LECTURE 11

Laplace's Equation

1.

We now turn to the last basic example of a second order linear PDE. The PDE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{11.1}$$

is called Laplace's equation. Solutions of Laplace's equation are often called *harmonic functions*. The corresponding inhomogeneous PDE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f(x,y) \tag{11.2}$$

is called Poisson's equation.

These PDEs arise in a variety of mathematical and physical contexts. For example, both the imaginary and real parts of an analytic function on $\mathbb{C} = \mathbb{R}^2$ satisfy Laplace's equation. Poisson's equation arises as the equation for the electric potential $\phi(x,y)$ at the point $(x,y) \in \mathbb{R}^2$ in the presense of a charge distribution prescribed by a function $f(x,y)$; or as the equation for a temperature distribution $\phi(x,y)$ in a thermal equilibrium problem.

Associated with the two physical interpretations mentioned above are two special types of boundary conditions.

Dirichlet Boundary Conditions

In a thermal equilibrium problem is seems reasonable to expect the equilibrium temperature distribution of a planar object to be completely determined by the temperature distribution imposed on its boundary. The corresponding mathematical problem would be phrased as follows: Let R be a closed region of the plane and let ∂R denote the boundary of R, find a (the) function $\phi(x,y)$ such that

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0 \quad , \quad \forall\,(x,y) \in R \\ \phi(x,y) &= \phi_o(x,y) \quad , \quad \forall\,(x,y) \in \partial R \end{aligned} \tag{11.3}$$

Such a PDE/BVP is called a Dirichlet problem.

Neumann Boundary Conditions

Consider the following physical problem: A planar object is surrounded by material capable of transfering heat at a prescribed rate $f(x,y)$; find the equilibrium temperature inside the object.

The corresponding mathematical problem would be phrased as follows: Let R be a closed region of the plane and let ∂R denote the boundary of R, find a (the) function $\phi(x,y)$ such that

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0 \quad , \quad \forall\,(x,y) \in R \\ \frac{\partial\phi}{\partial n}(x,y) &= kf(x,y) \quad , \quad \forall\,(x,y) \in \partial R \quad . \end{aligned} \tag{11.4}$$

(Here $\frac{\partial \phi}{\partial n}$ is the derivative of ϕ in the direction normal to the boundary.) Such a PDE/BVP is called a Neumann problem.

It should be remarked that the physical problem is ill-posed unless the total rate at which heat flows into the object is zero (otherwise it will never be in equilibrium); mathematically this would correspond to the condition

$$0 = \int_{\partial R} \frac{\partial \phi}{\partial n} ds = \int_{\partial R} f(x,y) ds = 0 \quad . \tag{11.5}$$

Indeed, (11.4) will have no solutions unless (11.5) is satisfied.

2. Simple Properties of Harmonic Functions

Let $\phi(x,y)$ be a solution of Laplace's equation in a region $R \subset \mathbb{R}^2$. Recall that, we may interprete $\phi(x,y)$ physically as the equilibrium temperature at the point (x,y) of some planar object of shape R. This physical interpretation suggests several properties which solutions of Laplace's equation might be expected to possess.

First of all, we should not expect $\phi(x,y)$ to possess any local maximum within R. For if the temperature of the object were higher at one point (x_o, y_o), then there would be continual heat flow away from this point. But then the temperature at (x_o, y_o) would not be constant and so we would not have equilibrium.

Let us now consider the properties of harmonic functions in a more formal manner. Let R be a closed region in the plane and let $\phi(x,y)$ be a solution of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad , \quad \forall\ (x,y) \in R \quad . \tag{11.6}$$

Then

$$\begin{aligned} 0 &= \int_R \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] dA \\ &= \int_R \left[\frac{\partial}{\partial x} \phi_x + \frac{\partial}{\partial y} \phi_y \right] dA \\ &= \int_R \nabla \cdot (\nabla \phi)\, dA \\ &= \int_{\partial R} \nabla \phi \cdot dn \\ &= \int_{\partial R} \frac{\partial \phi}{\partial n} ds \end{aligned} \tag{11.7}$$

In the fourth step we have simply applied Gauss's Theorem in $\mathbb{R}^2$ (the Divergence Theorem in the Plane). Note how this result proves the consistency condition (11.5).

