aligned} \tag{D.14}$$

The equation tells us that $\beta(x) = -\alpha(x)$. Making this substitution, we get from the second equation that

$$2c\alpha'(x) = 0$$

so

$$\alpha(x) = K \quad .$$

It would seem that this trivializes everything; however, the first two conditions in (D.14) are only imposed only for $x > 0$. We are therefore free to adjust the functions $\alpha(x)$ and $\beta(x)$ in the region where $x < 0$. The third equation, in fact, tells us that

$$\alpha(cx) + \beta(-cx) = \gamma(x)$$

which we can satisfy by extending the definition of $\beta(x)$ to $x < 0$

$$\beta(-x) = -K + \gamma\left(\frac{x}{c}\right) \qquad , \qquad \forall\ x > 0 \quad .$$

It is not necessary to extend the domain of $\alpha(x)$ to $x < 0$, since in the expression (D.12) for ϕ(x,t), the argument of α is always positive. Thus, we take

$$\begin{aligned} \alpha(\zeta) &= \begin{cases} K & , \quad \zeta > 0 \\ K & , \quad \zeta < 0 \end{cases} \\ \beta(\eta) &= \begin{cases} \gamma\left(-\frac{\eta}{c}\right) - K & , \quad \eta < 0 \\ -K & , \quad \eta > 0 \end{cases} \end{aligned} \tag{D.15}$$

Thus,

$$\Phi(\zeta,\eta) = \alpha(\zeta) + \beta(\eta) = \begin{cases} \gamma\left(-\frac{\eta}{c}\right) & , \quad \eta < 0 \\ 0 & , \quad \eta > 0 \end{cases}$$

and so the solution of (D.11) is

$$\phi(x,t) = \begin{cases} \gamma\left(t - \frac{x}{c}\right) & , \quad \chi - ct < 0 \\ 0 & , \quad x - ct > 0 \end{cases} \quad . \tag{D.16}$$

3. (Problem 3.12.1 in text)

(a) Let $u(x,t)$ satisfy the equation

$$u_{tt} = c^2 u_{xx} \quad , \quad c = costant \quad ,$$

in some region of the (x,t) plane. Show that the quantity $(u_t - cu_x)$ is constant along each straight line defined by $x - ct = constant$, and that $(u_t + cu_x)$ is constant along each straight line of the form $x + ct = constant$. These straight lines are called *characteristics*; we will refer to typical members of the two families as C_+ and C_- curves, respectively; thus $(x - ct = constant)$ is a C_+ curve.

Set

(D.17) $$\phi_+(x,t) = u_t\,(x,t) - cu_x\,(x,t) \quad .$$

Along a C_+ curve we have

(D.18) $$x = k_1 + ct$$

and so along such a curve

(D.19) $$\phi_+(x,t) = \phi_+(t) = u_t\,(k_1 + ct, t) - cu_x\,(k_1 + ct, t) \quad .$$

Differentiating ϕ_+ with respect to t we obtain

(D.20) $$\frac{d\phi_+}{dt} = cu_{tx} + u_{tt} - c^2 u_{xx} - cu_{tx}$$

(D.21) $$= u_{tt} - c^2 u_{xx}$$

(D.22) $$= 0$$

since u satisfies the wave equation. Therefore, ϕ_+ is constant along any curve of the form (D.18).

Similarly, if we set

(D.23) $$\phi_-(x,t) = u_t\,(x,t) + cu_x\,(x,t) \quad .$$

Then along the curve

(D.24) $$x = k_2 - ct$$

we have

$$\phi_-(x,t) = \phi_-(t) = u_t\,(k_2 - ct, t) + cu_x\,(k_2 - ct, t)$$

and so

(D.25) $$\frac{d\phi_-}{dt} = -cu_{tx} + u_{tt} - c^2 u_{xx} + cu_{tx}$$

(D.26) $$= u_{tt} - c^2 u_{xx}$$

(D.27) $$= 0$$

Thus, ϕ_- is constant along any curve of the form (D.24). □

(b) Let $u(x,0)$ and $u_t(x,0)$ be prescribed for all values of x between $-\infty$ and $+\infty$, and let (x_o, t_o) be some point in the (x,t) plane, with $t_o > 0$. Draw the C_+ and C_- curves through (x_o, t_o) and let A and B denote, respectively, their intercepts with the x-axis. Use the properties of C_+ and C_- derived in part (a) to determine $u_t\,(x_o, t_o)$ in terms of initial data at points $(A, 0)$ and $(B, 0)$. Using a similar technique to obtain $u_t\,(x_o, \tau)$ with $0 < \tau < t_o$, determine $u\,(x_o, t_o)$ by integration with respect to τ, and compare with Equation (8.22). Observe that this "method of characteristics" again shows that $u\,(x_o, t_o)$ depends only on that part of the initial data between points $(A, 0)$ and $(B, 0)$.

