LECTURE 15

Characteristics and the Classification of Second Order Linear PDEs

Let us now consider the case of a general second order linear PDE in two variables;

$$
\begin{aligned}
0 &= \textstyle\sum_{i,j=1}^{2} A_{ij}\phi_{x_i x_j} + \sum_{i-1}^{2} B_i \phi_{x_i} + C\phi + F \\
&= A_{11}\phi_{x_1x_1} + 2A_{12}\phi_{x_1x_2} + A_{22}\phi_{x_2x_2} + B_1\phi_{x_1} + B_2\phi_{x_2} + C\phi + F \quad ,
\end{aligned}
\tag{15.1}
$$

where

$$
\begin{aligned}
\phi_{x_i} &= \tfrac{\partial\phi}{\partial x_i} \\
\phi_{x_i x_j} &= \tfrac{\partial^2\phi}{\partial x_i \partial x_j} \quad .
\end{aligned}
\tag{15.2}
$$

Suppose we make a general coordinate transformation

$$
\begin{aligned}
y_1 &= \tilde{y}_1(x_1, x_2) \\
y_2 &= \tilde{y}_2(x_1, x_2) \\
x_1 &= \tilde{x}_1(y_1, y_2) \\
x_2 &= \tilde{x}_2(y_1, y_2)
\end{aligned}
\tag{15.3}
$$

in which the functions $\tilde{y}_a$, x_i, $a, i = 1, 2$, are at least twice differentiable and the Jacobian

$$
J\left[\frac{\partial\tilde{y}}{\partial x}\right] = \frac{\partial\tilde{y}_1}{\partial x_1}\frac{\partial\tilde{y}_2}{\partial x_2} - \frac{\partial\tilde{y}_2}{\partial x_1}\frac{\partial\tilde{y}_1}{\partial x_2}
\tag{15.4}
$$

is nowhere vanishing. Under such a coordinate transformation, equation (15.1) becomes

$$
0 = \textstyle\sum_{a,b=1}^{2} A'_{ab}\Phi_{y_a y_b} + \sum_{a=1}^{2} B'_a \Phi_{y_a} + C\Phi + F
\tag{15.5}
$$

where

$$
\begin{aligned}
A'_{ab} &= \textstyle\sum_{i,j=1}^{2} A_{ij}\tilde{y}_{a;x_i}\tilde{y}_{b;x_j} \\
&= A_{11}\tilde{y}_{a;x_1}\tilde{y}_{b;x_1} + A_{12}\left(\tilde{y}_{a;x_1}\tilde{y}_{b;x_2} + \tilde{y}_{a;x_2}\tilde{y}_{b;x_1}\right) + A_{22}\tilde{y}_{a;x_2}\tilde{y}_{b;x_2} \\
B'_a &= \textstyle\sum_{i,j=1}^{2} A_{ij}\tilde{y}_{a;x_i x_j} + \sum_{i=1}^{2} B_i \tilde{y}_{a;x_i} \\
&= A_{11}\tilde{y}_{a;x_1x_1} + 2A_{12}\tilde{y}_{a;x_1x_2} + A_{22}\tilde{y}_{a;x_2x_2} + B_1\tilde{y}_{a;x_1} + B_2\tilde{y}_{a;x_2} .
\end{aligned}
\tag{15.6}
$$

and

$$
\begin{aligned}
\tilde{y}_{a;x_i} &= \tfrac{\partial\tilde{y}_a}{\partial x_i} \\
\tilde{y}_{a;x_i x_j} &= \tfrac{\partial^2\tilde{y}_a}{\partial x_i \partial x_j} \quad .
\end{aligned}
\tag{15.7}
$$

Up to this point everything is completely general. Let us now explore and see what conditions to place on the coordinate transformation $y_a(x)$ in order that the coefficient

$$
A'_{11} = A_{11}\tilde{y}_{1;x_1}\tilde{y}_{1;x_1} + 2A_{12}\tilde{y}_{1;x_1}\tilde{y}_{1;x_2} + A_{22}\tilde{y}_{1;x_2}\tilde{y}_{1;x_2}
\tag{15.8}
$$

of $\Phi_{y_1 y_1}$ in (15.5) vanish identically.

