LECTURE 20

Green's Functions and Solutions of Laplace's Equation, I

In our discussion of Laplace's equation in three dimensions

$$0 = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{20.1}$$

I pointed out one solution of special importance, the so-called fundamental solution

$$\Phi(x, y, z) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \quad . \tag{20.2}$$

Note that due to the singularity at the point (0,0,0), the solution (20.2) is really only a solution for the region $\mathbb{R}^3 - (0,0,0)$. The nature of this solution when $r \to 0$ is worth examining a little closer.

In terms of spherical coordinates

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \psi &= \tan^{-1}\left(\frac{y}{x}\right) \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{aligned} \tag{20.3}$$

we have

$$\begin{aligned} \Phi(r, \psi, \theta) &= \frac{1}{r} \\ \nabla &= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\psi}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial \psi} \\ \nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial \theta}\right) + \frac{1}{r\sin(\theta)}\frac{\partial^2}{\partial \psi^2} \end{aligned} \tag{20.4}$$

where $\hat{r}$, $\hat{\theta}$, $\hat{\psi}$ are respectively, the unit vectors indicating the directions of tangent vectors to the corresponding coordinate curves.

Applying the gradient ∇ and the Laplacian ∇^2 to our solution (20.2) we get

$$\begin{aligned} \nabla \Phi &= \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = \frac{-\hat{\mathbf{r}}}{r^2} \\ \nabla^2 \Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right) = 0 \quad . \end{aligned} \tag{20.5}$$

However, we should note again that these formula are not really valid when $r = 0$ (since Φ is not continuous when $r = 0$, we certainly cannot evaluate derivatives of Φ when $r = 0$). To study the situation near $r = 0$, let $\epsilon > 0$ be a small positive parameter and define

$$\Phi_\epsilon = \frac{1}{r + \epsilon} \quad . \tag{20.6}$$

Since r is never negative, Φ_ϵ is perfectly regular thoughout $\mathbb{R}^3$, and obviously

$$\Phi = \lim_{\epsilon \to 0} \Phi_\epsilon . \tag{20.7}$$

Applying the Laplacian to Φ_ϵ yields

$$\begin{aligned}\nabla^2\Phi_\epsilon &= \tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial}{\partial r}\tfrac{1}{r+\epsilon}\right)\\ &= \tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(\tfrac{-r^2}{(r+\epsilon)^2}\right)\\ &= \tfrac{1}{r^2}\left(\tfrac{-2r(r+\epsilon)^2+2r^2(r+\epsilon)}{(r+\epsilon)^4}\right)\\ &= \tfrac{1}{r^2}\left(\tfrac{-2r^2-2r\epsilon+2r^2}{(r+\epsilon)^3}\right)\\ &= \tfrac{-2\epsilon}{r(r+\epsilon)^3}\quad .\end{aligned}\tag{20.8}$$

Now let us now consider the volume integral of $\nabla^2\Phi_\epsilon$ over $\mathbb{R}^3$. We have

$$\begin{aligned}\textstyle\int_{\mathbb{R}^3}\nabla^2\Phi_\epsilon dV &= \lim_{R\to\infty}\textstyle\int_0^R\int_0^{2\pi}\int_0^\pi \tfrac{-2\epsilon}{r(r+\epsilon)^3}r^2\sin(\theta)drd\psi d\theta\\ &= \lim_{R\to\infty}\textstyle\int_0^R \tfrac{-8\pi\epsilon r}{(r+\epsilon)^3}dr\\ &= \lim_{R\to\infty}\textstyle\int_\epsilon^{R+\epsilon}\tfrac{-8\pi\epsilon(\rho-\epsilon)}{\rho^3}d\rho\\ &= \lim_{R\to\infty}\left(\tfrac{8\pi\epsilon}{\rho}-\tfrac{4\pi\epsilon^2}{\rho^2}\right)\Big|_\epsilon^{R+\epsilon}\\ &= -4\pi\quad .\end{aligned}\tag{20.9}$$

Notice the result we obtain is **independent** of ϵ.

