LECTURE 24

Green's Functions for Wave Equations, Cont'd

Last time we looked for a Green's function suitable for calculating the solution of an inhomogeneous wave equation. We ended up with

$$G(\mathbf{r},t;\mathbf{r}_o,t_o)=\frac{1}{4\pi c^2|\mathbf{r}-\mathbf{r}_o|}\delta\left(t-t_o-\frac{|\mathbf{r}-\mathbf{r}_o|}{c}\right)\quad . \tag{24.1}$$

as a solution to

$$\begin{aligned}\left(\frac{\partial^2}{\partial t_o^2}-c^2\nabla^2_{\mathbf{r}_o}\right)G(\mathbf{r},t;\mathbf{r}_o,t_o)&=\delta(t-t_o)\,\delta(\mathbf{r}-\mathbf{r}_o)\\ G(\mathbf{r},t;\mathbf{r}_o,t_o)&=0\quad,\quad t<t_o\\ \frac{\partial G}{\partial t_o}(\mathbf{r},t;\mathbf{r}_o,t_o)&=0\quad,\quad t<t_o\quad .\end{aligned} \tag{24.2}$$

If $G(\mathbf{r},t;\mathbf{r}_o,t_o)$ satisfies (24.2), then the solution of

$$\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)\Phi(\mathbf{r},t)=f(\mathbf{r},t)\,,\quad\forall\,t>0\quad,\quad\mathbf{r}\in V_o\subset\mathbb{R}^3 \tag{24.3}$$

is

$$\begin{aligned}\Phi(\mathbf{r},t)&=\int_0^{t^+}\int_{V_o}G(\mathbf{r},t;\mathbf{r}_o,t_o)\,f(\mathbf{r}_o,t_o)\,dV_o dt_o\\ &\quad+\int_{V_o}\left(\Phi(\mathbf{r}_o,0)\frac{\partial G}{\partial t_o}(\mathbf{r},t;\mathbf{r}_o,0)-G(\mathbf{r},t;\mathbf{r}_o,0)\frac{\partial\Phi}{\partial t_o}(\mathbf{r}_o,0)\right)dV_o\\ &\quad+c^2\int_0^{t^+}\int_{\partial V_o}\left(G\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial G}{\partial n}\right)dS_o dt_o\quad .\end{aligned} \tag{24.4}$$

Let us now consider a specific example.

Example.

At time $t=0$ a small piece of charged material is ejected with velocity along the z-axis from an infinite plane of (perfectly) conducting material that is maintained at constant potential 0, and coincident with the (x,y)-plane. Assume that initially the scalar potential had constant value 0 everywhere within the halfspace

$$V_o=\left\{(x,y,z)\in\mathbb{R}^3\mid z>0\right\}$$

and find an expression for the scalar electromagnetic potential at the point $\mathbf{r}$ at time t.

The scalar electromagnetic potential is governed by equation

$$\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)\Phi(\mathbf{r},t)=\frac{4\pi\rho(\mathbf{r},t)}{\epsilon_o} \tag{24.5}$$

where $\rho(\mathbf{r},t)$ is the charge density at the point $\mathbf{r}$ at the time t. Our initial conditions imply

$$\begin{aligned}\Phi(\mathbf{r},0)&=0\\ \frac{\partial\Phi}{\partial t}(\mathbf{r},0)&=0\quad .\end{aligned}$$

We also have the following boundary conditions along the plane $z=0$.

$$\begin{aligned}\Phi(x,y,0,t)&=0\\ \frac{\partial\Phi}{\partial z}(x,y,0,t)&=0\quad .\end{aligned}$$

The second condition comes from the fact that the electric field, which is identifiable with $\nabla\Phi$ must vanish at the surface of a perfect conductor. But then

$$0 = \mathbf{E}\cdot\mathbf{n} = \nabla\phi\cdot\mathbf{n} = \left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y},\frac{\partial\Phi}{\partial z}\right)\cdot(0,0,1) = \frac{\partial\Phi}{\partial z} \quad .$$

We shall also assume that the solution $\Phi(\mathbf{r},t)$ vanishes sufficiently fast as $|\mathbf{r}|\to\infty$; more explicitly,

$$\lim_{vol\ V\to\infty}\int_{\partial V_o}\Phi(\mathbf{r},t)F(\mathbf{r})dS = 0$$

for any function $F(\mathbf{r})$. This is reasonable since the effect of any source that is confined within a bounded region of space can never propagate all the way to spatial infinity in any finite time. We thus need to solve the following PDE/BVP

$$\begin{array}{rcl} \left(\frac{\partial^2}{\partial t^2}-\lambda^2\nabla^2\right)\Phi(\mathbf{r},t) & = & \sigma\delta(x)\delta(y)\delta\left(z+vt\right) \\ \Phi(\mathbf{r},0) & = & 0 \\ \frac{\partial\Phi}{\partial t}\left(\mathbf{r},0\right) & = & 0 \\ \Phi\left(\mathbf{r},t\right)|_{z=0} & = & 0 \\ \lim_{|\mathbf{r}|\to\infty}F(\mathbf{r})\Phi(\mathbf{r},t) & = & 0 \quad . \end{array} \tag{24.6}$$

