LECTURE 16

# Characteristics and Cauchy Data

Let's again consider a general second order linear PDE in two variables, but now with Cauchy boundary conditions;

$$
\begin{aligned}
A_{11}\phi_{tt} + 2A_{12}\phi_{tx} + A_{22}\phi_{xx} + B_1\phi_t + B_2\phi_x + C\phi &= F \\
\phi(0,x) &= f(x) \\
\phi_t(0,x) &= g(x) \quad .
\end{aligned} \tag{16.1}
$$

Notice that once values of $\phi$ and $\phi_t$ are prescribed along the $x$-axis, then so are all their derivatives with respect to $x$;

$$
\begin{aligned}
\phi_x(0,x) &= f'(x) \\
\phi_{xx}(0,x) &= f''(x) \\
\phi_{xxx}(0,x) &= f'''(x) \\
&\vdots \\
\phi_{tx}(0,x) &= g'(x) \\
\phi_{txx}(0,x) &= g''(x) \\
\phi_{txxx}(0,x) &= g'''(x) \\
&\vdots
\end{aligned} \tag{16.2}
$$

Note, however, that the *Cauchy data* by itself

$$
\begin{aligned}
\phi(0,x) &= f(x) \\
\phi_t(0,x) &= g(x)
\end{aligned} \tag{16.3}
$$

does not impose any restrictions on the higher derivatives of $\phi$ and $\phi_t$ with respect to $t$.

Let us now rewrite the original PDE (16.1) in the form

$$
A_{11}\phi_{tt} = -2A_{12}\phi_{tx} - A_{22}\phi_{xx} - B_1\phi_t - B_2\phi_x - C\phi + F \tag{16.4}
$$

We now see that if $A_{11} \neq 0$, we can solve (16.4) for $\phi_{tt}$

$$
\phi_{tt} = \frac{-1}{A_{11}}\left[2A_{12}\phi_{tx} + A_{22}\phi_{xx} + B_1\phi_t + B_2\phi_x + C\phi - F\right]. \tag{16.5}
$$

Evaluating (16.5) along the $x$-axis we find

$$
\begin{aligned}
\phi_{tt}(0,x) &= \tfrac{-1}{A_{11}}\left[2A_{12}\phi_{tx} + A_{22}\phi_{xx} + B_1\phi_t + B_2\phi_x + C\phi - F\right]\Big|_{t=0} \\
&= -\tfrac{1}{A_{11}}\left[2A_{12}g'(x) + A_{22}f''(x) + B_1 g(x) + B_2 f'(x) + Cf(x) - F\right]
\end{aligned} \tag{16.6}
$$

Thus, so long as $A_{11} \neq 0$, we can use the PDE to express the values of $\phi_{tt}$ along the $x$-axis in terms of the the Cauchy data. Moreover, having determined $\phi_{tt}(0,x)$ as an explicit function of $x$, we can also compute all of the values of $\phi_{ttx}, \phi_{ttxx}, \phi_{ttxxx}, \ldots$ along the $x$-axis.

If we now differentiate (16.5) with respect to $t$ we get

$$
\phi_{ttt} = \frac{\partial}{\partial t}\left[\frac{-1}{A_{11}}\left[2A_{12}\phi_{tx} + A_{22}\phi_{xx} + B_1\phi_t + B_2\phi_x + C\phi - F\right]\right] \tag{16.7}
$$

Notice that the right hand side involves only first and second derivatives of $\phi$ with respect to $t$; so, in view of (16.2) and (16.6), $\phi_{ttt}$ is completely determined along the $x$-axis by the Cauchy data.

Continuing in this manner, we see that if $A_{11} \neq 0$, the original PDE and the Cauchy data can be used to calculate the values of the higher derivatives $\phi_{tt}, \phi_{ttt}, \phi_{tttt}, \ldots$ along $x$-axis. We can now use the calculated values of the derivatives of $\phi$ with respect to $t$ to construct a Taylor series solution of (16.1); viz.,

$$\phi(t,x) = \phi(0,x) + t\phi_t(0,x) + \frac{t^2}{2!}\phi_{tt}(0,x) + \cdots \tag{16.8}$$

In fact, we have:

THEOREM 16.1. ***(Cauchy-Kowalewski)*** *If $f(x), g(x)$ and all the functions $\frac{A_{12}}{A_{11}}, \frac{A_{22}}{A_{11}}, \frac{B_1}{A_{11}}, \frac{B_2}{A_{11}}, \frac{C}{A_{11}}, \frac{F}{A_{11}}$ have power series representations about the point $(0,x)$, then the procedure outlined above constructs a unique solution to the Cauchy problem*

$$\begin{array}{rcl} \sum_{i,j=1}^{n} A_{ij}\phi_{x_ix_j} + \sum_{i=1}^{2} B_i\phi_{x_i} + C\phi & = & F \\ \phi(0,x) & = & f(x) \\ \phi_t(0,x) & = & g(x) \end{array} \tag{16.9}$$

*in some neighborhood of $(0,x)$.*

Note that this algorithm for constructing a power series solution of the PDE/BVP (16.1) depended crucially on the hypothesis that $A_{11} \neq 0$. Indeed, if this condition does not hold, we have absolutely no means of determining the higher derivatives of $\phi$ with respect to $t$ in terms of the Cauchy data on the $x$-axis.

