LECTURE 21

Green's Identities

Let us recall Stokes' Theorem in n-dimensions.

THEOREM 21.1. *Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field over $\mathbb{R}^n$ that is of class C^1 on some closed, connected, simply connected n-dimensional region $D \subset \mathbb{R}^n$. Then*

$$\int_D \nabla \cdot \mathbf{F}\, dV = \int_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS$$

where ∂D is the boundary of D and $\mathbf{n}(\mathbf{r})$ is the unit vector that is (outward) normal to the surface ∂D at the point $\mathbf{r} \in \partial D$.

As a special case of Stokes' theorem, we may set

$$\mathbf{F} = \nabla \phi \tag{21.1}$$

with ϕ a C^2 function on D. We then obtain

$$\int_D \nabla^2 \phi\, dV = \int_{\partial D} \nabla \phi \cdot \mathbf{n}\, dS \quad . \tag{21.2}$$

Recall that the identity (21.2) was essential to the proof that any extrema of a solution ϕ of 2-dimensional Laplace's equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

must occur on the boundary of region. The analogous proposition about extrema for solutions of Laplace's equation in n-dimensions is also true and again it is relatively easy consequence of (21.2).

Another special case of Stokes' theorem comes from the choice

$$\mathbf{F} = \phi \nabla \psi \quad . \tag{21.3}$$

For this case, Stokes' theorem says

$$\int_D \nabla \cdot (\phi \nabla \psi)\; dV = \int_{\partial D} \phi \nabla \psi \cdot n dS \quad . \tag{21.4}$$

Using the identity

$$\nabla \cdot (\phi \mathbf{F}) = \nabla \phi \cdot \mathbf{F} + \phi \nabla \cdot \mathbf{F} \tag{21.5}$$

we find (21.4) is equivalent to

$$\int_D \nabla \phi \cdot \nabla \psi\, dV + \int_D \phi \nabla^2 \psi\, dV = \int_{\partial D} \phi \nabla \psi \cdot \mathbf{n}\, dS \quad . \tag{21.6}$$

Equation (21.6) is known as **Green's first identity**.

Reversing the roles of ϕ and ψ in (21.6) we obtain

$$\int_D \nabla \psi \cdot \nabla \phi\, dV + \int_D \psi \nabla^2 \phi\, dV = \int_{\partial D} \psi \nabla \phi \cdot \mathbf{n}\, dS \quad . \tag{21.7}$$

Finally, subtracting (21.7) from (21.6) we get

$$\int_D \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) \, dV = \int_{\partial D} \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot \mathbf{n}\, dS \quad . \tag{21.8}$$

Equation (21.8) is known as **Green's second identity**.

Now set

$$\psi(\mathbf{r}) = \frac{1}{|\mathbf{r}-\mathbf{r}_o| + \epsilon}$$

and insert this expression into (21.8). We then get

$$\begin{aligned}\int_D \phi \left(\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon}\right) dV \quad &= \quad \int_D \frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon}\nabla^2\phi \, dV \\ &\quad + \int_{\partial D}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_o|-\epsilon}\nabla\phi - \phi\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon}\right)\cdot\mathbf{n}dS\right) \quad .\end{aligned}$$

Taking the limit $\epsilon \to 0$ and using the identities

$$\begin{aligned}\lim_{\epsilon\to 0}\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon} \quad &= \quad -4\pi\delta^n\left(\mathbf{r}-\mathbf{r}_o\right) \\ \lim_{\epsilon\to 0}\frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon} \quad &= \quad \frac{1}{|\mathbf{r}-\mathbf{r}_o|} \\ \lim_{\epsilon\to 0}\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_o|+\epsilon} \quad &= \quad \nabla\frac{1}{|\mathbf{r}-\mathbf{r}_o|}\end{aligned}$$

we obtain

$$\begin{aligned}-4\pi\phi\left(\mathbf{r}_o\right) \quad &= \quad \int_D \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\nabla^2\phi\, dV \\ &\quad + \int_{\partial D}\left(\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\nabla\phi - \phi\left(\nabla\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\right)\cdot\mathbf{n}dS\right) \quad .\end{aligned} \tag{21.9}$$

Equation (21.9) is known as **Green's third identity**.

