LECTURE 19

Characteristics and First Order Equations

We shall now generalize the methods developed in the preceding lecture.

DEFINITION 19.1. *A partial differential equation in n variables, x_i is said to be* ***quasi-linear*** *if it is linear in the partial derivatives of the unknown function. Thus, a quasi-linear PDE is an equation of the form*

$$\sum_{i=1}^{n} F_i u_{x_i} = G \tag{19.1}$$

where the coefficients F_i and G are given functions of the coordinates x_i and the unknown function u. Note that we do not require the F_i or G to be linear in u.

DEFINITION 19.2. *A* ***characteristic curve*** *for a quasi-linear PDE*

$$\sum_{i=1}^{n} F_i u_{x_i} = G$$

is a curve , $: \mathbb{R} \to \mathbb{R}^{n+1}$ satisfying

$$\begin{array}{rcll} \frac{dx_i}{dt}(t) & = & F_i(\mathbf{x}(t), u(t)) \quad , \quad i = 1, \ldots, n & ; \\ \frac{dx_{n+1}}{dt}(t) & = & G(\mathbf{x}(t), u(t)) & . \end{array} \tag{19.2}$$

Let me now recall the existence and uniqueness theorem for systems of first order ODEs.

THEOREM 19.3. *Let $f_1, \ldots, f_m$ be a set of functions of $m+1$ variables $t, y_1, \ldots, y_n$ and suppose the f_i are continuous in a region $S \subseteq \mathbb{R}^m$ defined by*

$$\begin{aligned} |t - t_o| & \leq k_o \\ |y_1 - a_1| & \leq k_1 \\ & \vdots \\ |y_m - a_m| & \leq k_m \end{aligned}$$

and that moreover each function f_i satisfies a Lipschitz condition: there exists N_i such that

$$\left| f_i(t, y_1, \ldots, y_n) - f_i\left(\tilde{t}, \tilde{y}_1, \ldots, \tilde{y}_n\right) \right| \leq N_i \left(|y_1 - \tilde{y}_1| + \cdots + |y_m - \tilde{y}_m| \right)$$

for every pair of points $(t, y_1, \ldots, y_n)$, $\left(\tilde{t}, \tilde{y}_1, \ldots, \tilde{y}_n\right) \in S$. Then there exists an interval $I = \{t \in \mathbb{R} \mid |t - t_o| < h\}$ in which there exist one and only one set $\{y_1(t), \ldots, y_n(t)\}$ of continuous functions with continuous derivatives in I satisfying the differential equations

$$\begin{aligned} \frac{dy_1}{dt}(t) & = f_1(t, y_1(t), \ldots, y_m(t)) \\ & \vdots \\ \frac{dy_m}{dt}(t) & = f_m(t, y_1(t), \ldots, y_m(t)) \end{aligned}$$

and the initial conditions

$$\begin{aligned} y_1(t_o) & = a_1 \\ & \vdots \\ y_m(t_o) & = a_m \quad . \end{aligned}$$

COROLLARY 19.4. *Consider the quasi-linear PDE*

$$\sum_{i=1}^{n} F_i \phi_{x_i} = G \quad .$$

Let R be a region in $\mathbb{R}^{n+1}$ where the functions $F_1, \cdots, F_n$, and G are all Lipschitz. Then for all $\mathbf{y} \in R$ there is one and only one characteristic curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ passing through $\mathbf{y}$ at $t = 0$.

Here is another fundamental fact about characteristics.

PROPOSITION 19.5. *Suppose ϕ is a solution of a quasi-linear PDE*

$$\sum_{i=1}^{n} F_i \phi_{x_i} = G \quad .$$

Let $S = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid x_{n+1} = \phi(x_1, \cdots, x_n) \right\}$ be corresponding surface in $\mathbb{R}^{n+1}$. Then if a characteristic curve $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ passes through one point of S, it lies entirely in S.

