LECTURE 8

The Wave Equation

1.

A wave equation (in $1+1$ dimensions) is a partial differential equation of the form

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = f(x,t) \quad . \tag{8.1}$$

Such equations crop up in a variety of physical contexts; vibrating strings, electrical circuits, electromagnetism, and in general whereever some sort of oscillitory motion takes place. The function $f(x,t)$ is referred to as the driving term. Typically, it represents some external force function applied to an oscillitory system.

To start we shall look for solutions of the following boundary value problem:

$$\begin{aligned} \phi_{tt} - c^2 \phi_{xx} &= 0 && (8.2)\\ \phi(x,0) &= h(x) && (8.3)\\ \phi_t(x,0) &= p(x) \quad . && (8.4) \end{aligned}$$

The equation

$$\phi_{tt} - c^2 \phi_{xx} = 0 \tag{8.5}$$

is very exceptional for a PDE; because it is quite simple to write down its general solution. Set

$$\begin{matrix} \zeta &=& x + ct & \qquad & t &=& \frac{\zeta+\eta}{2} \\ \eta &=& x - ct & \qquad & x &=& \frac{\zeta-\eta}{2c} \end{matrix} \tag{8.6}$$

and write

$$\phi(x,t) = \Phi(\zeta,\eta) \quad .$$

Then the chain rule for partial differentiation yields

$$\begin{aligned} \phi_t &= c\Phi_\zeta - c\Phi_\eta && (8.7)\\ \phi_x &= \Phi_\zeta + \Phi_\eta && (8.8) \end{aligned}$$

and

$$\begin{aligned} \phi_{tt} &= c^2\Phi_{\zeta\zeta} - c^2\Phi_{\eta\zeta} - c^2\Phi_{\zeta\eta} + c^2\Phi_{\eta\eta} && (8.9)\\ \phi_{xx} &= \Phi_{\zeta\zeta} + \Phi_{\zeta\eta} + \Phi_{\eta\zeta} + \Phi_{\eta\eta} \quad . && (8.10) \end{aligned}$$

Thus,

$$\phi_{tt} - c^2\phi_{xx} = -4c^2\Phi_{\zeta\eta} \quad .$$

But the general solution of

$$\Phi_{\zeta\eta} = 0 \tag{8.11}$$

is

$$\Phi(\zeta,\eta) = \alpha(\zeta) + \beta(\eta) \quad .$$

Thus, the general solution of (8.5) is

(8.12) $$\phi(x,t) = \alpha\,(x+ct) + \beta\,(x-ct) \quad .$$

Thus, the general solution of the wave equation can be represented as the sum of an arbitrary function of $x+ct$ and an arbitrary function of $x-ct$.

In order to satisfy the boundary conditions in (8.3)-(8.4) then, all we need to do is find functions $\alpha(x+ct)$ and $\beta(x-ct)$ satisfying

(8.13) $$\alpha(x) + \beta(x) = f(x)$$

(8.14) $$c\alpha'(x) - c\beta'(x) = p(x) \quad .$$

To solve these equations, let

$$P(x) = \int_0^x p(\zeta)d\zeta \quad .$$

Equations (8.13)-(8.14) are then equivalent to

(8.15) $$\alpha(x) + \beta(x) = f(x)$$

(8.16) $$c\alpha(x) - c\beta(x) = P(x) + K \quad .$$

Solving the first equation for $\alpha(x)$ we get

(8.17) $$\alpha(x) = f(x) - \beta(x) \quad .$$

Inserting this expression for $\alpha(x)$ into the second equation yields

$$c\,(f(x) - \beta(x)) - c\beta(x) = P(x) + K$$

or

(8.18) $$\beta(x) = \frac{1}{2}f(x) - \frac{1}{2c}P(x) + K' \quad .$$

Inserting (8.18) into (8.17) yields

(8.19) $$\alpha(x) = \frac{1}{2}f(x) + \frac{1}{2c}P(X) - K' \quad .$$

Finally, we insert (8.18) and (8.19) into (8.12) to obtain

(8.20) $$\phi(x,t) = \frac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right]$$

(8.21) $$+\frac{1}{2c}\left[\int_0^{x+ct} p(\zeta)d\zeta - \int_0^{x-ct} p(\zeta)d\zeta\right]$$

or

(8.22) $$\boxed{\phi(x,t) = \tfrac{1}{2}\left[f\,(x+ct) + f\,(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} p(\zeta)\,d\zeta}$$

Interpretation of Solutions

Case 1. $\phi(x,0) = f(x)$, $\phi_t(x,0) = 0$.

In this case, we have

$$\phi(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) \quad .$$

If we think of $\phi(x,t)$ as representing the vertical displacement of an infinite horizontal string at the point x at time t, then the function $f(x)$ corresponds to an initial displacement; e.g., a plucking of the string at $t=0$. The disturbance then propagates along the string in both directions maintaining the same shape as the initial displacement (once the two components of the disturbance separate).

