LECTURE 12

Special Solutions of Laplace's Equation

1. Separation of Variables with Respect to Cartesian Coordinates

Suppose

$$\phi(x,y) = X(x)Y(y) \tag{12.1}$$

is a solution of

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad . \tag{12.2}$$

Then we must have

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \quad . \tag{12.3}$$

Applying the usual separation of variables argument, we find

$$\begin{aligned} \tfrac{d^2X}{dx^2} + \lambda^2 X &= 0 \quad \Rightarrow \quad X(x) = A\sin(\lambda) + B\cos(\lambda) \\ \tfrac{d^2Y}{dy^2} - \lambda^2 Y &= 0 \quad \Rightarrow \quad Y(y) = Ce^{-\lambda y} + De^{\lambda y} \quad . \end{aligned} \tag{12.4}$$

when the separation constant K is a positive real number $\lambda^2 > 0$. The possibility that the $K = 0$ should not be excluded, however. For this special case we would have

$$\begin{aligned} \tfrac{d^2X}{dx^2} &= 0 \quad \Rightarrow \quad X(x) = ax + b \quad , \\ \tfrac{d^2Y}{dy^2} &= 0 \quad \Rightarrow \quad Y(y) = cy + d \quad . \end{aligned} \tag{12.5}$$

Nor should we exclude the case when $K = -\lambda^2 < 0$:

$$\begin{aligned} \tfrac{d^2X}{dx^2} - \lambda^2 X &= 0 \quad \Rightarrow \quad X(x) = Ae^{-\lambda x} + Be^{\lambda x} \\ \tfrac{d^2Y}{dy^2} + \lambda^2 Y &= 0 \quad \Rightarrow \quad Y(y) = C\sin(\lambda y) + D\cos(\lambda y) \quad . \end{aligned} \tag{12.6}$$

Thus separation of variables yields three 4-parameter families of solutions.

$$\begin{aligned} \phi_0(x,y) &= axy + bx + cy + d \\ \phi_\lambda(x,y) &= Ae^{\lambda y}\sin(\lambda x) + Be^{-\lambda y}\sin(\lambda x) + Ce^{\lambda y}\cos(\lambda x) + De^{-\lambda y}\cos(\lambda x) \\ \phi_{i\lambda}(x,y) &= Ae^{\lambda x}\sin(\lambda y) + Be^{-\lambda x}\sin(\lambda y) + Ce^{\lambda x}\cos(\lambda y) + De^{-\lambda x}\cos(\lambda y) \end{aligned} \tag{12.7}$$

2. Separation of Variables with Respect to Polar Coordinates

If we make a change of variables to polar coordinates

$$\begin{aligned} x &= r\cos(\theta) & \qquad r &= \sqrt{x^2+y^2} \\ y &= r\sin(\theta) & \qquad \theta &= \tan^{-1}\left(\tfrac{y}{x}\right) \end{aligned} \tag{12.8}$$

then

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial \theta}\right) \\ &= \cos^2\frac{\partial^2}{\partial r^2} - 2\frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2} \\ &\quad + \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{\sin(\theta)\cos(\theta)}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial y^2} &= \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right) \\ &= \sin^2(\theta)\frac{\partial^2}{\partial r^2} + \frac{2\cos(\theta)\sin(\theta)}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2} \\ &\quad - \frac{\sin(\theta)\cos(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} - \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta}\end{aligned}$$

and

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad . \tag{12.9}$$

Thus, in polar coordinates Laplace's equation takes the form

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \quad . \tag{12.10}$$

If we set

$$\phi(r,\theta) = R(r)\Theta(\theta) \quad . \tag{12.11}$$

and plug (12.11) into (12.10) and then divide by $R(r)\Theta(\theta)$ then we obtain

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{\Theta''}{r^2\Theta} = 0$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} \quad .$$

Applying the separation of variables argument we now look for solutions of

$$\begin{aligned} r^2R'' + rR' - \lambda^2 R &= 0 \\ \Theta'' + \lambda^2\Theta &= 0 \end{aligned} \tag{12.12}$$

If $\lambda^2 \neq 0$, then the second equation has as solutions

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\theta\lambda) \quad . \tag{12.13}$$

