LECTURE 23

Green's Functions for Wave Equations

We shall now develop the theory of Green's functions for wave equations, i.e., for PDEs of the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\Phi(\mathbf{r},t) = f(\mathbf{r},t), \quad \forall\, t>0 \quad , \quad \mathbf{r}\in V_o\subset\mathbb{R}^3 \tag{23.1}$$

Let $G(\mathbf{r},t;\mathbf{r}_o,t_o)$ be any solution of

$$\left(\frac{\partial^2}{\partial t_o^2} - c^2\nabla^2_{\mathbf{r}_o}\right)G(\mathbf{r},t;\mathbf{r}_o,t_o) = \delta(\mathbf{r}-\mathbf{r}_o)\,\delta(t-t_o) \qquad . \tag{23.2}$$

Multiplying (23.1) by $G(\mathbf{r},t;\mathbf{r}_o,t_o)$ and (23.2) by $\Phi(\mathbf{r}_o)$ we get

$$\begin{aligned} G(\mathbf{r},t;\mathbf{r}_o,t_o)&\left(\tfrac{\partial^2}{\partial t_o^2} - c^2\nabla^2_{\mathbf{r}_o}\right)\Phi(\mathbf{r}_o,t_o) - \Phi(\mathbf{r}_o,t_o)\left(\tfrac{\partial^2}{\partial t_o^2} - c^2\nabla^2_{\mathbf{r}_o}\right)G(\mathbf{r},t;\mathbf{r}_o,t_o) \\ &= G(\mathbf{r},t;\mathbf{r}_o,t_o)\,f(\mathbf{r}_o,t_o) - \Phi(\mathbf{r}_o,t_o)\,\delta(\mathbf{r}-\mathbf{r}_o)\,\delta(t-t_o)\end{aligned} \tag{23.3}$$

Integrating both sides of (23.3) with respect to $\mathbf{r}_o$ over a volume V_o and over t_o from 0 to $t^+ = t+\epsilon$, we get

$$\begin{aligned}\int_0^{t^+}\int_{V_o}\left(G\tfrac{\partial^2\Phi}{\partial t^2} - \Phi\tfrac{\partial^2 G}{\partial t_o^2}\right)dV_o dt_o + c^2\int_0^{t^+}\int_{V_o}\left(\Phi\nabla^2_{\mathbf{r}_o}G - G\nabla^2_{\mathbf{r}_o}\Phi\right)dV_o dt_o \\ = \int_0^{t^+}\int_{V_o} G(\mathbf{r},t;\mathbf{r}_o,t_o)\,f(\mathbf{r}_o,t_o)\,dV_o dt_o - \Phi(\mathbf{r},t)\end{aligned} \tag{23.4}$$

Now

$$G\frac{\partial^2\Phi}{\partial t_o^2} - \Phi\frac{\partial^2 G}{\partial t_o^2} = \frac{\partial}{\partial t_o}\left(G\frac{\partial\Phi}{\partial t_o} - \Phi\frac{\partial G}{\partial t_o}\right) \tag{23.5}$$

and so the integration over t_o in the first term on the left hand side of (23.4) can be carried out by applying the fundamental theorem of calculus;

$$\int_0^{t^+}\int_{V_o}\left(G\frac{\partial^2\Phi}{\partial t^2} - \Phi\frac{\partial^2 G}{\partial t_o^2}\right)dV_o dt_o = \int_{V_o}\left(G\frac{\partial\Phi}{\partial t_o} - \Phi\frac{\partial G}{\partial t_o}\right)dV_o\Bigg|_0^{t^+} \quad . \tag{23.6}$$

Applying Green's second identity to the second term on the left hand side of (23.4) we get

$$c^2\int_0^{t^+}\int_{V_o}\left(\Phi\nabla^2_{\mathbf{r}_o}G - G\nabla^2_{\mathbf{r}_o}\Phi\right)dV_o dt_o = c^2\int_0^{t^+}\int_{\partial V_o}\left(\phi\frac{\partial G}{\partial n} - G\frac{\partial\Phi}{\partial n}\right)dS_o dt_o \tag{23.7}$$

Inserting (23.6) and (23.7) into (23.5), and moving things around a bit, we get

$$\begin{aligned}\Phi(\mathbf{r},t) \;=\; &\int_0^{t^+}\int_{V_o} G(\mathbf{r},t;\mathbf{r}_o,t_o)\,f(\mathbf{r}_o,t_o)\,dV_o dt_o \\ &+ \int_{V_o}\left(\Phi\tfrac{\partial G}{\partial t_o} - G\tfrac{\partial\Phi}{\partial t_o}\right)dV_o\Big|_0^{t^+} \\ &+ c^2\int_0^{t^+}\int_{\partial V_o}\left(G\tfrac{\partial\Phi}{\partial n} - \Phi\tfrac{\partial G}{\partial n}\right)dS_o dt_o\end{aligned} \tag{23.8}$$