We shall proceed as in the previous lecture; however, first let us simplify our notation a bit by setting

$$\begin{aligned} x &= x_1 \\ y &= x_2 \\ \tilde{\zeta} &= \tilde{y}_1 \\ \tilde{\eta} &= \tilde{y}_2 \end{aligned}$$

We then have

$$\begin{aligned} A'_{11} &= A_{11}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{12}\tilde{\zeta}_x\tilde{\zeta}_y + A_{22}\tilde{\zeta}_y\tilde{\zeta}_y \\ A'_{12} &= A_{11}\tilde{\zeta}_x\tilde{\eta}_x + A_{12}\tilde{\zeta}_x\tilde{\eta}_y + A_{21}\tilde{\zeta}_y\tilde{\eta}_x + A_{22}\tilde{\zeta}_y\tilde{\eta}_y \\ A'_{22} &= A_{11}\tilde{\eta}_x\tilde{\eta}_x + 2A_{12}\tilde{\eta}_x\tilde{\eta}_y + A_{22}\tilde{\eta}_y\tilde{\eta}_y \\ B'_1 &= A_{11}\tilde{\zeta}_{xx} + 2A_{12}\tilde{\zeta}_{xy} + A_{22}\tilde{\zeta}_{yy} + B_1\tilde{\zeta}_x + B_2\tilde{\zeta}_y \\ B'_2 &= A_{11}\tilde{\eta}_{xx} + 2A_{12}\tilde{\eta}_{xy} + A_{22}\tilde{\eta}_{yy} + B_1\tilde{\eta}_x + B_2\tilde{\eta}_y \end{aligned} \tag{15.9}$$

Suppose

$$\tilde{\zeta}(x,y) = const \tag{15.10}$$

is a curve in the xy-plane along which (15.8) vanishes identically and let

$$y = f(x)$$

be a (local) representation of this curve as the graph of a function f of x; so that (15.10) is equivalent to

$$\tilde{\zeta}\left(x, f(x)\right) = const \quad . \tag{15.11}$$

Differentiating (15.11) with respect to x yields

$$\frac{\partial\tilde{\zeta}}{\partial x} + f'\frac{\partial\tilde{\zeta}}{\partial y} = 0 \tag{15.12}$$

or

$$\frac{1}{f'}\frac{\partial\tilde{\zeta}}{\partial x} + \frac{\partial\tilde{\zeta}}{\partial y} = 0 \quad , \tag{15.13}$$

The condition that A'_{11} vanish along the curve (15.11) is

$$\begin{aligned} 0 &= A_{11}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{12}\tilde{\zeta}_x\tilde{\zeta}_y + A_{22}\tilde{\zeta}_y\tilde{\zeta}_y \\ &= A_{11}\left(-f'\tilde{\zeta}_y\right)^2 - 2A_{12}f'\left(\tilde{\zeta}_y\right)^2 + A_{22}\left(\tilde{\zeta}_y\right)^2 \\ &= \left(A_{11}\left(f'\right)^2 - 2A_{22}f' + A_{22}\right)\left(\zeta_y\right)^2 \quad . \end{aligned} \tag{15.14}$$

Thus, if f is chosen to satisfy the differential equation

$$A_{11}\left(f'\right)^2 - 2A_{22}f' + A_{22} = 0 \quad . \tag{15.15}$$

then the coefficient A'_{11} will vanish.

Now the general solution of a first order ODE will inevitably be of the form

$$f(x) = F(x,C) \tag{15.16}$$

where C is an arbitrary constant of integration. In order to recover a coordinate function $\tilde{\zeta}(x,y)$ corresponding to $f(x)$, let's first recall the graph of $f(x)$ is supposed to coincide with a level curve of $\tilde{\zeta}(x,y)$. In other words, the solution set of

$$y = f(x) = F(x,C) \tag{15.17}$$

is to coincide with the solution set of an equation of the form

$$\tilde{\zeta}(x,y) = \zeta_0. \tag{15.18}$$

Now suppose we can invert equation (15.16) to get C as a function of x and y,

$$C = \tilde{F}(x,y). \tag{15.19}$$

Then by taking

$$\tilde{\zeta}(x,y) = \tilde{F}(x,y) \tag{15.20}$$

$$\zeta_o = C \tag{15.21}$$

we obtain an equation of the form (15.18) that has the same solution set as (15.17). We thereby obtain a suitable coordinate function.