Notice that if ϕ satisfies Laplace's equation the first term on the right hand side vanishes and so we have

$$\begin{aligned}\phi\left(\mathbf{r}_o\right) \quad &= \quad \tfrac{-1}{4\pi}\int_{\partial D}\left(\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\nabla\phi - \phi\left(\nabla\tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\right)\cdot\mathbf{n}dS\right) \\ &= \quad \tfrac{1}{4\pi}\int_{\partial D}\left(\phi\tfrac{\partial}{\partial n}\tfrac{1}{\mathbf{r}-\mathbf{r}_o} - \tfrac{1}{|\mathbf{r}-\mathbf{r}_o|}\tfrac{\partial\phi}{\partial n}\right) dS \quad .\end{aligned} \tag{21.10}$$

Here $\frac{\partial}{\partial n}$ is the directional derivative corresponding to the surface normal vector $\mathbf{n}$. Thus, if ϕ satisfies Laplace's equation in D then its value at any point $\mathbf{r}_o \in D$ is completely determined by the values of ϕ and $\frac{\partial\phi}{\partial n}$ on the boundary of D.

1. Green's Functions and Solutions of Laplace's Equation, II

Recall the fundamental solutions of Laplace's equation in n-dimensions

$$\Phi_n\left(r,\psi,\theta_1,\ldots,\theta_{n-2}\right)=\begin{cases}\log|r| & , \quad \text{if } n=2\\ \frac{1}{r^{n-2}} & , \quad \text{if } n>2\end{cases} \quad . \tag{21.11}$$

Each of these solutions really only makes sense in the region $\mathbb{R}^n-\{\mathbf{O}\}$; for each possesses a singularity at the origin.

We studied the case when $n=3$, a little more closely and found that we could actually write

$$\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta^3\left(\mathbf{r}\right)=\begin{cases}0 & , \quad \text{if } \mathbf{r}\neq\mathbf{O}\\ \infty & , \quad \text{if } \mathbf{r}=\mathbf{O}\end{cases} \tag{21.12}$$

In fact, using similar arguments one can show that

$$\nabla^2\Phi(\mathbf{r})=-c_n\delta^n\left(\mathbf{r}\right) \tag{21.13}$$

where c_n is the surface area of the unit sphere in $\mathbb{R}^n$. Thus, the fundamental solutions can actually be regarded as solutions of an **inhomogeneous** Laplace equation where the driving function is concentrated at a single point.

Let us now set $n=3$ and consider the following PDE/BVP

$$\begin{array}{rcl}\nabla^2\Phi(\mathbf{r}) & = & f(\mathbf{r}) \quad , \quad r\in D\\ \Phi(\mathbf{r})|_{\partial D} & = & h(\mathbf{r})|_{\partial D}\end{array} \tag{21.14}$$

where D is some closed, connected, simply connected region in $\mathbb{R}^3$. Let $\mathbf{r}_o$ be some fixed point in D and set

$$G\left(\mathbf{r},\mathbf{r}_o\right)=\frac{-1}{4\pi\left|\mathbf{r}-\mathbf{r}_o\right|}+\phi_o(\mathbf{r},\mathbf{r}_o) \tag{21.15}$$

where $\phi_o(\mathbf{r},\mathbf{r}_o)$ is some solution of the homogeneous Laplace equation

$$\nabla^2\phi_o(\mathbf{r},\mathbf{r}_o)=0 \quad . \tag{21.16}$$

Then

$$\nabla^2 G\left(\mathbf{r},\mathbf{r}_o\right)=\delta^3\left(\mathbf{r}-\mathbf{r}_o\right) \quad . \tag{21.17}$$