Now consider any point $P = (x_o, y_o)$ in R and let (r,θ) be a polar coordinate system with origin P. Define $\psi(r,\theta)$ to be the value of $\phi(x,y)$ at the corresponding point:

$$\psi(r,\theta) \equiv \phi\left(x_o + r\cos(\theta), y_o + r\sin(\theta)\right) \quad .$$

Construct a circle C_ρ of radius ρ about P and consider the average value $\Psi(\rho)$ of $\psi(r,\theta)$ on this circle:

$$\Psi(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\rho,\theta) d\theta \quad . \tag{11.8}$$

Differentiating with respect to ρ and using $\frac{\partial \psi}{\partial \rho} = \frac{\partial \psi}{\partial n}$, we obtain

$$\frac{\partial \Psi}{\partial \rho} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \psi}{\partial \rho} d\theta = \frac{1}{2\pi\rho} \int_C \frac{\partial \psi}{\partial n} ds \tag{11.9}$$

But since ψ is a harmonic within C, equation (11.7) implies that the right hand side of (11.9) must vanish. In other words, $\frac{\partial \Psi}{\partial \rho} = 0$, and so the value of $\Psi(\rho)$ is independent of ρ. Since $\Psi(0) = \psi(0,\theta) = \phi(x_o, y_o)$ we have just proved the following:

PROPOSITION 11.1. *The value of a harmonic function at a point P is equal to the average of its values on the circumference of any circle centered about P.*

The following corollary follows by averaging the constant function $\Psi(\rho)$ over the entire disk of radius ρ.

COROLLARY 11.2. *The value of a harmonic function at a point P is equal to the average of its values over the area of any circle centered about P.*

These two essentially equivalent results are refered two collectively as the *Mean Value Theorem* for harmonic functions. Here is another easy result.

COROLLARY 11.3. *Let R be a closed simply connected domain in $\mathbb{R}^2$ and let ϕ be a solution of*

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} &= 0 \quad , \quad \forall\,(x,y)\in R \quad , \\ \phi(x,y) &= 0 \quad , \quad \forall\,(x,y)\in \partial R \quad . \end{aligned} \tag{11.10}$$

Then $\phi(x,y)=0$ for all $(x,y)\in R$.

PROOF. We first recall some basic theorems concerning the extrema of continuous functions of several variables.

THEOREM 11.4. *If R is a closed bounded subset of $\mathbb{R}^n$ and f is a continuous function on R, then f attains a maximal value and a minimal value on R.*

COROLLARY 11.5. *If a continuous function f has no local extrema in the interior of a closed region $R\subset\mathbb{R}^n$, then its maximal and minimal values on R must occur on the boundary of R.*

In the case at hand, we know that ϕ has no local extrema within R and that the values of ϕ on the boundary of R are restricted to zero. Therefore, 0 is both the maximal and minimal value of ϕ on R. Hence, $\phi(x,y)=0$ for all $(x,y)\in R$. ☐

THEOREM 11.6. *Let R be a closed simply connected domain in $\mathbb{R}^2$. Then there is a unique solution to the following Dirichlet problem*

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} &= f(x,y) \quad , \quad \forall\,(x,y)\in R \quad , \\ \phi(x,y) &= h(x,y) \quad , \quad \forall\,(x,y)\in \partial R \quad . \end{aligned} \tag{11.11}$$

Proof. Suppose ϕ_1 and ϕ_2 are two solutions of (11.11). Then there difference $\phi_1-\phi_2$ satisfies

$$\frac{\partial^2}{\partial x^2}(\phi_1-\phi_2)+\frac{\partial^2}{\partial y^2}(\phi_1-\phi_2) = \frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_1}{\partial y^2}-\frac{\partial^2\phi_2}{\partial x^2}-\frac{\partial^2\phi_2}{\partial y^2} \tag{11.12}$$

$$= f(x,y)-f(x,y) \tag{11.13}$$

$$= 0 \tag{11.14}$$

and

$$(\phi_1-\phi_2)\big|_{\partial R} = \phi_1|_{\partial R}-\phi_2|_{\partial R} \tag{11.15}$$

$$= h(x,y)\,|_{\partial R}-h(x,y)\,|_{\partial R} \tag{11.16}$$

$$= 0 \tag{11.17}$$

In other words $\psi=\phi_1-\phi_2$ must be a solution of

$$\begin{aligned} \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} &= 0 \quad , \quad \forall\,(x,y)\in R \quad , \\ \psi(x,y) &= 0 \quad , \quad \forall\,(x,y)\in \partial R \quad . \end{aligned}$$

By the Corollary above then, $0=\psi(x,y)=\phi_1(x,y)-\phi_2(x,y)$ for all $(x,y)\in R$. Hence $\phi_1=\phi_2$. ☐