APPENDIX B

Solutions to Problem Set 2

1. (Problem 2.2.2 in text)

(See Lecture 2)

2. (Problem 2.2.3 in text)

(See Lecture 2)

3. (Problem 2.2.5 in text)

Use the method of images to solve for $\phi(x,t)$ in the region $0 < x < L$, $t > 0$ satisfying

$$\begin{aligned} \phi_t - a^2\phi_{xx} &= 0 \\ \phi(x,0) &= f(x) \\ \phi(0,t) &= 0 \\ \phi_x(L,t) &= 0 \qquad . \end{aligned} \tag{B.1}$$

We start with the solution to the PDE/BVP

$$\begin{aligned} \phi_t - a^2\phi_{xx} &= 0 \\ \phi(x,0) &= F(x) \end{aligned} \tag{B.2}$$

on the interval $-\infty < x < +\infty$. In Problem 2.2.2 (and in lecture) the solution was shown to be

$$\phi(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} F(\zeta) \exp\left[-\frac{(x-\zeta)^2}{4a^2 t}\right] d\zeta \quad . \tag{B.3}$$

Our goal is to obtain a solution of (B.1) by defining a function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) = f(x) \quad , \quad 0 \le x \le L \quad , \tag{B.4}$$

$$0 = \phi(0,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} F(\zeta) \exp\left[-\frac{(x-\zeta)^2}{4a^2 t}\right] d\zeta \tag{B.5}$$

and

$$0 = \phi_x(L,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} - \left(\frac{L-\zeta}{2a^2 t}\right) F(\zeta) \exp\left[-\frac{(L-\zeta)^2}{4a^2 t}\right] d\zeta \tag{B.6}$$

Note that the integral on the right hand side will vanish automatically if the function $F(\zeta)$ were an odd function of ζ. (The integral of an odd function about symmetric limits vanishes identically).

Similarly, the integral of the right hand side would vanish automatically if $F(\zeta)$ is symmetric about $\zeta = L$ (because the rest of the integrand is already odd with respect to reflections about $\zeta = L$).

The question thus becomes how can we extend the function $f(x)$ to a function $F(x)$ that is odd with respect to reflections about $x = 0$ and even with respect to reflections about $x = L$.

This can be accomplished by defining $F(x)$ as the periodic function with period $4L$. We do this in two steps. First we define $g : [-2L, 2L] \to \mathbb{R}$ by

$$g(x) = \begin{cases} -f(2L+x) & -2L \le x \le -L \\ -f(-x) & , \quad -L \le x \le 0 \\ f(x) & , \quad 0 \le x \le L \\ f(2L-x) & , \quad L \le x \le 2L \end{cases} \tag{B.7}$$

and then define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = g(x - 4nL) \quad , \quad 4nL - 2L \le x \le 4nL + 2L \quad , \quad n \in \mathbb{Z} \quad . \tag{B.8}$$

The solution to (B.1) is thus given by (B.3) with $F(\chi)$ defined by (7) and (B.8).

4. Solution to Problem 2.6.2.

Referring to Problem 1.2.3, interprete the equation

$$\phi_t = a^2 \phi_{xx} + m(x) \cdot \delta(t) \tag{B.9}$$

for $-\infty < x < +\infty$, $t > 0$, where $m(x)$ is a given function, in terms of the release at $t = 0$ of heat energy distributed along the rod with an intensity proportional to $m(x)$. Use Laplace transforms to find $\phi(x,t)$ if $\phi(x, 0) = 0$ for all x. Next, replace $m(x)$ by $\delta(x - \zeta)$ (where ζ is a chosen point such that $-\infty < \zeta < +\infty$) and discuss the meaning of your results in this case.

In problem 1.2.3, the differential equation

$$\phi_t - \frac{k}{c\rho}\phi_{xx} + \frac{1}{c\rho}\beta(\phi - \phi_o) = \frac{1}{c\rho}h(x,t) \tag{B.10}$$

is interpreted as the follows. The unknown function $\phi(x, t)$ represents the temperature of a (1-dimensional) wire, with (linear) density ρ, specific heat c, and thermal conductivity k, at position x and time t. The term $\beta(\phi - \phi_o)$ represents the heat loss per unit length (β is a constant indicating the rate at which heat is lost to the surrounding environment and ϕ_o is the temperature of the environment). The function $h(x, t)$ is interpreted as the rate at which heat is being generated inside the wire at the point x and time t.

Comparing (B.9) and (B.10), we see that the PDE (B.9) would correspond to an infinite wire that loses no heat to its environment ($\beta = 0$) and that is heated at a rate of $c\rho m(x)\delta(t)$ at the point x at time t. Noting that the total amount of heat added to the wire at point x (over all time) is

$$\int_0^\infty c\rho m(x)\delta(t)dt = c\rho m(x) \quad .$$

and that the support of the generalized function $\delta(t)$ is concentrated at $t = 0$, we interprete the term $c\rho m(x)\delta(t)$ as representing the instantaneous addition of $c\rho m(x)$ units of heat to the wire at the point x at the time $t = 0$.

Let us now take the Laplace transform (with respect to t) of (1).

$$s\Phi(x,s) - \phi(x,0) - a^2\Phi_{xx}(x,s) = \int_0^\infty m(x)\delta(t)e^{-st}dt = m(x)e^{-s\cdot 0} = m(x).$$

Here we have set $\Phi(x,s) = \mathcal{L}[\phi(x,t)](s)$ and have implicitly assumed that $\phi(x,t)$ is sufficiently tame that $\mathcal{L}(\phi_{xx}) = \Phi_{xx}$. Imposing the initial condition $\phi(x, 0) = 0$ we thus arrive at the following second order ODE for $\Phi(x, s)$

$$\Phi_{xx} - \frac{s}{a^2}\Phi = m(x) \quad .$$