Now let us now look at equation (15.15) a little more carefully. We are assuming that $A_{11} \neq 0$; since otherwise there would be no point in making a coordinate transformation. (15.15) is thus a quadratic equation in f'. Using the quadratic formula we find that (15.15) is actually equivalent to

$$\left(f' - \frac{A_{12} + \sqrt{(A_{12})^2 - A_{11}A_{22}}}{A_{11}} \right) \left(f' - \frac{A_{12} - \sqrt{(A_{12})^2 - A_{11}A_{22}}}{A_{11}} \right) = 0 \quad ,$$

so either f satisfies

$$f' = \frac{A_{12} + \sqrt{(A_{12})^2 - A_{11}A_{22}}}{A_{11}} \tag{15.22}$$

or

$$f' = \frac{A_{12} - \sqrt{(A_{12})^2 - A_{11}A_{22}}}{A_{11}} \quad . \tag{15.23}$$

These differential equations are in practice very difficult to solve; for the coefficient functions on the right hand sides will in general be expressions in which f appears non-linearly; this is perhaps more obvious when one writes a more explicit expression for (15.22)

$$\frac{df}{dx}(x) = \frac{A_{12}(x,f(x)) + \sqrt{(A_{12}(x,f(x)))^2 + A_{11}(x,f(y))\,A_{22}(x,f(x))}}{A_{11}(x,f(x))} \quad .$$

However, even without explicitly solving equations (15.22), (15.23) we can identify three distinct cases.

Case I: $(A_{12})^2 - A_{11}A_{22} > 0$.

In this case the expressions inside the radicals in equations (15.22) and (15.23) are strictly positive functions, and so their square roots are well-defined real-valued functions of x. The general existence theorem for first order ODE's guarantees the existence of solutions of (15.22) and (15.23) so long as the original functions A_{ij} were smooth functions of x and y. Let f_+ be a solution of (15.22) and let f_- be a solution of (15.23). Since the equations (15.22) and (15.23) are distinct when $(A_{12})^2 - A_{11}A_{22} > 0$, f_+ and f_- will constituent distinct solutions of (15.15). We thus find two classes of curves

$$\begin{aligned} y &= f_+(x) + C_1 \\ y &= f_-(x) + C_2 \end{aligned}$$

which can be interpreted as the level curves of two coordinate functions $\tilde{y}_{1,\pm}(x,y)$ for which A'_{11} will vanish. This case is referred to as the **hyperbolic** case.

Case II. $(A_{12})^2 - A_{11}A_{22} = 0$

In this case, equations (15.22) and (15.23) reduces to a single differential equation

$$f' = \frac{A_{12}}{A_{11}}$$

and so we can expect a single family of curves

$$y = f(x) + C$$

which are interpretable as the level curves of a single coordinate function $\tilde{\zeta}(x,y)$ for which A'_{11} vanishes. This case is referred to as the **parabolic** case.

Case III. $(A_{12})^2 - A_{11}A_{22} < 0$

In this case, the expression inside the radical is negative definite, and so f', and hence f, must be a complex-valued function. But in this case, we will no longer be able to interprete the graph of f as a level surface of a coordinate function (our coordinates must be real-valued functions). Therefore, if the original coefficients satisfy the inequality

$$(A_{12})^2 - A_{11}A_{22} < 0$$

then there exists no coordinate system for which $A'_{11} = 0$. This case is referred to as the **elliptic** case.