In order for such solutions to be continuous across the ray $\theta = 0$ (i.e. so that $\Theta(2\pi) = \Theta(0)$) we must demand that $\lambda = n \in \mathbb{N} = \{1, 2, 3, \dots\}$. For such λ the first equation in (12.12) is an Euler-type equation which has as solutions

$$R(r) = Ar^n + Br^{-n} \quad . \tag{12.14}$$

If $\lambda^2 = 0$, then (12.12) reduces to

$$\begin{aligned} rR'' + R' &= 0 \\ \Theta'' &= 0 \end{aligned} \tag{12.15}$$

The general solution of the second equation is obviously

$$\Theta(\theta) = a\theta + b \quad . \tag{12.16}$$

To solve the first, we set $W = R'$ so that we can reduce it to the following first order ODE

$$-\frac{1}{r} = \frac{W'}{W} = \frac{d}{dr}\left(\ln(W)\right) \tag{12.17}$$

Integrating both sides of this equation yields

$$-\ln|r| = \ln(W) + C' \tag{12.18}$$

or

$$W = \frac{C}{r} \tag{12.19}$$

Replacing the left hand side of (12.19) R' and then integrating both sides of yields

$$R(r) = C\ln|r| + D \tag{12.20}$$

We thus arrive at the following two families of solutions

$$\phi_n(r,\theta) = Ar^n\cos(n\theta) + Br^n\sin(n\theta) + Cr^{-n}\cos(n\theta) + Dr^{-n}\sin(n\theta) \tag{12.21}$$

$$\phi_0(r,\theta) = a\ln|r|\theta + b\ln|r| + c\theta + d \tag{12.22}$$

From the solution $\phi_0(r,\theta)$ with $0 = a = c = d$, we can infer the existence of a solution of the form

$$\phi(x,y) = \ln\left|(x-x_o)^2 + (y-y_o)^2\right|$$

It can be easily checked that this is the only solution (up to a multiplicative and/or additive constant) of Laplace's equation that depends only the distance from the point (x_o, y_o). It is called the *fundamental solution* of Laplace's equation.

3. Polynomial Solutions

We have already seen that

$$\phi(x,y) = axy + bx + cy + d$$

is a solution of Laplace's equation. Let us now look to see if there are other polynomial solutions. Suppose

$$\phi(x,y) = Ax^2 + By^2$$

then

$$0 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = A + B \quad \Rightarrow \quad B = -A \quad .$$

Thus,

$$\phi(x,y) = A\left(x^2 - y^2\right)$$

is a solution of Laplace's equation.

Similarly, if we set

$$\phi(x,y) = Ax^3 + Bx^2y + Cxy^2 + Dy^3$$

then

$$0 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 6Ax + 2By + 2Cx + 6Dy \quad \Rightarrow \quad \begin{array}{rcl} C & = & -3A \\ B & = & -3D \end{array}$$

and so

$$\phi(x,y) = A\left(x^3 - 2xy^2\right) + B\left(y^3 - 3x^2y\right)$$

will be a solution of Laplace's equation.

Now let $\phi(x,y)$ be an arbitrary homogeneous polynomial of degree n:

$$\phi(x,y) = \sum_{i=0}^{n} a_i x^{n-i} y^i \quad . \tag{12.23}$$

Then if this is to be a solution of Laplace's equation we must have

$$
\begin{aligned}
0 &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} && (12.24)\\
&= \sum_{i=0}^{n}(n-i)(n-i-1)a_i x^{n-i-2}y^i + \sum_{i=0}^{n} i\,(i-1)\,a_i x^{n-i}y^{i-2} && (12.25)\\
&= \sum_{i=0}^{n-2}(n-i)\,(n-i-1)\,a_i x^{n-i-2}y^i + \sum_{i=0}^{n-2}(i+2)\,(i+1)\,a_{i+2}x^{n-2-i}y^i && (12.26)\\
&= \sum_{i=0}^{n-2}\left((n-i)(n-i-1)a_i + (i+2)(i+1)a_{i+2}\right)x^{n-2-i}y^i && (12.27)
\end{aligned}
$$

and so (12.23) will be a solution of Laplace's equation if the coefficients a_i satisfy the following recursion relation

$$
a_{i+2} = \frac{(n-i)(n-i-1)}{(i+1)(i+2)}a_i \quad . \tag{12.28}
$$

Note that these recursion relations imply that for each $n \in \mathbb{N}$ there are precisely two linearly independent homogeneous polynomial solutions of Laplace's equation (since (12.28) tells that all the coefficients a_i are completely determined by a_0 and a_1).

