APPENDIX A

Solutions to Problem Set 1

1. (Problem 1.2.4 in text)

Let the temperature ϕ inside a solid sphere be a function of the radial distance r from the center and the time t. Show that the 3-dimensional heat equation

$$\frac{\partial \phi}{\partial t} - a^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \right) = 0$$

when transformed to spherical coordinates reduces to

$$\frac{\partial \phi}{\partial t} = a^2 \left(\frac{\partial^2 \phi}{\partial r^2} + 2 \frac{\partial \phi}{\partial r} \right) .$$

Show also that a transformation of the form $\phi = r^{\alpha} \psi$, for a suitable choice of constant α reduces this equation to the form

$$\frac{\partial \phi}{\partial t} = C \frac{\partial^2 \phi}{\partial r^2} .$$

Discuss also the corresponding problem of 1-dimensional heat flow in a cylinder (consider here the transformation $\zeta = \ln |r|$).

□

2. (Problem 1.4.1 in text)

Using Separation of Variables, investigate solutions of of the Heat Equation

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \quad . \tag{A.1}$$

when the separation constant C is taken to be the square of a complex number. Which of these solutions compatible with the boundary conditions

$$\phi(x,0) = g(x) \tag{A.2}$$

$$\phi(0,t) = 0 \tag{A.3}$$

$$\phi(L,t) = 0 \tag{A.4}$$

$$\lim_{t \to \infty} \phi(x,t) = 0 \tag{A.5}$$

Suppose

$$\phi(x,t) = F(x)G(t).$$

If we plug this expression for $\phi(x,t)$ into the Heat Equation we find

$$\frac{\frac{dG}{dt}}{G} = a^2 \frac{\frac{d^2 F}{dx^2}}{F} .$$

Applying the usual Separation of Variables argument we thus arrive at the following pair of ordinary differential equations for $F(x)$ and $G(t)$:

$$\frac{\frac{dG}{dt}}{G} = C = a^2 \frac{\frac{d^2F}{dx^2}}{F}$$

or

$$\begin{aligned} \frac{d^2F}{dx^2} - \frac{C}{a^2}F &= 0 && \text{(A.6)} \\ \frac{dG}{dt} - CG &= 0. && \text{(A.7)} \end{aligned}$$

A priori, the separation constant C could be any complex number. Let us set $C = (\alpha + i\beta)^2$. By letting α range over the set of real numbers and β range over the set of non-negative real numbers, we can still obtain any complex number C we want. The general solution of

$$\frac{d^2F}{dx^2} - \left(\frac{\alpha + i\beta}{a}\right)^2 F = 0$$

is

$$\begin{aligned} F(x) &= c_1 \exp\left(\frac{(\alpha + i\beta)}{a}x\right) + c_2 \exp\left(-\frac{(\alpha + i\beta)}{a}x\right) \\ &= c_1 e^{\frac{\alpha x}{a}} e^{i\frac{\beta x}{a}} + c_2 e^{-\frac{\alpha x}{a}} e^{-i\frac{\beta x}{a}} \\ &= c_1 e^{\frac{\alpha x}{z}} \left(\cos\left(\frac{\beta x}{a}\right) + i\sin\left(\frac{\beta x}{a}\right)\right) + c_2^{-\frac{\alpha x}{z}} \left(\cos\left(\frac{\beta x}{a}\right) - i\sin\left(\frac{\beta x}{a}\right)\right). \end{aligned}$$

While is tempting to try to take the real and imaginary parts of these complex-valued solutions to obtain four families of real-valued solutions, this is not really permitted. For the differential equation satisfies by $F(x)$ is not invariant under complex conjugation (when $\alpha\beta \neq 0$), and so the complex conjugate of one solution is not necessarily a solution; hence the real and imaginary parts of a solution can not be separated without destroying the solution. We thus obtain two 2-parameter families of linearly independent complex-valued solutions:

$$\begin{aligned} f_{1,\alpha,\beta} &= e^{\left(\frac{\alpha+i\beta}{a}\right)x} \\ f_{2,\alpha,\beta} &= e^{-\left(\frac{\alpha+i\beta}{a}\right)x} \end{aligned}$$

