LECTURE 22

Green's Functions for the Modified Laplacian

Let D be a closed, connected, simply connected domain in $\mathbb{R}^n$ and consider the PDE/BVP

$$\begin{aligned} \nabla^2\phi(\mathbf{r}) - \alpha^2\phi(\mathbf{r}) &= f(\mathbf{r}) \qquad (\mathbf{r} \in D) \\ \phi(\mathbf{r})|_{\partial D} &= h(\mathbf{r})|_{\partial D} \end{aligned} \tag{22.1}$$

We seek to determine a Green's function for this problem; i.e., a function $G(\mathbf{r}, \mathbf{r}_o)$ satisfying

$$\begin{aligned} \nabla^2_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}_o) - \alpha^2 G(\mathbf{r}, \mathbf{r}_o) &= \delta^{(n)}(\mathbf{r} - \mathbf{r}_o) \\ \nabla^2_{\mathbf{r}_o} G(\mathbf{r}, \mathbf{r}_o) - \alpha^2 G(\mathbf{r}, \mathbf{r}_o) &= \delta^{(n)}(\mathbf{r} - \mathbf{r}_o) \\ G(\mathbf{r}, \mathbf{r}_o)|_{\mathbf{r}\in\partial D} &= 0 \end{aligned} \tag{22.2}$$

so that the solution of (22.1) can be written

$$\Phi(\mathbf{r}_o) = \int_D f(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_o)\, dV + \int_{\partial D} h(\mathbf{r}) \frac{\partial G}{\partial n} dS \tag{22.3}$$

To see that (22.2) actually implies (22.3), we simply apply the operator $\nabla^2_{\mathbf{r}_o} - \alpha^2$ to both sides of (22.3). This yields

$$\begin{aligned} \nabla^2_{\mathbf{r}_o}\Phi(\mathbf{r}_o) - \alpha^2\phi(\mathbf{r}_o) &= \int_D f(\mathbf{r}) \left(\nabla^2 - \alpha^2\right) G(\mathbf{r}, \mathbf{r}_o)\, dV \\ &\quad + \int_{\partial D} h(\mathbf{r}) \left(\nabla^2 - \alpha^2\right) \tfrac{\partial G}{\partial n} dS \\ &= \int_D f(\mathbf{r}) \delta^{(n)}(\mathbf{r} - \mathbf{r}_o)\, dV \\ &\quad + \int_{\partial D} h(\mathbf{r}) \nabla \left(\nabla^2 - \alpha^2\right) G \cdot \mathbf{n} dS \\ &= f(\mathbf{r}_o) + \int_{\partial D} h(\mathbf{r}) \left(\nabla\delta(\mathbf{r} - \mathbf{r}_o)\right) \cdot \mathbf{n} dS \\ &= f(\mathbf{r}_o) \quad \text{if } \mathbf{r}_o \notin \partial D \end{aligned} \tag{22.4}$$

To find a Green's function suitable for (22.1), we first look for solutions of

$$\nabla^2\Phi - \alpha^2\Phi = 0 \tag{22.5}$$

that depend only on the radial distance r. From Lecture 26, we know that the radial part of the Laplacian is

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial}{\partial r} \right) \quad . \tag{22.6}$$

We thus look for solutions of

$$\begin{aligned} 0 &= \tfrac{1}{r^{n-1}} \tfrac{\partial}{\partial r} \left(r^{n-1} \tfrac{\partial\phi}{\partial r} \right) - \alpha^2\phi \\ &= \phi'' + \tfrac{n-1}{r}\phi' - \alpha^2\phi \quad . \end{aligned} \tag{22.7}$$

Case 1: $n = 2$

When $n = 2$, (22.7) can be written

$$\phi'' + \frac{1}{r}\phi' - \alpha^2\phi = 0 \quad . \tag{22.8}$$

This is a second order linear ODE with a regular singular point at $r = 0$. Therefore, at least one solution can be found by assuming a generalized power series expansion. Substituting

$$\phi(r) = \sum_{n=0}^{\infty} a_n r^{\nu+n} \tag{22.9}$$

into (22.8) yields

$$\begin{aligned} 0 = & \sum_{n=0}^{\infty} (\nu+n)(\nu+n-1) a_n r^{\nu+n-2} \\ & + \sum_{n=0}^{\infty} (\nu+n) a_n r^{\nu+n-2} - \alpha^2 \sum_{n=0}^{\infty} a_n r^{\nu+n} \\ = & [\nu(\nu-1)+\nu] a_0 r^{\nu-2} + [(\nu+1)\nu + (\nu+1)] a_1 r^{\nu-1} \\ & + \sum_{n=0}^{\infty} \left[[(\nu+n+2)(\nu+n+1) + (\nu+n+2)] a_{n+2} - \alpha^2 a_n \right] r^{\nu+n} \end{aligned}$$