Similarly, we can construct a coordinate function $\tilde{\eta}(x_1, y)$ that will guarantee that the coefficient A'_{22} of $\Phi_{\eta\eta}$ in (15.5) will vanish. Let $y = g(x)$ be the graph of a function g of x corresponding to the curve

$$\tilde{\eta}(x, y) = const \tag{15.24}$$

and suppose $\tilde{\eta}(x, y)$ satisfies

$$0 = A'_{22} = A_{11}\tilde{\eta}_x\tilde{\eta}_x + 2A_{12}\tilde{\eta}_x\tilde{\eta}_y + A_{22}\tilde{\eta}_y\tilde{y}_{2;y} \quad . \tag{15.25}$$

Then, the condition

$$\tilde{\eta}\left(x, g(x)\right) = const \tag{15.26}$$

allows us to replace the terms $\tilde{\eta}_x$ in (15.25) by $-g'\tilde{\eta}_y$, to get

$$0 = \left(A_{11}\left(g'\right)^2 - 2A_{12}g' + A_{22}\right)\left(\tilde{\eta}_y\right)^2 = 0 \quad . \tag{15.27}$$

Thus, if $y = g(x)$ is a level curve of the coordinate function $\tilde{\eta}(x, y)$ and g satisfies the differential equation

$$A_{11}\left(g'\right)^2 - 2A_{12}g' + A_{22} = 0 \tag{15.28}$$

then the coefficient of $\Phi_{\eta\eta}$ in (15.5) will vanish.

Note that (15.28) is essentially the same differential equation as (15.15). The analysis of (15.28) proceeds just as before and we find we have three basic situations, distinguished by the sign of $(A_{12})^2 - A_{11}A_{22}$.

Hyperbolic Case: $(A_{12})^2 - A_{11}A_{22} > 0$

In this case, equations (15.15) and (15.28) have two independent solutions, $f_\pm$. One of these, say f_+, can be used to define a coordinate function $\tilde{\zeta}$ for which the coefficient A'_{11} vanishes identically, and the other can be used to define a coordinate function $\tilde{\eta}$ for which A'_{22} vanishes identically. One can thus define a coordinate system in which the original PDE (15.1) takes the form

$$2A'_{12}\Phi_{\zeta\eta} + B'_1\Phi_\zeta + B'_2\Phi_\eta + C\Phi + F = 0 \quad . \tag{15.29}$$

It is easy to see that in the hyperbolic case, the coefficient A'_{12} never vanishes, and so we can rewrite (15.29) as

$$\Phi_{\zeta\eta} + B''_1\Phi_\zeta + B''_2\Phi_\eta + C''\Phi + F' = 0 \quad . \tag{15.30}$$

Equation (15.30) is referred to as the *standard form of a hyperbolic PDE.*

Parabolic Case. $(A_{12})^2 - A_{11}A_{22} = 0$

In this case, we only find one real-valued function whose graph can be interpreted as the level curve of a coordinate function $\tilde{\zeta}$ for which A'_{11} vanishes identically. Lacking a second independent solution we can not find a second coordinate function $\tilde{\eta}$ for which A'_{22} vanishes identically. The coefficient A'_{12}, however, is

$$A'_{12} = A_{11}\tilde{\zeta}_x\tilde{\eta}_x + A_{12}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + A_{22}\tilde{\zeta}_y\tilde{\eta}_y \quad . \tag{15.31}$$

Using the relations

$$\tilde{\zeta}_x = -f'\tilde{\zeta}_y = \frac{A_{12}}{A_{11}}\tilde{\zeta}_y$$

and setting

$$\tilde{\eta}_x = -g'(x)\tilde{\eta}_y$$

we find

$$\begin{aligned}
A'_{12} &= \left(A_{11}\left(-\frac{A_{12}}{A_{11}}\right)(-g') + A_{12}\left(\left(-\frac{A_{12}}{A_{11}}\right) - g'\right) + A_{22}\right)\tilde{\zeta}_y\tilde{\eta}_y \\
&= \left(\frac{A_{11}A_{22} - (A_{12})^2}{A_{11}}\right)\tilde{\zeta}_y\tilde{\eta}_y \\
&= 0 \quad ;
\end{aligned}$$

and so actually by eliminating A'_{11} we simultaneously eliminate A'_{12} (in the parabolic case). Thus (15.1) reduces to

$$A'_{22}\Phi_{\eta\eta} + B'_1\Phi_\zeta + B'_2\Phi_\eta + C'\Phi + F' = 0 \quad . \tag{15.32}$$