PROOF. From Vector Calculus we know that the vector

$$\mathbf{n}\left(x_1, \cdots, x_n\right) = \left(\phi_{x_1}\left(x_1, \cdots, x_n\right), \cdots, \phi_{x_n}\left(x_1, \cdots, x_n\right), -1\right) \in \mathbb{R}^{n+1}$$

represents the direction (in $\mathbb{R}^{n+1}$) of the normal to the surface S above the point $(x_1, \cdots, x_n) \in \mathbb{R}^n$. We also know that the tangent vector (in $\mathbb{R}^{n+1}$) to a characteristic γ is given by

$$\dot{\gamma}(t) = \left(\frac{d\gamma_1}{dt}, \cdots, \frac{d\gamma_n}{dt}, \frac{d\gamma_{n+1}}{dt} \right) = \left(F_1\left(\gamma(t)\right), \cdots, F_n\left(\gamma(t)\right), G\left(\gamma(t)\right)\right) \quad .$$

But then if $\gamma(t) = \left(x_1, \cdots, x_n, \phi\left(x_1, \cdots, x_n\right)\right)$ we have

$$\dot{\gamma}(t) \cdot \mathbf{n}\left(x_1, \cdots, x_n\right) = F_1 \phi_{x_1} + \cdots F_n \phi_{x_n} - G = 0$$

by virtue of the original PDE. Thus, the tangent vector to any characteristic passing through a given point on a solution surface always lies in (the tangent plane to) the surface. Thus, if a characteristic passes through a solution surface it can never leave that surface. □

Let me now explain how these results allow us to construct solutions on first order quasi-linear PDEs.

Let Σ be an $(n-1)$-dimensional surface in $\mathbb{R}^n$ and consider an initial value problem of the form

$$\begin{array}{rcl} \sum_{i=1}^{n} F_i \phi_{x_i} & = & G \\ \phi(\mathbf{x})\big|_{\Sigma} & = & f(\mathbf{x})\big|_{\Sigma} \end{array} \tag{19.3}$$

Let $\tilde{\Sigma} = \left\{ \mathbf{y} \in \mathbb{R}^{n+1} \mid (y_1 \cdots y_n) \in \Sigma \quad \text{and} \quad y_n = f\left(y_1, \cdots, y_n\right) \right\}$ be the "lift" of the the initial data surface to $\mathbb{R}^{n+1}$. Obviously, $\tilde{\Sigma}$ lies in the solution surface. We shall assume that the characteristics on the solution surface are always always transverse to (i.e., their tangent vectors are never parallel to the tangent vectors of) Σ and that every characteristic on the solution surface passes through the Σ.

Our basic goal is to figure out the value of $\phi\left(x_1, \cdots, x_n\right)$ at arbitrary points $\left(x_1, \cdots, x_n\right) \in \mathbb{R}^n$.

Step 1. Solve the system of ordinary differential equations for a characteristic passing through an arbitrary point $(a_1, \ldots, a_n, f(a_1, ldots, a_n))$ on the lift $\tilde{\sigma}$ of the initial data curve at $t = 0$. More explicitly, solve

$$\begin{array}{rcl} \frac{d\gamma_1}{dt} & = & F_1 \\ & \vdots & \\ \frac{d\gamma_n}{dt} & = & F_n \\ \frac{d\gamma_{n+1}}{dt} & = & G \end{array} \tag{19.4}$$

subject to the initial conditions

$$\begin{aligned} \gamma_1(0) &= a_1 \\ &\vdots \\ \gamma_n(0) &= a_n \\ \gamma_{n+1}(0) &= f(a_1, \ldots, a_n) \end{aligned} \tag{19.5}$$

where $(a_1, \cdots, a_n)$ is an arbitrary point in Σ. This will give us an $(n-1)$-parameter family of characteristics

$$\mathcal{C} = \{\gamma_{a_1,a_2,\ldots,a_n}(t)\} \tag{19.6}$$

Step 2. Pick an arbitray point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and figure out which of the characteristics in $\mathcal{C}$ passes over $\mathbf{x}$ and the value of t when this occurs.

Step 3. Set

$$\phi(\mathbf{x}) = \gamma_{n+1}(t) \tag{19.7}$$

where γ_{n+1} is the $(n+1)^{th}$ component of the characteristic γ determined in Step 2, and t is the value of the parameter t when γ passes over $\mathbf{x}$.

Example 19.6. Consider the following PDE/BVP:

$$\begin{aligned} x\phi_x + y\phi_y &= 1 + y^2 \\ \phi(x,1) &= x + 1 \end{aligned} \tag{19.8}$$

In this example we have

$$\begin{aligned} n &= 2 \\ F_1 &= x \\ F_2 &= y \\ G &= 1 + y^2 \\ \Sigma &= \left\{(x,y) \in \mathbb{R}^2 \mid y = 1\right\} \end{aligned} \tag{19.9}$$

Step 1. We want to solve

$$\begin{aligned} \tfrac{dx}{dt} &= x \\ \tfrac{dy}{dt} &= y \\ \tfrac{dz}{dt} &= 1 + y^2 \end{aligned} \tag{19.10}$$

subject to the initial conditions

$$\begin{aligned} x(0) &= x_o \\ y(0) &= 1 \\ z(0) &= x_o + 1 \end{aligned} \tag{19.11}$$