The general solution of

$$\frac{dG}{dt} - (\alpha + i\beta)^2 G = 0$$

is given by

$$G(t) = A\exp\left((\alpha + i\beta)^2 t\right)$$

And so we obtain one 2-parameter family of complex-valued solutions of (A.7):

$$g_{\alpha,\beta}(t) = e^{(\bar{\alpha}+i\beta)^2 t}$$

We now obtain, in toto, two 2-parameter families of complex-valued solutions of the Heat Equation by taking all possible products of the functions $f_{i,\alpha,\beta}(x)$ and $g_{\alpha,\beta}(t)$:

$$\begin{aligned} \phi_{1,\alpha,\beta}(x,t) &= e^{(\bar{\alpha}+i\beta)^2 t} e^{\left(\frac{\alpha+i\beta}{a}\right)x} \\ \phi_{2,\alpha,\beta}(x,t) &= e^{(\bar{\alpha}+i\beta)^2 t} e^{-\left(\frac{\alpha+i\beta}{a}\right)x} \end{aligned}$$

Let us now decide which of these solutions is amenable to the boundary conditions.

First of all in order that our solutions be real-valued functions we must take either $\alpha = 0$ or $\beta = 0$. This leads us to restrict our attention to solutions of the form

$$\begin{aligned}
\phi_{1,\alpha}(x,t) &= e^{\alpha^2 t} e^{\frac{\alpha}{a}x} \\
\phi_{2,\alpha}(x,t) &= e^{\alpha^2 t} e^{-\frac{\alpha}{a}x} \\
\phi_{3,\beta}(x,t) &= e^{-\beta^2 t} \cos\left(\frac{\beta x}{a}\right) = \frac{1}{2}\left(\phi_{1,0,\beta}(x,t) + \phi_{2,0,\beta}(x,t)\right) \\
\phi_{4,\beta}(x,t) &= e^{-\beta^2 t} \sin\left(\frac{\beta x}{a}\right) = \frac{1}{2i}\left(\phi_{1,0,\beta}(x,t) - \phi_{2,0,\beta}(x,t)\right)
\end{aligned}$$

In order to satisfy the boundary condition (A.5) we eliminate those solutions that are proportional to $e^{\alpha^2 t}$.

In order to satisfy the boundary condition (A.3), we eliminate those solutions that are proportional to $\cos(\frac{\beta x}{a})$.

In order to satisfy the boundary condition (A.4) we demand further that

$$\frac{\beta L}{a} = n\pi$$

We are thus constricted to construct solutions from functions of the form

$$\phi_n(x,t) = e^{-\left(\frac{n\pi a}{L}\right)^2 t} \sin\left(\frac{2\pi x}{L}\right).$$

□

3. (Problem 1.4.2 in text)

Suppose we want to use the function

$$\Phi(x,t) = \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin\left(n\pi x/L\right)$$

where N is a chosen integer and the c_n are constants, to approximate solutions of the following problem: Find $\phi(x,t)$ satisfying the Heat Equation in the region $0 < x < L$, $0 < t$, with $0 = \phi(0,t) - \phi(L,t)$ and $\phi(x,0) = g(x)$. What would be a good way to determine the constants c_n. If we permit $N \to \infty$, what feature of the series would appear to ensure convergence for $t > 0$? (Hint: consider $\int_0^L \left[\phi(x,0) - \Phi(x,0)\right]^2 dx$.)

