LECTURE 14

Coordinate Transformations and the Method of Characteristics

Coordinate transformations enter the theory of PDE's in at least two different ways.

On the one hand it is sometimes useful to make a change in coordinates in order to exploit a particular symmetry of the system. For example, if one was attempting to determine the temperature distribution of a circular disk immersed in a heat bath, one would find the boundary conditions are most easily expressed in terms of polar coordinates.

Alternatively, as we have seen in our study of the wave equation, sometimes a change of variable (which afterall is just a coordinate transformation) simplifies a PDE in such a way that we can actually write down a general solution.

We will now explore what happens to a general second order linear PDE under a change of coordinates.

Suppose we start with a general second order linear PDE in n variables x_i:

(14.1) $$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\phi_{ij} + \sum_{i=1}^{n} B_i\phi_i + C\phi + F = 0 \quad .$$

where the coefficients A_{ij}, B_i, C, G are prescribed functions of the variables x_i and as usual

$$\begin{aligned} \phi_{ij} &= \frac{\partial^2\phi}{\partial x_i \partial x_j} \\ \phi_i &= \frac{\partial \phi}{\partial x_i} \quad . \end{aligned}$$

Let y_a, $a = 1, \ldots, n$ be another set of coordinates for $\mathbb{R}^n$, related to the original coordinates by

(14.2) $$\begin{aligned} y_a &= \tilde{y}_a\left(x_1, \ldots, x_n\right) \\ x_i &= \tilde{x}_i\left(y_1, \ldots, y_n\right) \end{aligned}$$

where $\tilde{y}_1, \ldots, \tilde{y}_n, \tilde{x}_1, \ldots, \tilde{x}_n$ are differentiable functions satisfying

(14.3) $$\begin{aligned} y_a &= \tilde{y}_a\left(\tilde{x}_1(y_1, \ldots, y_n), \ldots, \tilde{x}_n(y_1, \ldots, y_n)\right) \\ x_i &= \tilde{x}_i\left(\tilde{y}_1(x_1, \ldots, x_n), \ldots, \tilde{y}_n(x_1, \ldots, x_n)\right) \end{aligned}$$

and for which the Jacobian

(14.4) $$J\left[\frac{\partial y}{\partial x}\right] = \begin{vmatrix} \frac{\partial \tilde{y}_1}{\partial x_1} & \ldots & \frac{\partial \tilde{y}_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial \tilde{y}_1}{\partial x_n} & \ldots & \frac{\partial \tilde{y}_n}{\partial x_n} \end{vmatrix}$$

does not vanish.

Define $\Phi(y_1, \ldots, y_n)$ by

(14.5) $$\Phi(y) = \phi\left(\tilde{x}(y)\right).$$

In view of the relations (14.3), Φ satisfies

(14.6) $$\phi(x) = \Phi\left(\tilde{x}(y)\right) \quad .$$

(At this point I'm condensing my notation a bit; e.g., equation (14.6) should be interpreted as $\phi(x_1, \ldots, x_n) = \phi(\tilde{y}_1(x_1, \ldots, x_n), \ldots, \tilde{y}_n(x_1, \ldots, x_n)))$.

The chain rule for functions of several variables is

$$\frac{\partial \phi}{\partial x_i}(x) = \sum_{a=1}^{n} \frac{\partial \Phi}{\partial y_a}(\tilde{y}(x)) \frac{\partial \tilde{y}_a}{\partial x_i}(x) \quad . \tag{14.7}$$

Reiterating the chain rule we get

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) = \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{\partial^2 \Phi}{\partial y_a \partial y_b}(\tilde{y}(x)) \frac{\partial \tilde{y}_a}{\partial x_i} \frac{\partial \tilde{y}_b}{\partial x_j} + \sum_{a=1}^{n} \frac{\partial \Phi}{\partial y_a}(\tilde{y}(x)) \frac{\partial^2 \tilde{y}_a}{\partial x_i \partial x_j}. \tag{14.8}$$

We can condense our notation a little more by rewriting (14.7) and (14.8) as

$$\begin{array}{rcl} \phi_i & = & \sum_{a=1}^{n} \Phi_a \tilde{y}_{a;i} \\ \phi_{ij} & = & \sum_{a=1}^{n} \sum_{b=1}^{n} \Phi_{ab} \tilde{y}_{a;i} \tilde{y}_{b;j} + \sum_{a=1}^{n} \Phi_a \tilde{y}_{a;ij} \end{array} \tag{14.9}$$

If we now substitute the expressions (14.9) into the original PDE (14.1) we get

$$\begin{array}{rcl} 0 & = & \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \left(\sum_{a=1}^{n} \sum_{b=1}^{n} \Phi_{ab} \tilde{y}_{a;i} \tilde{y}_{b;j} + \sum_{a=1}^{n} \Phi_a \tilde{y}_{a;ij} \right) \\ & & \quad + \sum_{i=1}^{n} B_i \left(\sum_{a=1}^{n} \Phi_a \tilde{y}_{a;i} \right) + C\Phi + F \\ & = & \sum_{a=1}^{n} \sum_{b=1}^{n} A'_{ab} \Phi_{ab} + \sum_{a=1}^{n} B' \Phi_a + C\Phi + F \end{array} \tag{14.10}$$

where

$$\begin{array}{rcl} A'_{ab} & = & \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \tilde{y}_{a,i} \tilde{y}_{b,j} \\ B'_a & = & \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \tilde{y}_{a;ij} + \sum_{i=1}^{n} B_i \tilde{y}_{a;i} \end{array} \quad . \tag{14.11}$$

At first glance we seem to have only obfuscated matters: we have taken a general second order linear PDE, performed a coordinate transformation, and ended up with an equation of exactly the same form, but whose coefficients are considerably more complicated. However, the point to bear in mind is that the coordinate transformation $x_i \to y_a$ is completely arbitrary at this point. In fact, it may well happen that by a judicious change of coordinates the PDE will in fact simplify.