The first two equations are trivially integrated to yield

$$\begin{aligned} x(t) &= C_1 e^t \\ y(t) &= C_2 e^t \end{aligned} \tag{19.12}$$

In order to satisfy the initial conditions, we must take $C_1 = x_o$ and $C_2 = 1$. Substituting $y(t) = e^t$ into the ODE for $z(t)$ we get

$$\frac{dz}{dt} = 1 + e^{2t}$$

which has as its general solution

$$z(t) = t + \frac{1}{2}e^{2t} + C_3 \quad . \tag{19.13}$$

In order to satisfy the initial condition $z(0) = x_o + 1$ we must take $C_3 = x_o + \frac{1}{2}$.

Thus, the characteristics lying in the solution surface and passing above the points $(x_o, 1)$ will be curves of the form

$$\gamma_{x_o}(t) = \left(x_o e^t, e^t, t + \frac{1}{2}e^{2t} + \left(x_o + \frac{1}{2}\right)\right) \quad . \tag{19.14}$$

Step 2. We now will try to determine which of the characteristics γ_{x_o} will pass over a given point (x, y). Setting

$$\begin{aligned} x &= x_o e^t \\ y &= e^t \end{aligned}$$

and solving for x_o and t we get

$$\begin{aligned} t &= \ln|y| \\ x_o &= e^{-t}x = e^{-\ln|y|}x = \tfrac{x}{y} \end{aligned} \tag{19.15}$$

Step 3. We now set

$$\begin{aligned} \phi(x,y) &= \left(\gamma_{x_o}\right)_z(t) \\ &= t + \frac{1}{2}e^{2t} + x_o + \frac{1}{2} \\ &= \ln|y| + \frac{1}{2}e^{2\ln|y|} + \frac{x}{y} + \frac{1}{2} \\ &= \ln|y| + \frac{1}{2}y^2 + \frac{x}{y} + \frac{1}{2} \end{aligned}$$

Thus, the solution to our PDE/BVP is

$$\boxed{\phi(x,y) = \ln|y| + \frac{1}{2}y^2 + \frac{x}{y} + \frac{1}{2}}$$

Example 19.7. Solve the following Cauchy problem.

$$x^2\phi_x + \phi\phi_y = 1 \tag{19.16}$$
$$\phi(x, 1-x) = 0 \tag{19.17}$$

in the region $x > 0$.

The differential equation for the characteristics is

$$\begin{aligned} \tfrac{dx}{dt} &= x^2 \\ \tfrac{dy}{dt} &= \phi \\ \tfrac{d\phi}{dt} &= 1 \quad . \end{aligned} \tag{19.18}$$

The last equation is easily integrated to produce

$$\phi(t) = t + c_1 \quad . \tag{19.19}$$

Inserting this expression for $\phi(t)$ into the second equation and integrating we get

$$y(t) = \frac{1}{2}t^2 + c_1 t + c_2 \quad . \tag{19.20}$$

Finally, the general solution to the first equation is

$$x(t) = \frac{1}{c_{3-t}} \quad . \tag{19.21}$$

We know demand that when $t = 0$ the point $,_{c_1,c_2,c_2}(0)$ lie on the line $x + y = 1$ in the plane $u = 0$. This requires

$$\begin{aligned} \tfrac{1}{c_3} + c_2 &= 1 \\ c_1 &= 0 \quad . \end{aligned} \tag{19.22}$$

Thus we can restrict attention to characteristics of the form

$$,_{0,1-\frac{1}{c_3},c_3}(t) = \left(\frac{1}{c_3 - t}, \frac{1}{2}t^2 + 1 - \frac{1}{c_3}, t \right) \quad . \tag{19.23}$$

If we now demand that such a characteristic pass over the point (x, y) we get

$$\begin{aligned} x &= \tfrac{1}{c_3 - t} \\ y &= \tfrac{1}{2}t^2 + 1 - \tfrac{1}{c_3} \quad . \end{aligned} \tag{19.24}$$

These equations we can solve to express c_3 and t in terms of x and y - the result is not very pretty however. Nevertheless, it is clear that we can compute $\phi(x, y)$ as

$$\phi(x, y) = ,_{0,1-\frac{1}{c_3},c_3}(t)$$

with c_3 and t determined by (19.24).

Homework: 6.4.2, 6.4.4