In fact, if $A_{11} = 0$, there may exist more than one independent solution with the same Cauchy data along the $x-$axis; or they may exist no solution for a given set of Cauchy data along $x$-axis.

To see this, consider the following simple PDE/BVP

$$\begin{array}{rcl} \phi_{tx} & = & 0 \\ \phi(0,x) & = & f(x) \\ \phi_t(0,x) & = & g(x) \end{array} . \tag{16.10}$$

The general solution of the first equation in (16.10) is

$$\phi(t,x) = \alpha(t) + \beta(x) . \tag{16.11}$$

Plugging this expression into the Cauchy boundary conditions yields

$$\beta(x) = f(x) + \alpha(0) \tag{16.12}$$

$$\alpha'(t) = g(x) \tag{16.13}$$

From the first equation we see that the Cauchy data determines determines the function $\alpha$ only up to an constant. Thus, it would appear that we have an infinite number of solutions of (16.10). On the other hand, Equation (16.13) implies that both $\alpha'(t)$ and $g(x)$ must be constants (via a Separation of Variables type argument). We thus see that unless $g(x)$ is equal to a constant we will be lead to an inconsistency. In summary, there will exist no solution to (16.10) unless $g(x)$ is constant, and if $g(x)$ is constant then there exist infinitely many solutions.

We summarize these difficulties by saying that the Cauchy problem is not *well posed* (for Cauchy data along the $t$-axis) if $A_{11} = 0$.

Let us now consider the more general case of a Cauchy problem where the Cauchy data is specified along some curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ in the $tx$-plane;

$$\begin{array}{rcl} \sum_{i,j=1}^{n} A_{ij}\phi_{x_ix_j} + \sum_{i=1}^{2} B_i\phi_{x_i} + C\phi & = & F \\ \phi(\gamma(t)) & = & f(t) \\ \frac{\partial\phi}{\partial n}(\gamma(t)) & = & g(t) \end{array} , \tag{16.14}$$

where $\frac{\partial}{\partial n}$ is directional derivative normal to the curve $\gamma$.

If we reinterpret the curve $\gamma$ as a coordinate curve $y_1(t,x)=0$ and interpret the directional derivative $\frac{\partial}{\partial n}$ as the partial derivative with respect to this coordinate, then we can cast the generalized Cauchy problem (16.14) back into the original form (16.1);

$$\begin{array}{rcl} \sum_{i,j=1}^{n} A'_{ij}\Phi_{y_iy_j} + \sum_{i=1}^{2} B'_i\Phi_{y_i} + C\Phi & = & F \\ \phi(0,y_2) & = & \tilde{f}(y_2) \\ \phi_{y_1}(0,y_2) & = & \tilde{g}(y_2) \end{array} \quad . \tag{16.15}$$

Applying the Cauchy-Kowalewski theorem to (16.12) we can conclude that if $\tilde{f}(y_2), \tilde{g}(y_2)$ and all the functions $\frac{A'_{12}}{A'_{11}}, \frac{A'_{22}}{A'_{11}}, \frac{B'_1}{A'_{11}}, \frac{B'_2}{A'_{11}}, \frac{C}{A'_{11}}, \frac{F}{A'_{11}}$ have power series representations about the point $(0,y_2)$, then the procedure outlined above constructs a unique solution to the Cauchy problem (16.14) in some neighborhood of $(0,y_2)$.

In particular, if $\gamma$ is curve along which $A'_{11}$ vanishes then the Cauchy problem will be ill posed. Now let us recall our discussion of the coordinate transformations and the classification of second order linear PDEs.

If the PDE in (16.14) is hyperbolic in a region $R \subset \mathbb{R}^2$, then by definition, the original functions $A_{11}$, $A_{12}$, $A_{22}$ satisfy

$$(A_{12})^2 - A_{11}A_{22} > 0$$

and there exists two distinct families of curves along which $A'_{11}$ will vanish. Thus, if a PDE is hyperbolic in a region $R$, there will exist two special curves $\gamma_{\pm}$, through a given point $x \in \mathbb{R}$, for which the Cauchy problem will be ill posed; the two coordinate curves for which $A'_{11}$ vanishes. These curves are called **characteristics**. We conclude that the Cauchy problem is well posed along a curve $\gamma \in R$ if and only if $\gamma$ never coincides with one of the characteristics $\gamma_{\pm}$.

If the PDE in (16.14) is parabolic in a region $R \subset \mathbb{R}^2$, then by definition, the original functions $A_{11}$, $A_{12}$, $A_{22}$ satisfy

$$(A_{12})^2 - A_{11}A_{22} = 0$$

and there exists only distinct family of coordinate curves along which $A'_{11}$ will vanish. Thus, if a PDE is parabolic in $R$ there will exist only one characteristic curve $\gamma$ passing through a given point $x \in \mathbb{R}$, and so the Cauchy problem is well-posed along every curve through $x$ except $\gamma$.

If the PDE in (16.14) is elliptic in a region $R \subset \mathbb{R}^2$, then by definition, the original functions $A_{11}$, $A_{12}$, $A_{22}$ satisfy

$$(A_{12})^2 - A_{11}A_{22} < 0$$

and there exists no curve $\gamma$ along which $A'_{11}$ will vanish. Thus, if a PDE is elliptic in $R$, the Cauchy problem is always well-posed in $R$.

**Homework:** 5.7.1, 5.7.2, 5.7.3(b), 5.7.3(d), 5.7.6