Now recall Green's third identity

$$\int_D\left(\Phi\nabla^2\Psi-\Psi\nabla^2\Phi\right)\,dV=\int_{\partial D}\left(\Phi\nabla\Psi-\Psi\nabla\Phi\right)\cdot\mathbf{n}\,dS \quad . \tag{21.18}$$

If we replace ψ in (21.18) by $G\left(\mathbf{r},\mathbf{r}_o\right)$ we get

$$\begin{array}{rcl}\Phi\left(\mathbf{r}_o\right) & = & \int_D\Phi(\mathbf{r})\delta^3\left(\mathbf{r}-\mathbf{r}_o\right)dV\\ & = & \int_D\Phi\nabla^2 G dV\\ & = & \int_D G\nabla^2\Phi\,dV+\int_{\partial D}\left(\Phi\nabla G-G\nabla\Phi\right)\cdot\mathbf{n}\,dS\\ & = & \int_D Gf\,dV+\int_{\partial D}\left(h\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right)\,dS\\ & = & \int_D Gf\,dV+\int_{\partial D}h\frac{\partial G}{\partial n}dS-\int_{\partial D}G\frac{\partial\Phi}{\partial n}dS \quad .\end{array} \tag{21.19}$$

Up to this point we have only required that the function ϕ_o satisfies Laplace's equation. We will now make our choice of ϕ_o more particular; we shall choose $\phi_o(\mathbf{r},\mathbf{r}_o)$ to be the unique solution of Laplace's equation in D satisfying the boundary condition

$$\left.\frac{1}{4\pi\left|\mathbf{r}-\mathbf{r}_o\right|}\right|_{\partial D}=\phi_o(\mathbf{r},\mathbf{r}_o)|_{\partial D} \tag{21.20}$$

so that

$$G\left(\mathbf{r},\mathbf{r}_o\right)|_{\partial D}=0 \quad .$$

Then the last integral on the right hand side of (21.19) vanishes and so we have

$$\Phi(\mathbf{r}_o) = \int_D G(\mathbf{r},\mathbf{r}_o) f(\mathbf{r})\, dV + \int_{\partial D} h(\mathbf{r}) \frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)\, dS \quad . \tag{21.21}$$

Thus, once we find a solution $\phi_o(\mathbf{r},\mathbf{r}_o)$ to the homogenenous Laplace equation satisfying the boundary condition (21.20), we have a closed formula for the solution of the PDE/BVP (21.14) in terms of integrals of $G(\mathbf{r},\mathbf{r}_o)$ times the driving function $f(\mathbf{r})$, and of $\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)$ times the function $h(r)$ describing the boundary conditions on Φ. Note that the Green's function $G(\mathbf{r},\mathbf{r}_o)$ is fixed once we fix ϕ_o which in turn depends only on the nature of the boundary of the region D (through condition (21.20)).

Example

Let us find the Green's function corresponding to the interior of sphere of radius R centered about the origin. We seek to find a solution of ϕ_o of the homogenous Laplace's equation such that (21.20) is satisfied. This is accomplished by the following trick.

Suppose $\Phi(r,\psi,\theta)$ is a solution of the homogeneous Laplace equation inside the sphere of radius R centered at the origin. For $r > R$, we define a function

$$\tilde{\Phi}(r,\psi,\theta) = \frac{R}{r}\Phi\left(\frac{R^2}{r},\psi,\theta\right) \quad . \tag{21.22}$$

I claim that $\tilde{\Phi}(r,\psi,\theta)$ so defined also satisfies Laplace's equation in the region exterior to the sphere.