Example 14.1. To see how such a judicious choice of coordinates might be found, let us consider anew the case of the homogeneous wave equation:

$$\phi_{tt} - c^2 \phi_{xx} = 0 \quad . \tag{14.12}$$

Under a general coordinate transformation

$$\begin{array}{rcl} \zeta & = & \tilde{\zeta}(x,t) \\ \eta & = & \tilde{\eta}(x,t) \end{array} \tag{14.13}$$

the wave equation becomes

$$A' \phi_{\zeta\zeta} + B' \phi_{\zeta\eta} + C' \phi_{\eta\eta} + D' \phi_\zeta + E' \phi_\eta = 0 \tag{14.14}$$

where

$$\begin{array}{rcl} A' & = & \left(\tilde{\zeta}_t\right)^2 - c^2 \left(\tilde{\zeta}_x\right)^2 \\ B' & = & \tilde{\zeta}_t \tilde{\eta}_t + \tilde{\zeta}_x \tilde{\eta}_x \\ C' & = & \left(\tilde{\eta}_t\right)^2 - c^2 \left(\tilde{\eta}_x\right)^2 \\ D' & = & \tilde{\zeta}_{tt} - c^2 \tilde{\zeta}_{xx} \\ E' & = & \tilde{\eta}_{tt} - c^2 \tilde{\eta}_{xx} \end{array} \quad . \tag{14.15}$$

This PDE is certainly more complicated than (14.12); however, we are still free to choose $\tilde{\zeta}$ and $\tilde{\eta}$ as we please. Now note that if

$$\begin{aligned} \tilde{\zeta}_t &= \pm c\tilde{\zeta}_x \\ \tilde{\eta}_t &= \pm c\tilde{\eta}_x \end{aligned} \tag{14.16}$$

then both A' and C' will vanish. Suppose $\tilde{\zeta}_+(x,t)$ is a solution of

$$\tilde{\zeta}_{+,t} = c\tilde{\zeta}_{+,x} \tag{14.17}$$

and consider the level curve in the xt-plane defined by

$$\tilde{\zeta}_+(x,t) = \zeta_o \quad . \tag{14.18}$$

Let $x = f(t)$ be a local representation of this curve (as the graph of the function f of t). Then we have

$$\tilde{\zeta}_+\left(f(t), t\right) = \zeta_o \tag{14.19}$$

and so

$$\frac{\partial\tilde{\zeta}}{\partial x}\frac{df}{dt} + \frac{\partial\tilde{\zeta}}{\partial t} = 0 \quad . \tag{14.20}$$

Comparing (14.20) with (14.17) we see that

$$\begin{aligned} 0 &= \tilde{\zeta}_{+,x}\tfrac{df}{dt} + \tilde{\zeta}_{+,t} \\ &= \tilde{\zeta}_{+,x}\tfrac{df}{dt} + c\tilde{\zeta}_{+,x} \\ &= \left(\tfrac{df}{dt} + c\right)\tilde{\zeta}_{+,x} \end{aligned} \tag{14.21}$$

or

$$\frac{df}{dt} = -c \tag{14.22}$$

and so the level curve of $\tilde{\zeta}_+(x,t)$ must be equivalent to the graph of a function of the form

$$x = f(t) = -ct + constant \quad . \tag{14.23}$$

To make things as simple as possible, we can define $\tilde{\zeta}_+(x,t)$ to be precisely the constant on the right hand side of (14.23). Thus, we take

$$\tilde{\zeta}_+(x,t) = x + ct \quad . \tag{14.24}$$

Similarly, one can show that

$$\eta_-(x,t) = x - ct$$

is a solution of

$$(\tilde{\eta}_t) = -c\left(\tilde{\eta}_x\right) \quad .$$

Therefore, by setting

$$\begin{aligned} \tilde{\zeta}(x,t) &= x + ct \\ \eta(x,t) &= x - ct \end{aligned} \tag{14.25}$$

we can arrange it so that the coefficients A' and C' in (14.14) vanish. In fact, for this particular choice of $\tilde{\zeta}$ and $\tilde{\eta}$, the coefficients D' and E' also vanish. Hence under the coordinate transformation (14.24) the wave equation reduces to

$$(1 - c^2)\Phi_{\zeta\eta} = 0 \quad . \tag{14.26}$$

As we have seen, one can easily write down the general solution of (14.26) as

$$\Phi(\zeta,\eta) = \alpha(\zeta) + \beta(\eta) \quad , \tag{14.27}$$

α and β being arbitrary functions. Converting back to our original coordinates we find the general solution of (14.1) to be

$$\phi(x,t) = \alpha(x+ct) + \beta(x-ct) \quad . \tag{14.28}$$

Let me now summarize what we have accomplished. In the course of trying to discover a coordinate system in which the wave equation might be simplest, we uncovered two distinguished types of curves

$$\begin{aligned} \tilde{\zeta}(x,t) &= x + ct \\ \tilde{\eta}(x,t) &= x - ct. \end{aligned} \tag{14.29}$$

These curves are referred to as the *characteristic curves* of the wave equation and they define a coordinate system for which the wave equation takes its simplest form. Thus, associated with the wave equation, there is a preferred geometry that coming directly from the functional form of the PDE (14.12).

Homework Problem: 3.12.1