Case 2. $\phi(x,t) = 0$, $\phi_t(x,0) = p(x)$.

In this case, we have

$$\phi(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} p(\tau)d\tau \quad . \tag{8.23}$$

Thus, the displacement of the string at point x at time t is given by the integral of $p(x)$ between the points $x-ct$ and $x+ct$. Note how the displacement $\phi(x,t)$ depends only on points that lie within the "light cone" centered at (x,t).

2. Uniqueness of Solutions of Wave Equation with Cauchy Boundary Conditions

Consider the homogeneous wave equation representing a string of length L with fixed endpoints and whose intial transverse displacement at the point x is given by h(x) and whose initial transverse velocity at the point x is given by $p(x)$. The PDE/BVP corresponding to this system is

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= 0 \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= h(x) \\ \phi_t(x,0) &= p(x) \end{aligned} \tag{8.24}$$

Let $\psi(x,t)$ be any solution of (8.24) and set

$$I_\psi(t) = \frac{1}{2}\int_0^L \left[\frac{1}{c^2}\left(\psi_t(x,t)\right)^2 + \left(\psi_x(x,t)\right)^2\right] dx \quad .$$

We then have

$$\begin{aligned} \frac{d}{dt}I_\psi(t) &= \int_0^L \left[\tfrac{1}{c^2}\psi_t(x,t)\psi_{tt}(x,t) + \psi_x(x,t)\psi_{xt}(x,t)\right] dx \\ &= \int_0^L \left[\tfrac{1}{c^2}\psi_t(x,t)\left(c^2\psi_{xx}(x,t)\right) + \psi_x(x,t)\psi_{xt}(x,t)\right] dx \\ &= \int_0^L \psi_t(x,t)\tfrac{\partial}{\partial x}\psi_x(x,t)dx + \int_0^L \psi_x(x,t)\psi_{xt}(x,t)dx \\ &= \psi_t(x,t)\psi_x(x,t)\big|_0^L - \int_0^L \psi_{tx}(x,t)\psi_x(x,t)dx + \int_0^L \psi_x(x,t)\psi_{xt}(x,t)dx \\ &= 0 \end{aligned} \tag{8.25}$$

(To reach the fourth line, we integrated the first integral on the third line by parts. The first term on the fourth line vanishes since the boundary conditions $\psi(0,t) = 0$, $\psi(L,t) = 0$ imply

$$\psi_t(0,t) = 0 = \psi_t(L,t) \quad , \quad \forall\, t \quad .$$

The two remaining terms cancel one another.)

Now consider the following PDE/BVP

$$\begin{aligned} \phi_{tt} - c^2\phi_{xx} &= f(x,t) \\ \phi(0,t) &= \alpha(t) \\ \phi(L,t) &= \beta(t) \\ \phi(x,0) &= H(x) \\ \phi_t(x,0) &= P(x) \end{aligned} \tag{8.26}$$

and suppose that $\phi_1(x,t)$ and $\phi_2(x,t)$ are two solutions to (8.26). Then

$$\Phi(x,t) = \phi_1(x,t) - \phi_2(x,t)$$

satisfies

$$\phi_{tt} - c^2\phi_{xx} = 0 \tag{8.27}$$

$$\phi(0,t) = 0 \tag{8.28}$$

$$\phi(L,t) = 0 \tag{8.29}$$

$$\phi(x,0) = 0 \tag{8.30}$$

$$\phi_t(x,0) = 0 \tag{8.31}$$

Therefore, in light of (8.25). we must have

$$0 = \frac{d}{dt} I_\Phi(t) = \frac{d}{dt}\left[\frac{1}{2}\int_0^L \left[\frac{1}{c^2}\left(\Phi_t(x,t)\right)^2 + \left(\Phi_x(x,t)\right)^2\right] dx\right] \quad .$$

It follows that

$$I_\Phi(t) = const$$

In fact, the initial conditions $\Phi_t(x,0) = 0$, $\Phi(x,0) = 0$, imply that

$$\Phi_t(x,0) = \Phi_x(x,0) = 0 \qquad \Rightarrow \qquad I_\Phi(0) = 0 \quad .$$

Thus, we have

$$0 = \frac{1}{2}\int_0^L \left[\frac{1}{c^2}\left(\Phi_t(x,t\right)^2 + \left(\Phi_x(x,t\right)^2\right] dx \quad .$$

Note the integrand is the sum of two squares. Therefore, it can only vanish if

$$\Phi_t(x,t) = 0 \tag{8.32}$$

$$\Phi_x(x,t) = 0 \tag{8.33}$$

for all x and t. But this then implies

$$\Phi(x,t) = \phi_1(x,t) - \phi_2(x,t) = constant \quad .$$

But since $\phi_1 = \phi_2$ on the boundary, we must have this *constant* equal to zero. Hence,

$$\phi_1(x,t) = \phi_2(x,t)$$

and so the solution to (8.26) if it exists is unique.

□

Homework: 3.4.3, 3.4.4