4. Series Solutions

Consider the following PDE/BVP

$$
\begin{aligned}
\phi_{rr} + \tfrac{1}{r}\phi_r + \tfrac{1}{r^2}\phi_{\theta\theta} &= 0\\
\phi(R,\theta) &= f(\theta)
\end{aligned} \tag{12.29}
$$

which is just Laplace's equation in polar coordinates with Dirichlet boundary conditions imposed on the boundary of the circle $r = R$.

To construct a solution of (12.29) we shall first expand the solution $\phi(r,\theta)$ in a series of θ-dependent eigenfuctions:

$$
\phi(r,\theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{\infty}\left(a_n(r)\cos(n\theta) + b_n(r)\sin(n\theta)\right) \quad . \tag{12.30}
$$

We note that the trigonometric functions $\cos(n\theta)$ and $\sin(n\theta)$ are eigenfunctions of the Sturm-Liouville problem with differential equation

$$
\Theta'' + \lambda^2\Theta = 0
$$

(compare with (12.12) and boundary conditions

$$
\Theta(0) = \Theta(2\pi) \quad .
$$

Inserting (12.30) into (12.29) we obtain

$$
\begin{aligned}
0 \;=\; & a_0'' + \tfrac{1}{r}a_0' + \textstyle\sum_{n=1}^{\infty}\left(\left(a_n'' + \tfrac{1}{r}a_n'\right)\cos(n\theta) + \left(b_n'' + \tfrac{1}{r}b_n'\right)\sin(n\theta)\right)\\
& + \textstyle\sum_{n=1}^{\infty}\left(-n^2 a_n\cos(n\theta) - n^2 b_n\sin(n\theta)\right)
\end{aligned} \tag{12.31}
$$

Multiplying both sides by $\cos(n\theta)$ or $\sin(n\theta)$, integrating with respect to θ from 0 to 2π, and employing the identities

$$
\begin{aligned}
\int_0^{2\pi} \cos(n\theta)d\theta &= \begin{cases} 2\pi & , \quad n=0 \\ 0 & , \quad n \neq 0 \end{cases} \\
\int_0^{2\pi} \sin(n\theta)\cos(m\theta)d\theta &= 0 \quad , \quad \forall\, n,m \in \mathbb{Z} \\
\int_0^{2\pi} \cos(n\theta)\cos(m\theta)d\theta &= \pi\delta_{nm} \\
\int_0^{2\pi} \sin(n\theta)\sin(m\theta)d\theta &= \pi\delta_{nm}
\end{aligned}
\tag{12.32}
$$

we obtain

$$
\begin{aligned}
a_n'' + \tfrac{1}{r}a_n' - \tfrac{n^2}{r^2}a_n &= 0 \quad , \quad n = 0,1,2,3,\dots \\
b_n'' + \tfrac{1}{r}b_n' - \tfrac{n^2}{r^2}b_n &= 0 \quad , \quad n = 1,2,3,\dots
\end{aligned}
\tag{12.33}
$$

These are all Euler type ODEs which have as their general solution

$$
\begin{aligned}
R(r) &= Ar^n + Br^{-n} \quad , \quad n \neq 0 \\
R(r) &= A + B\ln|r| \quad , \quad n = 0
\end{aligned}
\tag{12.34}
$$

In order for our solution to be regular at the origin we must exclude solutions proportional to r^{-n} and $\ln|r|$. We therefore take

$$
\begin{aligned}
a_n(r) &= A_n r^n \quad , \quad n = 0,1,2,\cdots \\
b_n(r) &= B_n r^n \quad , \quad n = 1,2,3,\cdots
\end{aligned}
\tag{12.35}
$$

to be the appropriate solutions of (12.33). Hence, (12.25) becomes

$$
\phi(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right) \quad . \tag{12.36}
$$

To fix the constants A_n, B_n we now impose the boundary condition at $r = R$;

$$
f(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n R^n \cos(n\theta) + B_n R^n \sin(n\theta)\right) \quad . \tag{12.37}
$$

Multiplying both sides of (12.37) by $\cos(m\theta)$ or $\sin(m\theta)$ and integrating from 0 to 2π then yields

$$
\begin{aligned}
A_n &= \tfrac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \quad , \quad n = 0,1,2,3,\dots \\
B_n &= \tfrac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \quad , \quad n = 1,2,3,\dots
\end{aligned}
\tag{12.38}
$$

Homework: Problem 4.5.4