It is not hard to show that the coefficient A'_{22} does not vanish in this case, and so (15.32) can be rewritten in the form

$$\Phi_{\eta\eta} + B''_1\Phi_\zeta + B''_2\Phi_\eta + C''\Phi + F' = 0 \quad . \tag{15.33}$$

Equation (15.33) is referred to as the *standard form of a parabolic PDE.*

Elliptic Case. $(A_{12})^2 - A_{11}A_{22} < 0$

In this case, there is no choice of real coordinates that will allow the elimination of A'_{11} and A'_{22}. However, can arrange it so that $A'_{11} = A'_{22}$ and $A'_{12} = 0$. This is done as follows.

From (15.6) we have

$$\begin{aligned}
A'_{11} &= A_{11}\tilde{\zeta}_x\tilde{\zeta}_x + 2A_{12}\tilde{\zeta}_x\tilde{\zeta}_y + A_{22}\tilde{\zeta}_y\tilde{\zeta}_y \\
A'_{12} &= A_{11}\tilde{\zeta}_x\tilde{\eta}_x + A_{12}\left(\tilde{\zeta}_x\tilde{\eta}_y + \tilde{\zeta}_y\tilde{\eta}_x\right) + A_{22}\tilde{\zeta}_y\tilde{\eta}_y \\
A'_{22} &= A_{11}\tilde{\eta}_x\tilde{\eta}_x + 2A_{12}\tilde{\eta}_x\tilde{\eta}_y + A_{22}\tilde{\eta}_y\tilde{\eta}_y
\end{aligned} \tag{15.34}$$

Suppose we represent the level curves

$$\begin{aligned}
\tilde{\zeta}(x,y) &= const \\
\tilde{\eta}(x,y) &= const
\end{aligned} \tag{15.35}$$

as graphs of functions $f_1(x)$ and $f_2(x)$, respectively. Then we have

$$\begin{aligned}
\tilde{\zeta}_x &= -f'_1\tilde{\zeta}_y \\
\tilde{\eta}_x &= -f'_2\tilde{\eta}_y
\end{aligned} \tag{15.36}$$

Then (15.34) can be written as

$$\begin{aligned}
A'_{11} &= \left[A_{11}(f'_1)^2 - 2f'_1A_{12} + A_{22}\right]\tilde{\zeta}_y\tilde{\zeta}_y \\
A'_{12} &= \left[A_{11}f'_1f'_2 - A_{12}\left(f'_1 + f'_2\right) + A_{22}\right]\tilde{\zeta}_y\tilde{\eta}_y \\
A'_{22} &= \left[A_{11}(f'_2)^2 - 2f'_2A_{12} + A_{22}\right]\tilde{\eta}_y\tilde{\eta}_y
\end{aligned} \tag{15.37}$$

A'_{12} will thus vanish if

$$0 = A_{11}f'_1f'_2 - A_{12}\left(f'_1 + f'_2\right) + A_{22}$$

for f'_2 we get

$$f'_2 = \frac{A_{12}f'_1 - A_{22}}{A_{11}f'_1 - A_{12}} \quad . \tag{15.38}$$

The requirement that that $A'_{11} = A'_{22}$, then leads to

$$
\begin{aligned}
0 &= \left[A_{11}(f'_1)^2 - 2f'_1A_{12} + A_{22}\right]\tilde{\zeta}_y\tilde{\zeta}_y \\
&\qquad + \left[A_{11}(f'_2)^2 - 2f'_2A_{12} + A_{22}\right]\tilde{\eta}_y\tilde{\eta}_y \\
&= \left[A_{11}(f'_1)^2 - 2f'_1A_{12} + A_{22}\right]\tilde{\zeta}_y\tilde{\zeta}_y \\
&\quad + \left[A_{11}\left[\tfrac{A_{12}f'_1 - A_{22}}{A_{11}f'_1 - A_{12}}\right]^2 - 2\left[\tfrac{A_{12}f'_1 - A_{22}}{A_{11}f'_1 - A_{12}}\right]A_{12} + A_{22}\right]\tilde{\eta}_y\tilde{\eta}_y
\end{aligned} \tag{15.39}
$$

