LECTURE 10

The Wave Equation and Laplace Transforms

Before discussing the application of Laplace transforms to the solution the Wave Equation, let me first state and prove a simple proposition about the inverse Laplace transforms of exponential functions.

PROPOSITION 10.1.

$$\begin{aligned} \mathcal{L}^{-1}\left[e^{-\alpha s}\right] &= \delta(t-\alpha) \\ \mathcal{L}^{-1}\left[\frac{1}{s}e^{-\alpha s}\right] &= \theta(t-\alpha) \equiv \begin{cases} 0 & if\, t-\alpha<0 \\ 1 & if\, t-\alpha>0 \end{cases} \end{aligned}$$

PROOF. The first formula "makes sense" since

$$\mathcal{L}\left[\delta(t-\alpha\right] \equiv \int_0^\infty \delta(t-\alpha)e^{-st}dt = e^{-s\alpha} \quad .$$

The second formula makes sense since

$$\begin{aligned} \mathcal{L}\left[\theta(t-\alpha)\right] &\equiv \int_0^\infty \theta(t-\alpha)e^{-st}dt \\ &= \int_\alpha^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-ts}\Big|_\alpha^\infty \\ &= \frac{1}{s}e^{-\alpha s} \quad . \end{aligned}$$

□

Now consider the PDE/BVP

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= 0 \\ \phi(x,0) &= h(x) \\ \phi_t(x,0) &= p(x) \quad . \end{aligned} \tag{10.1}$$

Setting

$$\Phi(x,s) = \int_0^\infty e^{-st}\phi(x,t)dt \tag{10.2}$$

and taking the Laplace transform of the wave equation, and applying the identity

$$\mathcal{L}\left[\phi_{tt}(x,t)\right] = s^2\Phi(x,s) - s\phi(x,0) - \phi_t(x,0) \quad ,$$

we find that the equations (10.1) are equivalent to

$$s^2\Phi(x,t) - sh(x) - p(x) - c^2\mathcal{L}\left[\phi_{xx}(x,t)\right] = 0 \quad . \tag{10.3}$$

Assuming that the (eventual) solution ϕ is sufficiently well-behaved to allow us to reverse the order of partial differentiation with respect to x and the Laplace transform with respect to t, we can rewrite (10.3) as

$$\Phi_{xx}(x,s) - \frac{s^2}{c^2}\Phi(x,s) = g(x,s) \tag{10.4}$$

where

$$g(x,s) = -\frac{s}{c^2}h(x) - \frac{1}{c^2}p(x) \quad .$$

Fixing s and regarding (10.4) as an ODE with respect to the variable x, we obtain

$$\begin{aligned} y_1(x) &= e^{-\frac{s}{c}x} \\ y_2(x) &= e^{\frac{s}{c}x} \end{aligned}$$

as solutions of the corresponding homogeneous problem. The Wronskian of y_1 and y_2 is

$$W[y_1,y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = \frac{2s}{c} \quad .$$

According to the Method of Variation of Parameters, the general solution of

$$\begin{aligned} \Phi(x,s) &= \left(A(s) - \int_0^x \frac{y_2(\zeta)g(\zeta,s)}{W[y_1,y_2](\zeta)}d\zeta\right) y_1(x) + \left(B(s) + \int_0^x \frac{y_1(\zeta)g(\zeta,s)}{W[y_1,y_2](\zeta)}d\zeta\right) y_2(x) \\ &= \left(A(s) - \frac{c}{2s}\int_0^x e^{\frac{s\zeta}{c}} g(\zeta,s)d\zeta\right) e^{-\frac{sx}{c}} + \left(B(s) + \frac{c}{2s}\int_0^x e^{-\frac{s\zeta}{c}} g(x,\zeta)d\zeta\right) e^{\frac{sx}{c}} \quad . \end{aligned} \tag{10.5}$$

