LECTURE 17

Characteristics, Discontinuities, and Signal Propagation

Recall that in our discussion of the wave equation with Cauchy data along the x-axis

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} &= 0 \\
\phi(0,x) &= f(x) \\
\phi_t(0,x) &= p(x)
\end{aligned}
\tag{17.1}
$$

we found that the value of the general solution in the region $t > 0$, $-\infty < x < +\infty$ was given by

$$
\phi(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} p(\zeta)d\zeta \tag{17.2}
$$

at a point (x_o, t_o) and so depended only on the Cauchy data lying within the backwards (light) cone formed by the characteristic curves through (x_o, t_o). In particular if $p(x) = 0$ for all $x \in \mathbb{R}$, and $f(x) = 0$ for all $x \notin [a,b]$ we must expect that the solution (17.2) to vanish at all points in the xt-plane outside the region

$$
R = \{(x,t) \mid t > 0, a - ct \leq x \leq b + ct\} \quad .
$$

Note that the region R is the subset of the upper half plane that can be reached by characteristics passing through the interval $[a,b]$ on the x-axis. The boundary of R consists of a characteristic passing through $(a,0)$ and a characteristic passing through $(b,0)$. Since $\phi(x,t)$ is nonzero only inside R, and definitely zero outside R, one might ask if whether or not the solution of the wave equation can develop discontinuities across a characteristic.

It should be pointed out that this is **not** a ill-posed question; in order to make sense of equations (17.1) all we require is that $\frac{\partial^2\phi}{\partial t^2}$, $\frac{\partial^2\phi}{\partial x^2}$ exist throughout the upper half plane - we do not necessarily have to require also that ϕ and/or all of its derivatives are continuous. Indeed, the existence of partial derivatives with respect to one set of coordinates is not sufficient to prove the existence of partial derivatives with respect to another choice of coordinates or even sufficient to prove the continuity of the function itself. Thus, the PDE/BVP (17.1) might be a legitimate problem even if ϕ or some directional derivative of ϕ is discontinuous.

This observation raises the following problem: assuming the validity of (17.1), under what circumstances is it possible to find a coordinate system for which $\Phi_{y_1y_1}$ is discontinuous.

Consider a general second order linear PDE in two variables

$$
\sum_{i,j=1}^{2} A_{ij}\phi_{x_ix_j} + \sum_{i=1}^{2} B_i\phi_{x_i} + C\phi + F = 0 \quad . \tag{17.3}
$$

Let , be a curve in the (x_1,x_2)-plane defined by $\tilde{y}_1(x_1,x_2) = 0$; let $\tilde{y}_2(x_1,x_2)$ be an auxiliary coordinate, chosen so that the Jacobian of the coordinate transformation $(x_1,x_2) \rightarrow (y_1,y_2)$ does not vanish; and let Φ denote the re-expression of $\phi(x_1,x_2)$ in terms of the new coordinates y_1 and y_2. We shall assume that $\Phi(y_1,y_2)$ is continuous and posseses continuous first and second derivatives except perhaps the single exception that $\Phi_{y_1y_1}$ is discontinuous across , .

We also assume equation (17.3) to hold on each side of , . Making a change of coordinates $(x_1,x_2) \rightarrow (y_1,y_2)$ and denoting values of $\Phi_{y_1y_1}$ and its derivatives on one or the other side of , by superscripts $\Phi^{\pm}_{y_1y_1}$, we can write

$$\begin{array}{rcl} A'_{11}\Phi^+_{y_1y_1} + 2A'_{12}\Phi_{y_1y_2} + A'_{22}\Phi_{y_2y_2} + \sum_{i=1}^2 B'_i\Phi_{y_i} + C'\Phi + F' & = & 0 \quad , \\ A'_{11}\Phi^-_{y_1y_1} + 2A'_{12}\Phi_{y_1y_2} + A'_{22}\Phi_{y_2y_2} + \sum_{i=1}^2 B'_i\Phi_{y_i} + C'\Phi + F' & = & 0 \quad . \end{array} \tag{17.4}$$

Subtracting these two equations and taking the limit as $y_1 \to 0$, we find that we must have

$$\lim_{y_1 \to 0} A'_{11}\left(\Phi^+_{y_1y_1} - \Phi^-_{y_1y_1}\right) = 0 \quad . \tag{17.5}$$

Thus, if $\Phi_{y_1y_1}$ is discontinuous we can only maintain self-consistency in the limit $y_1 \to 0$ when

$$A'_{11} = A_{11}\tilde{y}_{1;x_1}\tilde{y}_{1;x_1} + 2A_{12}\tilde{y}_{1;x_1}\tilde{y}_{1;x_2} + A_{22}\tilde{y}_{1;x_2}\tilde{y}_{1;x_2} = 0 \tag{17.6}$$

But (17.6) is precisely the condition that the coordinate curve defined by $\tilde{y}_1(x_1, x_2) = 0$ is a characteristic. We conclude that if (17.3) is to be satisfied everywhere and a discontinuity in a double directional derivative occurs, then it must occur along a characteristic.

Such discontinuities often have a physical interpretation. For example, the wave equation

$$\frac{\partial^2\phi}{\partial t^2} - \nu^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0$$

describes the propagation of sound in space ($\phi(x, y, z, t)$ represents the amplitude of the pressure wave at the point (x, y, z) at time t). As is well known, a supersonic plane produces a shock wave that might be (and is, in fact) interpretable as a discontinuity of the function describing the air pressure. The discussion above then predicts that such a discontinuity must occur along a characteristic. This is precisely where the shock waves lie.