In order to solve (15.39) for f'_1 we need to know something about the relationship between $\tilde{\zeta}_y$ and $\tilde{y}_{2;y}$. It will be sufficient for our purposes to simply set

$$\tilde{\zeta}_y = \tilde{\eta}_y \quad .$$

This will have the effect of restricting the set of solutions of (15.39) - but afterall, we only need one solution. Equation (15.31) now becomes

$$
\begin{aligned}
0 &= A_{11}(f'_1)^2 - 2f'_1A_{12} + A_{22} - A_{11}\left[\tfrac{A_{12}f'_1 - A_{22}}{A_{11}f'_1 - A_{12}}\right]^2 \\
&\quad +2\left[\tfrac{A_{12}f'_1 - A_{22}}{A_{11}f'_1 - A_{12}}\right]A_{12} - A_{22}
\end{aligned} \tag{15.40}
$$

Solving (15.40) for f'_1 yields

$$f'_1 = \frac{A_{12}}{A_{11}} \pm \frac{\sqrt{A_{11}A_{22} - A_{12}^2}}{A_{11}} \tag{15.41}$$

or

$$f'_1 = \frac{A_{12} \pm \sqrt{(A_{12})^2 - A_{11}A_{22}}}{A_{11}}. \tag{15.42}$$

The latter solutions (15.42) can be disgarded - for the expression inside the radical is negative by hypothesis and so these solutions will not be real-valued. Plugging (15.41) into (15.38) yields

$$f'_2 = \frac{A_{12}}{A_{11}} \mp \frac{\sqrt{A_{11}A_{22} - A_{12}^2}}{A_{11}} \tag{15.43}$$

Note that in the equations (15.41) and (15.43), the expressions inside the radicals are positive in this case and so the differential equations (15.41) and (15.43) **are** solvable in terms of real-valued functions and hence are interpretable as functions whose graphs correspond to the level curves of coordinate functions $\tilde{y}_1$ and $\tilde{\eta}$ for which $A'_{12} = 0$ and $A'_{11} = A'_{11}$.

Summarizing, in the elliptic case, where $(A_{12})^2 - A_{11}A_{22} < 0$, we cannot find coordinates for which $A'_{11} = 0$ or $A'_{22} = 0$; however, we can find coordinates functions for which $A'_{12} = 0$ and $A'_{11} = A'_{22}$. Doing so, we obtain

$$A'_{11}\Phi_{\zeta\zeta} + A'_{11}\Phi_{\eta\eta} + B'_1\Phi_\zeta + B'_2\Phi_\eta + C'\Phi + F' = 0 \quad . \tag{15.44}$$

Since $A'_{11} \neq 0$, we can divide (15.44) by A'_{11}, thus obtaining

$$\Phi_{\zeta\zeta} + \Phi_{\eta\eta} + B''_1\Phi_\zeta + B''_2\Phi_\eta + C''\Phi + F'' = 0 \quad . \tag{15.45}$$

Equation (15.45) is the *standard form of an elliptic PDE.*

1. Summary: Classification of 2nd Order Linear PDEs

Hyperbolic Case: $(A_{12})^2 - A_{11}A_{22} > 0$

In this case, we found two distinct real-valued solutions $f'_{\pm}$ of the relation

$$A_{11}(f')^2 - 2f'A_{22} + A_{22} = 0 \quad . \tag{15.46}$$

One of these, say f'_{+}, once integrated could be used to define a coordinate function $\tilde{\zeta}$ for which the coefficient A'_{11} vanishes identically, and the other can be used to define a coordinate function $\tilde{\eta}$ for which A'_{22} vanishes identically. One can thus construct a coordinate system in which the original PDE (15.1) takes the form

$$2A'_{12}\Phi_{\zeta\eta} + B'_1\Phi_{\zeta} + B'_2\Phi_{\eta} + C\Phi + F = 0 \quad .$$

Observing that, in the hyperbolic case, the coefficient A'_{12} never vanishes, and so we rewrote (15.29) as

$$\Phi_{\zeta\eta} + B''_1\Phi_{\zeta} + B''_2\Phi_{\eta} + C''\Phi + F' = 0 \quad . \tag{15.47}$$