Imposing specific boundary conditions on the Green's function $G(\mathbf{r},t;\mathbf{r}_o,t)$ can simplify the evaluation of (23.8). For example, if we impose the boundary conditions

$$\begin{aligned} G(\mathbf{r},t;\mathbf{r}_o,t_o) &= 0 \quad , \quad t<t_o \\ \tfrac{\partial G}{\partial t_o}(\mathbf{r},t;\mathbf{r}_o,t_o) &= 0 \quad , \quad t<t_o\end{aligned} \tag{23.9}$$

then the upper limit t^+ does not contribute to the evaluation of the second term. We thus have

$$(23.10) \quad \begin{aligned} \Phi(\mathbf{r},t) &= \int_0^{t^+}\int_{V_o} G(\mathbf{r},t;\mathbf{r}_o,t_o)\, f(\mathbf{r}_o,t_o)\, dV_o dt_o \\ &+\int_{V_o}\left(\Phi(\mathbf{r}_o,0)\frac{\partial G}{\partial t_o}(\mathbf{r},t;\mathbf{r}_o,0) - G(\mathbf{r},t;\mathbf{r}_o,0)\frac{\partial \Phi}{\partial t_o}(\mathbf{r}_o,0)\right) dV_o \\ &+c^2\int_0^{t^+}\int_{\partial V_o}\left(G\frac{\partial\Phi}{\partial n} - \Phi\frac{\partial G}{\partial n}\right) dS_o dt_o \end{aligned}$$

Thus, $\Phi(\mathbf{r},t)$ is completely specified in terms of the Green's function $G(\mathbf{r},t;\mathbf{r}_o,t_o)$, the values of Φ and $\frac{\partial\Phi}{\partial t}$ at time $t=0$, and the values of Φ on the surface bounding V.

Let us now consider the problem of actually finding a function $G(\mathbf{r},t;\mathbf{r}_o,t_o)$ satisfying

$$(23.11) \quad \begin{aligned} \left(\frac{\partial^2}{\partial t_o^2} - c^2\nabla^2_{\mathbf{r}_o}\right) G(\mathbf{r},t;\mathbf{r}_o,t_o) &= \delta(t-t_o)\,\delta(\mathbf{r}-\mathbf{r}_o) \\ G(\mathbf{r},t;\mathbf{r}_o,t_o) &= 0 \quad , \quad t<t_o \\ \frac{\partial G}{\partial t_o}(\mathbf{r},t;\mathbf{r}_o,t_o) &= 0 \quad , \quad t<t_o \quad . \end{aligned}$$

Such a Green's function would be suitable for solving a Cauchy problem in which the Cauchy data is specified on the surface $t=\tau$.

We now take the Laplace transform of both sides of the PDE in (1) with respect to t_o. The Laplace transform of the left hand side is

$$\begin{aligned} \mathcal{L}\left[\frac{\partial^2 G}{\partial t_o^2} - c^2\nabla^2_{\mathbf{r}_o}G\right] &= s^2\mathcal{L}[G] - sG|_{t_o=0} - \left.\frac{\partial G}{\partial t_o}\right|_{t_o=0} - c^2\mathcal{L}[G] \\ &= s^2\mathcal{L}[G] - c^2\nabla^2_{\mathbf{r}_o}\mathcal{L}[G] \end{aligned}$$

in view of the boundary conditions in (23.11). The Laplace transform of the right hand side of the PDE in (23.11) is

$$\begin{aligned} \mathcal{L}[\delta(t-t_o)\,\delta(\mathbf{r}-\mathbf{r}_o)] &= \int_0^\infty e^{-st_o}\delta(t-t_o)\,\delta(\mathbf{r}-\mathbf{r}_o)\, dt_o \\ &= e^{-st}\delta(\mathbf{r}-\mathbf{r}_o) \end{aligned}$$

We thus get from (23.11)

$$(23.12) \quad \left(s^2 - c^2\nabla^2_{\mathbf{r}_o}\right)\mathcal{G} = e^{-st}\delta(\mathbf{r}-\mathbf{r}_o)$$

where $\mathcal{G} = \mathcal{L}[G]$. Now (23.12) is equivalent to

$$(23.13) \quad \left(\nabla^2_{\mathbf{r}_o} - \frac{s^2}{c^2}\right)\mathcal{G} = -\frac{e^{-st}}{c^2}\delta(\mathbf{r}-\mathbf{r}_o) \quad ,$$

and so $\mathcal{G}$ should correspond to $-\frac{e^{-st}}{c^2}$ times the Green's function for the modified Laplacian $\left(\nabla^2 - \frac{s^2}{c^2}\right)$. Thus,

$$(23.14) \quad \begin{aligned} \mathcal{G}(\mathbf{r},t;\mathbf{r}_o,s) &= \left(-\frac{e^{-st}}{c^2}\right)\left(\frac{-e^{-\frac{s}{c}|\mathbf{r}-\mathbf{r}_o|}}{4\pi|\mathbf{r}-\mathbf{r}_o|}\right) \\ &= \frac{e^{-s\left(t+\frac{|\mathbf{r}-\mathbf{r}_o|}{c}\right)}}{4\pi c^2} \end{aligned}$$

Recalling that the inverse Laplace transform of $e^{-s\tau}$ is $\delta(t_o-\tau)$, we get

$$(23.15) \quad G(\mathbf{r},t;\mathbf{r}_o,t_o) = \frac{1}{4\pi c^2|\mathbf{r}-\mathbf{r}_o|}\delta\left(t-t_o-\frac{|\mathbf{r}-\mathbf{r}_o|}{c}\right) \quad .$$