LECTURE 9

Series Solutions of The Wave Equation

1.

Suppose we're given the following PDE/BVP (partial differential equation/boundary value problem);

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= f(x,t) \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= h(x) \\ \phi_t(x,0) &= p(x) \end{aligned} \tag{9.1}$$

corresponding to a string of length L, fixed at both ends, driven by a varying force $f(x,t)$, with a given initial shape $h(x)$ and a given initial transverse velocity $p(x)$.

We intend to solve this problem by means of an expansion of the form

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\beta_n(x) \quad , \tag{9.2}$$

where $\beta_n(x)$ some suitably chosen complete set of functions for the interval $(0,L)$. The criteria by which we *suitably choose* the functions $\beta_n(x)$ is the same as in Chapter 2; we choose the $\beta_n(x)$ to be the eigenfunctions of the Sturm-Liouville problem coming from separation of variables (for the homogeneous problem) and the boundary conditions at time $t = 0$.

First, let's separate variables. Let

$$\phi(x,t) = F(x)G(t) \quad . \tag{9.3}$$

Inserting this expression into

$$\phi_{tt} - c^2\phi_{xx} = 0$$

we get

$$F''(t)G(x) - c^2F(t)G''(x) = 0$$

or

$$\frac{F''(t)}{F(t)} = c^2\frac{G''(x)}{G(x)} \quad .$$

Thus, the function $G(x)$ should satisfy

$$\Lambda = c^2\frac{G''(x)}{G(x)}$$

or

$$G''(x) - \frac{\Lambda}{c^2}G(x) = 0 \quad . \tag{9.4}$$

In a Sturm-Louiville problem

$$\frac{d}{dx}\left(p(x)y\right) + \left(q(x) + \lambda(x)r(x)\right)$$

the functions $p(x)$ and $r(x)$ and the eigenvalues are required to be positive. Therefore, we require

$$-\frac{\Lambda}{c^2} = \lambda^2$$

and rewrite (9.4) as

$$y'' + \lambda^2 y = 0 \quad . \tag{9.5}$$

The eigenvalues and eigenfunctions corresponding to the boundary conditions

$$\begin{aligned} y(0) &= 0 \\ y(L) &= 0 \end{aligned} \tag{9.6}$$

(the analogs of the boundary conditions at $t = 0$ in (9.1)), are

$$\lambda_n = \frac{n\pi}{L} \tag{9.7}$$

and

$$\beta_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad . \tag{9.8}$$

We thus set

$$\phi(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right) \quad . \tag{9.9}$$

Let

$$\begin{aligned} f_n(t) &= \tfrac{2}{L}\int_0^L f(x,t)\sin\left(\tfrac{n\pi x}{L}\right) dx \\ h_n &= \tfrac{2}{L}\int_0^L h(x)\sin\left(\tfrac{n\pi x}{L}\right) dx \\ p_n &= \tfrac{2}{L}\int_0^L p(x,t)\sin\left(\tfrac{n\pi x}{L}\right) dx \end{aligned} \tag{9.10}$$

so that

$$\begin{aligned} f(x,t) &= \textstyle\sum_{n=1}^{\infty} f_n(t)\sin\left(\tfrac{n\pi x}{L}\right) \\ h(x) &= \textstyle\sum_{n=1}^{\infty} h_n \sin\left(\tfrac{n\pi x}{L}\right) \\ p(x) &= \textstyle\sum_{n=1}^{\infty} p_n \sin\left(\tfrac{n\pi x}{L}\right) \end{aligned} \tag{9.11}$$

Plugging the expressions (9.11) into (9.1), and matching the coefficients of $\sin\left(\frac{n\pi x}{L}\right)$ on all sides, we get

$$\begin{aligned} \alpha_n''(t) + \tfrac{c^2 n^2 \pi^2}{L^2}\alpha(t) &= f_n(t) \\ \alpha_n(0) &= h_n \\ \alpha_n'(0) &= p_n \quad . \end{aligned} \tag{9.12}$$

The homogeneous equation corresponding to the ODE in (9.12) is

$$y'' + \frac{c^2 n^2 \pi^2}{L^2} y = 0 \quad . \tag{9.13}$$

The functions

$$\begin{aligned} y_1(t) &= \cos\left(\tfrac{n\pi ct}{L}\right) \\ y_2(t) &= \sin\left(\tfrac{n\pi ct}{L}\right) \end{aligned} \tag{9.14}$$

are two linearly independent solutions to (9.13) and so the general solution to the ODE in (9.12) is

$$\begin{aligned} \alpha_n(t) &= y_1(t)\left[A_n - \int_0^t \tfrac{y_2(\zeta) f_n(\zeta)}{W[y_1,y_2](\zeta)} d\zeta\right] + y_2(t)\left[B_n + \int_0^t \tfrac{y_1(\zeta) f_n(\zeta)}{W[y_1,y_2](\zeta)} d\zeta\right] \\ &= \cos\left(\tfrac{n\pi ct}{L}\right)\left[A_n + \int_0^t \frac{\sin\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \\ &\quad + \sin\left(\tfrac{n\pi ct}{L}\right)\left[B_n - \int_0^t \frac{\cos\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \quad . \end{aligned}$$

In order to satisfy the initial conditions in (9.12) we must have

$$A_n = h_n \tag{9.15}$$

$$B_n = \frac{L}{n\pi c} p_n \tag{9.16}$$

Thus,

$$\begin{aligned} \alpha_n(t) &= \cos\left(\tfrac{n\pi ct}{L}\right)\left[h_n + \int_0^t \frac{\sin\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \\ &\quad + \sin\left(\tfrac{n\pi ct}{L}\right)\left[\tfrac{L}{\pi nc} p_n - \int_0^t \frac{\cos\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \end{aligned} \tag{9.17}$$

The solution to

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= f(x,t) \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= h(x) \\ \phi_t(x,0) &= p(x) \end{aligned} \tag{9.18}$$

is thus given by

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\beta_n(x) \quad , \tag{9.19}$$

where

$$\beta_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad . \tag{9.20}$$

and the coefficients $\alpha_n(t)$ are determined by

$$\begin{aligned} \alpha_n(t) &= \cos\left(\tfrac{n\pi ct}{L}\right)\left[h_n + \int_0^t \frac{\sin\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \\ &\quad + \sin\left(\tfrac{n\pi ct}{L}\right)\left[\tfrac{L}{\pi nc} p_n - \int_0^t \frac{\cos\left(\frac{n\pi c\zeta}{L}\right) f_n(\zeta)}{\frac{2n\pi c}{L}} d\zeta\right] \end{aligned} \tag{9.21}$$

and

$$f_n(t) = \frac{2}{L}\int_0^L f(x,t)\sin\left(\frac{n\pi x}{L}\right) dx \tag{9.22}$$

$$h_n = \frac{2}{L}\int_0^L h(x)\sin\left(\frac{n\pi x}{L}\right) dx \tag{9.23}$$

$$p_n = \frac{2}{L}\int_0^L p(x)\sin\left(\frac{n\pi x}{L}\right) dx \quad . \tag{9.24}$$