To prove this, it suffices to show that

$$0 = r^2\nabla\tilde{\Phi} \;=\; \tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial\tilde{\Phi}}{\partial r}\right) + \tfrac{1}{\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)\tfrac{\partial\tilde{\Phi}}{\partial\theta}\right) + \tfrac{1}{\sin^2(\theta)}\tfrac{\partial^2\tilde{\Phi}}{\partial\psi^2} \tag{21.23}$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{\Phi}}{\partial r}\right) = -\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\tilde{\Phi}}{\partial\theta}\right) - \frac{1}{\sin^2(\theta)}\frac{\partial^2\tilde{\Phi}}{\partial\psi^2} \quad . \tag{21.24}$$

Set

$$u = \frac{R^2}{r} \quad . \tag{21.25}$$

so that

$$\begin{aligned} r &= \tfrac{R^2}{u} \\ \tilde{\Phi}(r,\psi,\theta) &= \tfrac{u}{R}\Phi(u,\psi,\theta) \\ \tfrac{\partial}{\partial r} &= -\tfrac{du}{dr}\tfrac{\partial}{\partial u} = -\tfrac{R^2}{r^2}\tfrac{\partial}{\partial u} = -\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u} \end{aligned} \tag{21.26}$$

and so

$$\begin{aligned} \tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial\tilde{\Phi}}{\partial r}\right) &= \left(-\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u}\right)\left(\tfrac{R^4}{u^2}\left(-\tfrac{u^2}{R^2}\tfrac{\partial}{\partial u}\right)\left(\tfrac{u}{R}\Phi\right)\right) \\ &= \tfrac{u^2}{R}\tfrac{\partial}{\partial u}\left(\tfrac{\partial}{\partial u}(u\Phi)\right) \\ &= \tfrac{u^2}{R}\left(u\tfrac{\partial^2\Phi}{\partial u^2} + 2\tfrac{\partial\Phi}{\partial u}\right) \\ &= \tfrac{u}{R}\left(\tfrac{\partial}{\partial u}\left(u^2\tfrac{\partial\Phi}{\partial u}\right)\right) \\ &= -\tfrac{u}{R}\left(\tfrac{1}{\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)\tfrac{\partial\Phi}{\partial\theta}\right) + \tfrac{1}{\sin^2(\theta)}\tfrac{\partial^2\Phi}{\partial\psi^2}\right) \\ &= -\left(\tfrac{1}{\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin(\theta)\tfrac{\partial\tilde{\Phi}}{\partial\theta}\right) + \tfrac{1}{\sin^2(\theta)}\tfrac{\partial^2\tilde{\Phi}}{\partial\psi^2}\right) \end{aligned} \tag{21.27}$$

$\square$

Notice that

$$\lim_{r\to R}\tilde{\Phi}(r,\psi,\theta) = \Phi(r,\psi,\theta) \tag{21.28}$$

This transform is called *Kelvin inversion.*

Now let return to the problem of finding a Green's function for the interior of a sphere of radius. Let

$$\tilde{\mathbf{r}} = \mathbf{r}\left(\frac{R^2}{r}, \psi, \theta\right) = \frac{R^2}{r^2}\mathbf{r} \quad . \tag{21.29}$$

In view of the preceding remarks, we know that the functions

$$\begin{aligned} \Phi_1(\mathbf{r}) &= \frac{1}{|\mathbf{r}-\mathbf{r}_o|} \\ \Phi_2(\mathbf{r}) &= \frac{R}{r}\frac{1}{|\tilde{\mathbf{r}}-\mathbf{r}_o|} = \tilde{\Phi}_1(\mathbf{r}) \end{aligned} \tag{21.30}$$

will satisfy, respectively,

$$\begin{aligned} \nabla^2\Phi_1(\mathbf{r}) &= -4\pi\delta^3(\mathbf{r}-\mathbf{r}_o) \\ \nabla^2\Phi_2(\mathbf{r}) &= -\frac{4\pi R}{r}\delta^3\left(\frac{R^2\mathbf{r}}{r^2}-\mathbf{r}_o\right) \quad . \end{aligned} \tag{21.31}$$