Since one assumes that $a_0 \neq 0$, in order for the term proportional to $r^{\nu-2}$ to vanish we must have

$$0 = \nu(\nu-1) + \nu = \nu^2 \tag{22.10}$$

Thus, we must take $\nu = 0$. In order to make the term proportional to $r^{\nu-1}$ vanish we must take $a_1 = 0$. The remaining terms will then vanish so long as the recursion relations

$$\begin{aligned} 0 &= [(0+n+2)(0+n+1) + (0+n+2)] a_{n+2} - \alpha^2 a_n \\ &= \left(n^2+4n+4\right) a_{n+2} - \alpha^2 a_n \end{aligned} \tag{22.11}$$

are satisfied. The relations (22.11) are equivalent to

$$a_{n+2} = \frac{\alpha^2}{(n+2)^2} a_n \tag{22.12}$$

Thus, setting $a_0 = 1$, and recalling $a_1 = 0$, we get

$$\begin{aligned} a_2 &= \frac{\alpha^2}{2^2} \\ a_3 &= 0 \\ a_4 &= \frac{\alpha^4}{2^2 4^2} = \frac{\alpha^4}{(2^4)(1)^2(2)^2} \\ a_5 &= 0 \\ a_6 &= \frac{\alpha^6}{2^2 4^2 6^2} = \frac{\alpha^6}{(2^6)(1)^2(2)^2(3)^2} \\ &\vdots \\ a_{2s} &= \frac{\alpha^{2s}}{2^{2s}(s!)^2} \\ a_{2s+1} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \phi_1(r) &= \sum_{s=0}^{\infty} \frac{\alpha^{2s}}{2^{2s}(s!)^2} r^{2s} \\ &= \sum_{s=0}^{\infty} \frac{1}{(s!)^2} \left(\frac{\alpha r}{2}\right)^{2s} \end{aligned} \tag{22.13}$$

is one solution of (22.8). This function is better known as the *modified Bessel function of the first kind* $I_0(\alpha r)$. More generally,

$$I_\nu(\alpha r) = \sum_{n=0}^{\infty} \frac{1}{n!(n+\nu)!} \left(\frac{\alpha r}{2}\right)^{2s+\nu} \tag{22.14}$$

is the solution of

$$\frac{1}{r}\frac{d}{dx}\left(r\frac{d\phi}{dx}\right) - \left(\alpha^2 + \frac{\nu^2}{r^2}\right)\phi = 0 \tag{22.15}$$

that is regular at $r = 0$. (Equation (22.15) is the just the ODE with respect to r that one gets by applying separation of variables to (22.1).)

In order to write down a Green's function for (22.1) we need to find a second linearly independent solution of (22.8). Unfortunately, there was only one root to the indicial equation (22.10) and so the second solution will not be obtainable from a generalized power series expansion. To find a second solution we must resort to some other technique; e.g, Reduction of Order (see Sec. 3.4 of Boyce and DiPrima). We thus set

$$\phi_2(r) = v(r)\phi_1(r) \tag{22.16}$$

where

$$\begin{aligned} v(r) &= \int^r \frac{1}{(\phi_1(s))^2}\exp\left[-\int^s p(t)dt\right]ds \\ &= \int^r \frac{1}{(\phi_1(s))^2}e^{-\ln|s|}ds \\ &= \int^r \frac{1}{s(\phi_1(s))^2}ds \end{aligned} \tag{22.17}$$

One can now insert the power series expansion (22.13) of $\phi_1(r)$ into (22.17) to get an asymptotic expansion of $\phi_2(r)$. To get a rough idea as to how this might be carried out, let's try to identify the dominant term as $r \to 0$. From (22.13) we have

$$\phi_1(r) = 1 + \frac{\alpha^2}{4}r^2 + \cdots$$

thus,

$$\frac{1}{\phi_1(s)} = 1 - \frac{\alpha^2}{4}s^2 + \cdots$$

and

$$\frac{1}{(\phi_1(s))^2} = 1 - \frac{\alpha^2}{2}s^2 + \ldots \quad .$$

Hence,

$$\begin{aligned} v(r) &= \int^r \frac{1}{s}\left[1 - \frac{\alpha^2}{2}s^2 + \cdots\right]ds \\ &= \log|r| - \frac{\alpha^2}{4}r^2 + \cdots \end{aligned}$$

and so

$$\begin{aligned} \phi_2(r) &= \left[\log|r| - \tfrac{\alpha^2}{4}r^2 + \cdots\right]\left[1 + \tfrac{\alpha^2}{4}r^2 + \cdots\right] \\ &= \log|r| + \cdots \end{aligned} \tag{22.18}$$

In fact, one can actually sum the series to show that the right hand side of (22.18) can be represented (up to a scalar factor) as

$$\int_0^\infty \cos(\alpha r\sinh(t))\,dt = \int_0^\infty \frac{cos(\alpha rt)}{\sqrt{1+t^2}}dt \tag{22.19}$$

This integral is known as *the modified Bessel function of the second kind* $K_0(\alpha r)$. Thus, the general solution of (22.8) can be written as

$$\phi(r) = AK_0(\alpha r) + BI_0(\alpha r) \quad . \tag{22.20}$$

We now want to form from K_0 and I_0 the Green's function for (1); i.e., a function $G(\mathbf{r},\mathbf{r}_o)$ such that

$$\begin{aligned} \nabla^2 G(\mathbf{r},\mathbf{r}_o) &= \delta(\mathbf{r}-\mathbf{r}_o) \\ G(\mathbf{r},\mathbf{r}_o)|_{\partial D} &= 0 \end{aligned} \tag{22.21}$$