According to the Fourier Theorem, every continuous function $f(x)$ on the interval $[0, L]$ has a Fourier (Sine) Series Representation:

$$f(x) = \sum_{n=0}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx.$$

Therefore, if fix the coefficients c_n to be

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) g(x)dx,$$

then

$$\Phi(x,t) = \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin\left(n\pi x/L\right)$$

being constructed as a linear combination of solutions of the Heat Equation, clearly satisfies the Heat Equation. Moreover,

$$\begin{aligned}
\Phi(0,t) &= \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin(0) = 0 \\
\Phi(L,t) &= \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin(n\pi) = 0
\end{aligned}$$

So $\Phi(x,t)$ also satisfies the differential equation and the boundary conditions at $x = 0$ and $x = L$, however it does not quite satisfy the boundary condition at $t = 0$ since

$$\phi(x,0) = g(x) \neq \sum_{n=1}^{N} c_n \sin\left(n\pi x/L\right) = \Phi(x,0)$$

in general. Indeed,

$$\begin{aligned}
\phi(x,0) - \Phi(x,0) &= \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) - \sum_{n=0}^{N} c_n \sin\left(\frac{n\pi x}{L}\right) \\
&= \sum_{n=N}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)
\end{aligned}$$

To clarify our notation, $\phi(x,t)$ is the solution of the Heat Equation vanishing at $x = 0$ and $x = L$ and satisfying

$$\phi(x,0) = g(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right);$$

and $\Phi(x,t)$ is the solution of the Heat Equation vanishing at $x = 0$ and $x = L$ and satisfying

$$\Phi(x,0) = G(x) = \sum_{n=1}^{N} c_n \sin\left(n\pi x/L\right)$$

To get a handle on the difference between $\phi(x,t)$ and $\Phi(x,t)$ we shall investigate the difference between $g(x)$ and $G(x)$. If we can show that in the limit $N \to \infty$, $G(x) \to g(x)$, then we can conclude that $\lim_{n\to\infty} \Phi(x,t)$ satisfies exactly the same boundary conditions as $\phi(x,t)$, and so $\Phi(x,t) = \phi(x,t)$. Now $G(x)$ is a manifestly continuous function, so if $g(x)$ is continuous, the square of the difference between $g(x)$ and $G(x)$ is a manifestly positive continuous function. But then we have necessarily

$$0 \leq \int_0^L \left(g(x) - G(x)\right)^2 dx$$

and furthermore

$$0 = \int_0^L \left(g(x) - G(x)\right)^2 dx \quad \text{if and only if} \quad g(x) = G(x).$$

But

$$
\begin{aligned}
0 &\leq \int_0^L (g(x) - G(x))^2 \, dx \\
&= \int_0^L \left(\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{N} c_n \sin(n\pi x/L) \right)^2 dx \\
&= \int_0^L \left(\sum_{n=N}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \right)^2 dx \\
&= \int_0^L \sum_{n=N}^{\infty} \sum_{m=N}^{\infty} c_n c_m \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \\
&= \sum_{n=N}^{\infty} \sum_{m=N}^{\infty} c_n c_m \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx \\
&= \sum_{n=N}^{\infty} \sum_{m=N}^{\infty} \frac{2}{L} c_n c_m \delta_{mn} \\
&= \frac{2}{L} \sum_{n=N}^{\infty} c_n^2
\end{aligned}
$$

Since the series

$$\sum_{n=1}^{\infty} c_n^2 = \int_0^L [g(x)]^2 dx < \infty$$

converges, necessarily we must have

$$\lim_{N\to\infty} \sum_{n=N}^{\infty} c_n^2 = 0.$$

Hence,

$$\lim_{N\to\infty} \int_0^L (g(x) - G(x))^2 \, dx = 0.$$

Now let us compare the difference between the solution $\phi(x,t)$ that satisfies our boundary conditions exactly, and the solution $\Phi(x,t)$ that satisfies our boundary conditions only approximately:

$$
\begin{aligned}
\phi(x,t) - \Phi(x,t) &= \sum_{n=1}^{\infty} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin(n\pi x/L) - \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin(n\pi x/L) \\
&= \sum_{n=N}^{\infty} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin(n\pi x/L)
\end{aligned}
$$

Note that this error term is negliable for large N not only because $\lim_{n\to\infty} c_n = 0$, but also because the factors

$$\exp\left[-n^2\pi^2 a^2 t/L^2\right]$$

go to zero extremely rapidly for large n (for any $t > 0$). □