In order to ensure good (bounded) behavior of $\Phi(x,s)$ as $x \to \pm\infty$, we must take

$$\begin{aligned} A(s) &= \frac{c}{2s}\int_0^{-\infty} e^{\frac{s(\zeta-x)}{c}} g(\zeta,s)d\zeta \\ B(s) &= -\frac{c}{2s}\int_0^{+\infty} e^{\frac{s(x-\zeta)}{c}} g(\zeta,s)d\zeta \end{aligned} \tag{10.6}$$

so (10.5) becomes

$$\begin{aligned} \Phi(x,s) &= -\frac{c}{2s}\left[\int_{-\infty}^x e^{\frac{s(\zeta-x)}{c}} g(\zeta,s)d\zeta - \int_x^\infty e^{\frac{s(x-\zeta)}{c}} g(\zeta,s)d\zeta\right] \\ &= -\frac{c}{2s}\int_{-\infty}^{+\infty} e^{-\frac{s}{c}|x-\zeta|} g(\zeta,s)d\zeta \\ &= -\frac{c}{2s}\int_{-\infty}^{+\infty} e^{-\frac{s}{c}|x-\zeta|}\left[-\frac{s}{c^2}h(\zeta) - \frac{1}{c^2}p(\zeta)\right]d\zeta \\ &= \frac{1}{2c}\int_{-\infty}^{+\infty} e^{-\frac{s}{c}|x-\zeta|}\left[h(\zeta) + \frac{1}{s}p(\zeta)\right]d\zeta \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \phi(x,t) &= \mathcal{L}^{-1}\left[\frac{1}{2c}\int_{-\infty}^{+\infty} e^{-\frac{s}{c}|x-\zeta|}\left[h(\zeta) + \frac{1}{s}p(\zeta)\right]d\zeta\right] \\ &= \frac{1}{2c}\int_{-\infty}^{+\infty} h(\zeta)\mathcal{L}^{-1}\left[e^{-\frac{s}{c}|x-\zeta|}\right]d\zeta + \frac{1}{2c}\int_{-\infty}^{+\infty} p(\zeta)\mathcal{L}^{-1}\left[\frac{1}{s}e^{-\frac{s}{c}|x-\zeta|}\right]d\zeta \quad . \end{aligned} \tag{10.7}$$

Now as we have shown above

$$\begin{aligned} \mathcal{L}^{-1}\left[e^{-\alpha s}\right] &= \delta(t-\alpha) \\ \mathcal{L}^{-1}\left[\frac{1}{s}e^{-\alpha s}\right] &= \theta(t-\alpha) \equiv \begin{cases} 0 & \text{if } t-\alpha < 0 \\ 1 & \text{if } t-\alpha > 0 \end{cases} \quad . \end{aligned}$$

So taking the inverse Laplace transform of (10.7) (and assuming this operation commutes with integration over ζ), we get

$$\begin{aligned} \phi(x,t) &= \frac{1}{2c}\int_{-\infty}^{\infty} h(\zeta)\delta\left(t - \frac{1}{c}|x-\zeta|\right)d\zeta + \frac{1}{2c}\int_{-\infty}^{+\infty} p(\zeta)\theta\left(t - \frac{1}{c}|x-\zeta|\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} h\left(\frac{\zeta'}{c}\right)\delta\left(t - \left|\frac{x}{c} - \zeta'\right|\right)d\zeta' + \frac{1}{2c}\int_{-\infty}^{+\infty} p(\zeta)\,\theta\left(t - \frac{1}{c}|x-\zeta|\right)d\zeta \end{aligned}$$

where in the last step, we have simply made a change of variable $\zeta \to \zeta' = c\zeta$. Only the points ζ' where

$$t - \left|\frac{x}{c} - \zeta'\right| = 0 \qquad \Rightarrow \qquad \zeta' = \frac{x}{c} \pm ct$$

contribute to the first integral. And only the points ζ where

$$t - \frac{1}{c}\left|x - \zeta\right| > 0 \qquad \Rightarrow \qquad x - ct < \zeta < x + ct$$

contribute to the second interval. Thus,

$$\begin{aligned} \phi(x,t) &= \frac{1}{2}\left[h\left[\frac{\frac{x}{c} + ct}{c}\right] + h\left[\frac{\frac{x}{c} - t}{c}\right]\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} p\left(\zeta\right) d\zeta \\ &= \frac{1}{2}\left[h\left[x + ct\right] + h\left[x - ct\right]\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} p\left(\zeta\right) d\zeta \qquad . \end{aligned}$$

$\square$