Equation (15.47) is referred to as the *standard form for a hyperbolic PDE.*

Parabolic Case. $(A_{12})^2 - A_{11}A_{22} = 0$

In this case, we could find only one real-valued solution f' of (15.46). Using this solution we could construct a coordinate function $\tilde{\zeta}$ for which A'_{11} vanished identically, but lacking a second independent solution we could not find a second coordinate function $\tilde{\eta}$ for which A'_{22} vanishes identically. However, we also showed that by choosing $\tilde{\zeta}$ so that A'_{11} vanished identically, we simultaneously forced A'_{12} to vanish identically. Thus (15.1) was reduced to

$$A'_{22}\Phi_{\eta\eta} + B'_1\Phi_{\zeta} + B'_2\Phi_{\eta} + C'\Phi + F' = 0 \quad .$$

Then observing that the coefficient A'_{22} could not vanish as well, we finally reexpressed (15.1) as

$$\Phi_{\eta\eta} + B''_1\Phi_{\zeta} + B''_2\Phi_{\eta} + C''\Phi + F' = 0 \quad . \tag{15.48}$$

Equation (15.48) is referred to as the *standard form for a parabolic PDE.*

Elliptic Case. $(A_{12})^2 - A_{11}A_{22} < 0$

In this case, there is no choice of real root f' of (15.46). Since we could not interprete the graph of a complex valued function as coinciding with a level surface of a coordinate function $\tilde{y}_i$, we concluded that in the elliptic case one can not find coordinates $\tilde{\zeta}$ and $\tilde{\eta}$ such that the either of the coefficients A'_{11} or A'_{22} vanish. However, we could arrange it so that $A'_{12} = 0$ and $A'_{11} = A'_{22} \neq 0$. In so doing we convert (15.1) to the form

$$A'_{11}\Phi_{\zeta\zeta} + A'_{11}\Phi_{\eta\eta} + B'_1\Phi_{\zeta} + B'_2\Phi_{\eta} + C'\Phi + F' = 0 \quad .$$

Since $A'_{11} \neq 0$, we could then divide by A'_{11}, to get

$$\Phi_{\zeta\zeta} + \Phi_{\eta\eta} + B''_1\Phi_{\zeta} + B''_2\Phi_{\eta} + C''\Phi + F'' = 0 \quad . \tag{15.49}$$

Equation (15.49) is the *standard form of an elliptic PDE.*

Of course, if this classification of PDE's is to really make any real sense, it should be independent of the choice of coordinates (that is, we should make sure that if we change coordinates we do not change the a PDE's type). Using the relations

$$A'_{ab} = \sum_{i,j=1}^{2} A_{ij}\tilde{y}_{a;x_i}\tilde{y}_{b;x_j}$$

one finds that

$$(A'_{12})^2 - A'_{11}A'_{22} = \left((A_{12})^2 - A_{11}A_{22}\right)\left(\tilde{\zeta}_x\tilde{\eta}_y - \tilde{\eta}_x\tilde{\zeta}_y\right)^2 . \tag{15.50}$$

The expression

$$\left(\tilde{\zeta}_x\tilde{\eta}_y - \tilde{\eta}_x\tilde{\zeta}_y\right)^2$$

is the square of the Jacobian of the coordinate transformation. Since the Jacobian is assumed to vanish nowhere, this expression is always positive. Thus, the left hand side of (15.50) is positive, negative or zero precisely when $(A_{12})^2 - A_{11}A_{22}$ is, respectively, positive, negative or zero. Thus, this classification of second order linear PDE's is independent of the choice of coordinates.

EXAMPLE 15.1. The wave equation

$$\frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} = F(x,t)$$

is an example of a hyperbolic PDE.

EXAMPLE 15.2. The heat equation

$$\frac{\partial\phi}{\partial t} - \lambda^2\frac{\partial^2\phi}{\partial x^2} = F(x,t)$$

is an example of an parabolic PDE.

EXAMPLE 15.3. Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = F(x,y)$$

is an example of an elliptic PDE.