Let

(D.28) $$k_\pm = x_o \mp ct_o$$

and set

$$c_{\pm} = \left\{ (x,t) \in \mathbb{R}^2 \mid x \mp ct = k_{\pm} \right\} \tag{D.29}$$

From part (a) we know that

$$\begin{aligned} \phi_+ &= u_t(x,t) - cu_x(x,t) \\ \phi_- &= u_t(x,t) + cu_x(x,t) \end{aligned} \tag{D.30}$$

are, respectively, constant along the lines c_+ and c_-.

At the point $(A, 0)$ where the line c_+ intersects the x-axis we have

$$\phi_+ = u_t(A,0) - cu_x(A,0) \tag{D.31}$$

and so the constant ϕ_+ is completely determined by the Cauchy data at the point $(A, 0)$.

Similarly, at the point $(B, 0)$ where the line c_- intersects the x-axis we have

$$\phi_- = u_t(B,0) + cu_x(B,0) \tag{D.32}$$

and so the constant ϕ_- is completely determined by the Cauchy data at the point $(B, 0)$.

Using (D.31) and (D.32) we can rewrite equations (D.30) as

$$\begin{aligned} u_t(A,0) - cu_x(A,0) &= u_t(x_o,t_o) - cu_x(x_o,t_o) \\ u_t(B,0) + cu_x(B,0) &= u_t(x_o,t_o) + cu_x(x_o,t_o) \end{aligned} \tag{D.33}$$

Adding the second equation to the first and then dividing by 2 we obtain

$$u_t(x_o,t_o) = \frac{1}{2}\left(u_t(A,0) + u_t(B,0) - cu_x(A,0) + cu_x(B,0)\right) \quad . \tag{D.34}$$

We can be a even more explicit than this. For the value of A is precisely $k_+ = x_o - ct_o$, and the value of B is precisely $k_- = x_o + ct_o$. Thus,

$$\begin{aligned} u_t(x_o,t_o) &= \tfrac{1}{2}\left(u_t(x_o - ct_o, 0) + u_t(x_o + ct_o, 0)\right) \\ &\quad + \tfrac{c}{2}\left(-u_x(x_o - ct_o, 0) + u_x(x_o + ct_o, 0)\right) \end{aligned} \tag{D.35}$$

This equation is perfectly valid for any choice of x_o and t_o, and so we can write

$$\begin{aligned} u_t(x_o,t) &= \tfrac{1}{2}\left(u_t(x_o - ct, 0) + u_t(x_o + ct, 0)\right) \\ &\quad + \tfrac{c}{2}\left(-u_x(x_o - ct, 0) + u_x(x_o + ct, 0)\right) \end{aligned} \tag{D.36}$$

Integrating both sides with respect to t from 0 to t_o we obtain

$$\begin{aligned} u(x_o,t_o) - u(x_o,0) &= \tfrac{1}{2}\int_0^{t_o} u_t(x_o - ct, 0)\,dt + \tfrac{1}{2}\int_0^{t_o} u_t(x_o + ct, 0)\,dt \\ &\quad - \tfrac{c}{2}\int_0^{t_o} u_x(x_o - ct, 0)\,dt + \tfrac{c}{2}\int_0^{t_o} u_x(x_o + ct, 0)\,dt \end{aligned} \tag{D.37}$$

If we make a change of variables $\zeta = x_o - ct$ in the first and third integrals and a change of variables $\zeta = x_o + ct$ in the second and fourth integrals, the (D.37) becomes

$$
\begin{aligned}
u(x_o, t_o) - u(x_o, 0) &= -\frac{1}{2c}\int_{x_o}^{x_o - ct_o} u_t(\zeta, 0)\, d\zeta + \frac{1}{2c}\int_{x_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta && \text{(D.38)}\\
&\quad + \frac{1}{2}\int_{x_o}^{x_o - ct_o} u_x(\zeta, 0)\, d\zeta + \frac{1}{2}\int_{x_o}^{x_o + ct_o} u_x(\zeta, 0)\, d\zeta && \text{(D.39)}\\
& && \text{(D.40)}\\
&= \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta && \text{(D.41)}\\
&\quad + \frac{1}{2}\, u(\zeta, 0)\Big|_{x_o}^{x_o - ct_o} + \frac{1}{2}\, u(\zeta, 0)\Big|_{x_o}^{x_o + ct_o} && \text{(D.42)}\\
& && \text{(D.43)}\\
&= \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta && \text{(D.44)}\\
&\quad + \frac{1}{2}\left(u(x_o - ct_o) + u(x_o + ct_o)\right) + u(x_o, 0) && \text{(D.45)}
\end{aligned}
$$

or

$$
\boxed{u(x_o, t_o) = \frac{1}{2}\left(u(x_o - ct_o) + u(x_o + ct_o)\right) + \frac{1}{2c}\int_{x_o - ct_o}^{x_o + ct_o} u_t(\zeta, 0)\, d\zeta}
$$

which is precisely Equation (8.22). □