However, notice that the support of $\nabla^2\Phi_2(\mathbf{r})$ lies completely outside the sphere. Therefore, in the interior of the sphere, Φ_2 is a solution of the homogenous Laplace equation. We also know that on the boundary of the sphere that we have

$$\Phi_1(\mathbf{r}) = \Phi_2(\mathbf{r}) \quad . \tag{21.32}$$

Thus, the function

$$\begin{aligned} G(\mathbf{r},\mathbf{r}_o) &= \frac{R}{r}\frac{1}{4\pi|\tilde{\mathbf{r}}-\mathbf{r}_o|} - \frac{1}{4\pi|\mathbf{r}-\mathbf{r}_o|} \\ &= \frac{1}{4\pi\left|\frac{R}{r}\mathbf{r}-\frac{r}{R}\mathbf{r}_o\right|} - \frac{1}{4\pi|\mathbf{r}-\mathbf{r}_o|} \end{aligned} \tag{21.33}$$

thus satisfies

$$\nabla^2_{\mathbf{r}}G(\mathbf{r},\mathbf{r}_o) = \delta^3(\mathbf{r}-\mathbf{r}_o) \tag{21.34}$$

for all $\mathbf{r}$ inside the sphere and

$$G(\mathbf{r},\mathbf{r}_o) = 0 \tag{21.35}$$

or all $\mathbf{r}$ on the boundary of the sphere. Thus, the function $G(\mathbf{r},\mathbf{r}_o)$ defined by (21.33) is the Green's function for Laplace's equation within the sphere.

Now consider the following PDE/BVP

$$\begin{aligned} \nabla^2\Phi(\mathbf{r}) &= f(\mathbf{r}) \quad , \qquad r \in B \\ \Phi(R,\psi,\theta) &= 0 \quad . \end{aligned} \tag{21.36}$$

where B is a ball of radius R centered about the origin. According to the formula (21.21) and (21.33), the solution of (21.36) is given by

$$\begin{aligned} \Phi(\mathbf{r}_o) &= \int_B G(\mathbf{r},\mathbf{r}_o)f(\mathbf{r})\,dV + \int_{\partial B} h(\psi,\theta)\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}_o)\,dS \\ &= \int_B G(\mathbf{r},\mathbf{r}_o)f(\mathbf{r})\,dV \end{aligned}$$

To arrive at a more explicit expression, we set

$$\begin{aligned} \mathbf{r}_o &= (r\cos(\psi)\sin(\theta), r\sin(\psi)\sin(\theta), r\cos(\theta)) \\ \mathbf{r} &= (\rho\cos(\alpha)\sin(\beta), \rho\sin(\alpha)\sin(\beta), \rho\cos(\beta)) \quad . \end{aligned}$$

Then

$$\begin{aligned} dV &= \rho^2\sin^2(\theta)\,d\rho\,d\alpha\,d\beta \\ dS &= \rho^2\sin^2(\theta)\,d\alpha\,d\beta \end{aligned}$$

and after a little trigonometry one finds

$$\begin{aligned} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}_o|} &= \frac{1}{4\pi\sqrt{r^2+\rho^2-2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}} \\ \frac{1}{4\pi\left|\frac{R}{r}\mathbf{r}_o-\frac{r}{R}\mathbf{r}_o\right|} &= \frac{R}{4\pi\sqrt{R^4+r^2\rho^2-2R^2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}}. \end{aligned}$$

Thus,

$$\begin{aligned}\Phi(r,\psi,\theta) &= \int_0^R\int_0^{2\pi}\int_0^{\pi}\frac{Rf(r,\psi,\theta)r^2\sin(\theta)drd\theta d\psi}{4\pi\sqrt{R^4+r^2\rho^2-2R^2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}}\\ &\quad -\int_0^R\int_0^{2\pi}\int_0^{\pi}\frac{f(r,\psi,\theta)r^2\sin(\theta)drd\theta d\psi}{4\pi\sqrt{r^2+\rho^2-2r\rho\left(\cos(\psi-\alpha)\sin(\theta)\sin(\beta)+\cos(\theta)\cos(\beta)\right)}}\end{aligned}$$

Homework: 9.3.1, 9.3.9