Thus, we have discovered a sequence of functions f_ϵ

$$f_\epsilon(\mathbf{r}) = \frac{-1}{4\pi}\nabla^2\Phi_\epsilon(\mathbf{r}) = \frac{\epsilon}{2\pi r(r+\epsilon)^3}\tag{20.10}$$

for which

$$\lim_{\epsilon\to 0} f_\epsilon(\mathbf{r}) = \begin{cases} 0 & ,\quad \text{if}\mathbf{r}\neq(0,0,0)\\ \infty & ,\quad \text{if } \mathbf{r}=(0,0,0)\end{cases}\tag{20.11}$$

and for which

$$\int_{\mathbb{R}^3} f_\epsilon(\mathbf{r})dV = 1 \qquad \forall\epsilon\neq 0\tag{20.12}$$

But properties (20.11) and (20.12) are exactly the properties that we demand for a sequence of functions to define a three-dimensional delta-function. (See Lecture 7.)

Indeed, let $g(\mathbf{r})$ be a differentiable function on $\mathbb{R}^3$ and consider the limit

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} g(\mathbf{r})f_\epsilon(\mathbf{r})dV \quad .$$

According to (20.11) the support of $f_\epsilon(\mathbf{r})$ for small ϵ is concentrated around the origin. For example, if we set

$$\epsilon = 10^{-6} f_\epsilon(r) < \frac{10^{-6}}{2\pi}\quad ,\quad \forall\, r>1$$

and if we set $\epsilon = 10^{-30}$

$$f_\epsilon(r) < \frac{10^{-6}}{2\pi}\quad ,\quad \forall\, r > 10^{-6}\quad .$$

In the limit the support of $f_\epsilon(\mathbf{r})$ the integrand is precisely the origin $\mathbf{O}$. Thus,

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} g(\mathbf{r})f_\epsilon(\mathbf{r})dV = \lim_{\epsilon\to 0}\int_{\mathbb{R}^3} g(\mathbf{O})f_\epsilon(\mathbf{r})dr = g(\mathbf{O})\quad .$$

And so we set

$$\delta^3(\mathbf{r}) = \lim_{\epsilon\to 0}\left(\frac{-1}{4\pi}\nabla^2\left(\frac{1}{r+\epsilon}\right)\right)\tag{20.13}$$

with the understanding that the limit is to be taken only after integrating. By an abuse of notation one sometimes writes

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{r}) \tag{20.14}$$

or even more generally,

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_o|} = -4\pi\delta^3\left(\mathbf{r}-\mathbf{r}_o\right) \quad . \tag{20.15}$$

Okay, so what is the point of all this? Consider the non-homogeneous equation

$$\nabla^2\Phi = -4\pi g(\mathbf{r}) \tag{20.16}$$

with $g(\mathbf{r})$ decaying faster than $\frac{1}{r^{2+\epsilon}}$ as $r \to \infty$. Multiplying both sides of (20.16) by

$$\frac{1}{|\mathbf{r}-\mathbf{r}_o|}$$

and integrating over $\mathbb{R}^3$ we get

$$\begin{aligned}
\int_{\mathbb{R}^3}\frac{-4\pi}{|\mathbf{r}-\mathbf{r}_o|}g(\mathbf{r})dV &= \int_{\mathbb{R}^3}\frac{1}{|\mathbf{r}-\mathbf{r}_o|}\nabla^2\Phi(\mathbf{r})dV \\
&= -\int_{\mathbb{R}^3}\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}_o|}\right)\cdot\nabla\Phi(\mathbf{r})dV \\
&= \int_{\mathbb{R}^3}\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r}_o|}\right)\Phi(\mathbf{r})dV \\
&= \int_{\mathbb{R}^3}-4\pi\delta^3\left(\mathbf{r}-\mathbf{r}_o\right)\Phi(\mathbf{r})dV \\
&= -4\pi\Phi\left(\mathbf{r}_o\right) \quad .
\end{aligned}$$

(In the second and third steps we have used an integration by parts formula coming from Gauss's theorem). We thus have the following solution to (20.16)

$$\Phi(\mathbf{r}) = \int_{\mathbb{R}^3}\frac{g(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}dV \quad .$$

Note how the integral kernel $G\left(\mathbf{r},\mathbf{r}'\right) = \frac{1}{|\mathbf{r}'-\mathbf{r}|}$ is used to construct the solution $\Phi(\mathbf{r})$ directly from the "source function" $g(\mathbf{r})$. More generally, an integral kernel that interpolates between source functions (inhomogeneous terms) and solutions of a nonhomogeneous PDE is referred to as Green's function for the PDE.