Now since $K_0(\alpha r)$ is a well-defined solution of (22.5) so long as $r \neq 0$, we have

$$\lim_{\epsilon \to 0} \left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right) = 0 \quad , \quad \text{if } r \neq 0 \quad . \tag{22.22}$$

Let us now investigate the nature of the limit as $\epsilon \to 0$ when $r = 0$. Because of the logarithmic singularity of $K_0(\alpha r)$

$$K_0(\alpha r) \quad \sim \quad -\log\left|\frac{1}{r}\right| \qquad \text{as } r \to 0 \quad . \tag{22.23}$$

we have

$$\begin{aligned} \left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right) \sim & \left(\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr} - \alpha^2\right)\left(-\log|r|\right) \\ = & \quad \tfrac{1}{(r+\epsilon)^2} - \tfrac{1}{r(r+\epsilon)} - \alpha^2 \log|r+\epsilon| \\ = & \quad \tfrac{r-(r+\epsilon)}{r(r+\epsilon)^2} - \alpha^2 \log|r+\epsilon| \\ = & \quad \tfrac{-\epsilon}{r(r+\epsilon)^2} - \alpha^2 \log|r+\epsilon| \end{aligned} \tag{22.24}$$

If we now integrate $\left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right)$ over a disc of radius $R > 0$, we get

$$\begin{aligned} \int_D \left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right) dA &= \int_0^R \int_0^{2\pi} \left[\frac{-\epsilon}{r(r+\epsilon)^2} - \alpha^2 \log|r+\epsilon|\right] r dr d\theta \\ &= 2\pi \int_\epsilon^{R-\epsilon} \left(\frac{-\epsilon}{\rho^2} - \alpha^2\left(\rho-\epsilon\right)\log|\rho|\right) \\ &= 2\pi \left[\frac{\epsilon}{\rho}\right]\Bigg|_\epsilon^{R-\epsilon} \\ &\quad - \alpha^2 \left[\frac{\rho^2}{4}\left(2\log|\rho| - 1\right) - \epsilon\left(\rho\log|\rho| - \rho\right)\right]\Bigg|_\epsilon^{R-\epsilon} \end{aligned}$$

Since the support of the integrand in the limit $\epsilon \to 0$ resides at the point $r = 0$, we can ignore the behavior at the upper limit. Thus,

$$\begin{aligned} \lim_{\epsilon\to 0} \int_D \left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right) dA &= \lim_{\epsilon\to 0} 2\pi \left[\frac{-\epsilon}{\rho}\right]\Bigg|_{\rho=\epsilon} \\ &\quad + \lim_{\epsilon\to 0} 2\pi\alpha^2 \left[\frac{\rho^2}{4}\left(2\log|\rho| - 1\right)\right]\Bigg|_{\rho=\epsilon} \\ &\quad - \lim_{\epsilon\to 0} 2\pi\alpha^2 \left[\epsilon\left(\rho\log|\rho| - \rho\right)\right]\Big|_{\rho=\epsilon} \\ &= -2\pi + 0 + 0 \\ &= -2\pi \end{aligned}$$

We thus conclude that

$$\lim_{\epsilon \to 0} \left(\nabla^2 - \alpha^2\right) K_0\left(\alpha(r+\epsilon)\right) = -2\pi\delta\left(\mathbf{r}\right) \tag{22.25}$$

and so we write

$$\left(\nabla^2 - \alpha^2\right) K_0\left(\alpha r\right) = -2\pi\delta\left(\mathbf{r}\right)$$

or even

$$\left(\nabla^2 - \alpha^2\right) K_0\left(\alpha\left|\mathbf{r} - \mathbf{r}_o\right|\right) = -2\pi\delta\left(\mathbf{r} - \mathbf{r}_o\right) \tag{22.26}$$

Thus, a Green's function for (22.1) should have the from

$$G(\mathbf{r}, \mathbf{r}_o) = -\frac{1}{2\pi} K_0\left(\alpha\left|\mathbf{r} - \mathbf{r}_o\right|\right) + \phi_o\left(\mathbf{r}, \mathbf{r}_o\right) \tag{22.27}$$

where $\phi(\mathbf{r}, \mathbf{r}_o)$ is a solution of the homogeneous equation corresponding to (22.1) satisfying the boundary conditions

$$\phi_o\left(\mathbf{r}, \mathbf{r}_o\right)\big|_{\partial D} = \frac{1}{2\pi} K_0\left(\alpha\left|\mathbf{r}-\mathbf{r}_o\right|\right)\Big|_{\partial D} \quad . \tag{22.28}$$

Homework: Work out the details of the case when $n = 3$.

Hint: At some point you will find it convenient (if not necessary) to impose regularity conditions on the asymptotic behavior of the Green's function as $r \to \infty$.