Setting

$$V_o = \left\{\mathbf{r}=(x,y,z)\in\mathbb{R}^3 \mid z<0\right\} \quad .$$

plugging into (24.3) yields

$$\begin{array}{rcl} \Phi(\mathbf{r},t) & = & \int_0^{t^+}\int_{V_o}G\left(\mathbf{r},t;\mathbf{r}_o,t_o\right)f\left(\mathbf{r}_o,t_o\right)dV_o dt_o \\ & & +\int_{V_o}\left(\Phi\left(\mathbf{r}_o,0\right)\frac{\partial G}{\partial t_o}\left(\mathbf{r},t;\mathbf{r}_o,0\right)-G\left(\mathbf{r},t;\mathbf{r}_o,0\right)\frac{\partial\Phi}{\partial t_o}\left(\mathbf{r}_o,0\right)\right)dV_o \\ & & +c^2\int_0^{t^+}\int_{\partial V_o}\left(G\frac{\partial\Phi}{\partial n}-\Phi\frac{\partial G}{\partial n}\right)dS_o dt_o \\ & = & \int_0^{t^+}\int_{V_o}\frac{1}{4\pi c^2|\mathbf{r}-\mathbf{r}_o|}\delta\left(t-t_o-\frac{|\mathbf{r}-\mathbf{r}_o|}{c}\right)\sigma\delta(x_o)\delta(y_o)\delta\left(z_o-vt_o\right)dV_o dt_o \end{array} \tag{24.7}$$

(the second and third integrals vanish identically due to the boundary conditions on Φ.)

Equation (24.7) may be written more explicitly as

$$\begin{array}{rcl} \Phi\left(x,y,z,t\right) & = & \frac{\sigma}{4\pi c^2}\int_0^{t^+}dt_o\int_{-\infty}^{+\infty}dx_o\int_{-\infty}^{+\infty}dy_o\int_{-\infty}^{0}dz_o\left[\frac{1}{\left[(x-x_o)^2+(y-y_0)^2+(z-z_o)^2\right]}\right. \\ & & \times\delta\left(t-t_o-\frac{1}{c}\left[(x-x_o)^2+(y-y_0)^2+(z-z_o)^2\right]\right) \\ & & \times\delta(x_o)\delta(y_o)\delta\left(z_o-vt_o\right) \\ & = & \frac{\sigma}{4\pi c^2}\int_0^{t^+}dt_o\frac{1}{\sqrt{x^2+y^2+(z-vt_o)^2}}\delta\left(t-t_o-\frac{1}{c}\sqrt{x^2+y^2+(z-vt_o)^2}\right) \end{array}$$

Making a change of variables

$$\begin{aligned}
\tau &= t_o + \frac{1}{c}\sqrt{x^2 + y^2 + (z - vt_o)^2} \\
t_o &= \frac{c^2\tau + vz + \sqrt{(c^2 - v^2)(x^2 + y^2) + c^2(z - v\tau)^2}}{c^2 - v^2} \\
dt_0 &= \left[\frac{c^2\left[\sqrt{(c^2 - v^2)\,x^2 + y^2 + c^2\,(z + v\tau)^2} - v\,(z - v\tau)\right]}{(c^2 - v^2)\sqrt{(c^2 - v^2)\,(x^2 + y^2) + c^2\,(z - v\tau)^2}}\right] d\tau \\
\frac{1}{\sqrt{x^2 + y^2 + (z - vt_0)^2}} &= \left[\frac{c^2 - v^2}{c\left[-vz - v^2\tau + \sqrt{(c^2 - v^2)\,(x^2 + y^2) + c^2\,(z - v\tau)^2}\right]}\right]
\end{aligned}$$

we get

$$\begin{aligned}
\Phi\,(x, y, z, t) &= \frac{\sigma}{4\pi c^2}\int_0^{\tau^+} \frac{c\delta\,(t - \tau)\,d\tau}{\sqrt{(c^2 - v^2)\,(x^2 + y^2) + c^2\,(z - v\tau)^2}} \\
&= \frac{\sigma}{4\pi c}\frac{1}{\sqrt{(c^2 - v^2)\,(x^2 + y^2) + c^2\,(z - vt)^2}}
\end{aligned}$$