

Distributions

1. Definition and Examples of Distributions

Let $C_o^\infty(\mathbb{R}^n)$ denote the space of all C^∞ complex-valued functions on \mathbb{R}^n with compact support. We regard $C_o(\mathbb{R}^n)$ as an infinite dimensional vector space over \mathbb{C} .

The space $\mathcal{D}(\mathbb{R}^n)$ of **distributions** on \mathbb{R}^n is the space of all linear functionals on $C_o(\mathbb{R}^n)$; that is to say, an element $T \in \mathcal{D}(\mathbb{R}^n)$ is a (continuous) map from $C_o(\mathbb{R}^n)$ to \mathbb{C} such that

$$(7.1) \quad T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \quad , \quad \forall \alpha, \beta \in \mathbb{C} \quad , \quad \forall f, g \in C_o(\mathbb{R}^n) \quad .$$

I should admit at this point that I'm purposely avoiding the definition of the appropriate topology on $C_o(\mathbb{R}^n)$; nevertheless a standard (albeit technical) definition does exist.

EXAMPLE 7.1. The prototypical example of a distribution is as follows.

Let g be any element of $C_o^\infty(\mathbb{R}^n)$ then we can associate with g the linear map $T_g : C_o^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ defined by

$$(7.2) \quad T_g(f) = \int_{\mathbb{R}^n} f(x)g(x) d^n x \quad .$$

The integral on the right hand side is guaranteed to converge since the integrand is smooth and has only compact support. The linearity of T_g follows from the corresponding property of convergent integrals:

$$\int_{\mathbb{R}^n} g(x) (\alpha f_1(x) + \beta f_2(x)) d^n x = \alpha \int_{\mathbb{R}^n} g(x)f_1(x) d^n x + \beta \int_{\mathbb{R}^n} g(x)f_2(x) d^n x \quad .$$

In this way every element of $C_o^\infty(\mathbb{R}^n)$ can be regarded as an element of $\mathcal{D}(\mathbb{R}^n)$.

But $\mathcal{D}(\mathbb{R}^n)$ includes much more. If g is any piecewise continuous function (actually the analog of such objects in the multivariable case) then the right hand side of (7.2) still makes sense (all we need for convergence is for f to have compact support and this is guaranteed by the fact that we pick f from $C_o^\infty(\mathbb{R}^n)$).

EXAMPLE 7.2. Let $x_o \in \mathbb{R}^n$. It is easy to see that the map $T_{x_o} : C_o^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ defined by

$$(7.3) \quad T_{x_o}(f) = f(x_o)$$

is a linear functional on $C_o^\infty(\mathbb{R}^n)$.

Similarly, let x_o be any point of \mathbb{R}^n and let $i \in \{1, 2, 3, \dots, n\}$. One easily verifies that the map $T_{x_o, i} : C_o^\infty(\mathbb{R}^n) \rightarrow \mathbb{C}$ defined by

$$(7.4) \quad T_{x_o, i}(f) = \left. \frac{\partial f}{\partial x^i} \right|_{x_o}$$

is also a (continuous) linear functional on $C_o^\infty(\mathbb{R}^n)$ and hence a distribution.

2. The Delta Function

Consider the following 1-parameter family of functions $\delta_a : \mathbb{R} \rightarrow \mathbb{R}$.

$$(7.5) \quad \delta_a(x) = \begin{cases} \frac{1}{2a} & ; |x| < a \\ 0 & ; |x| \geq a \end{cases} .$$

So long as $a \neq 0$, this function is well defined and, in fact, piecewise continuous as a function of x for all $x \in \mathbb{R}$.

Now let $f(x)$ be a continuous function on the real line and consider the integral

$$(7.6) \quad \int_{-\infty}^{+\infty} f(x)\delta_a(x)dx = \int_{-a}^a \frac{f(x)}{2a}dx .$$

This again is well defined for all $a \neq 0$, so we can ask if the limit

$$\lim_{a \rightarrow 0} \int_{-\infty}^{+\infty} f(x)\delta_a(x)dx = \lim_{a \rightarrow 0} \int_{-a}^{+a} \frac{f(x)}{2a}dx$$

exists. By the Mean Value Theorem, for any finite a there exists an $x_1 \in (-a, a)$ such that

$$f(x_1) = \frac{1}{2a} \int_{-a}^a f(x)dx .$$

When one takes the limit $a \rightarrow 0$, $x_1 \rightarrow 0$, and so

$$(7.7) \quad \lim_{a \rightarrow 0} \int_{-\infty}^{+\infty} f(x)\delta_a(x)dx = \lim_{a \rightarrow 0} \int_{-a}^{+a} \frac{f(x)}{2a}dx = f(0) .$$

Writing

$$(7.8) \quad \delta(x) = \lim_{a \rightarrow 0} \delta_a(x) ,$$

we then have

$$(7.9) \quad \int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0) ,$$

or, more generally,

$$(7.10) \quad \int_{-\infty}^{+\infty} f(x)\delta(x - x_o)dx = f(x_o) .$$

(We simply made a change of variables $x \rightarrow x - x_o$ in passing from (7.9) to (7.10).) The “function” $\delta(x - x_o)$ so defined is called the Dirac delta function. Comparing (7.10) with (7.3) we see that the distribution defined by integrating $\delta(x - x_o)$ against a function coincides with the distribution T_{x_o} on \mathbb{R} .

However, recalling the original definition (1) of $\delta_a(x)$, it is clear that the definition (7.7)

$$(7.11) \quad \lim_{a \rightarrow 0} \delta_a(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

does not really lead to any legitimate function. Of course, the limit (7.7) still makes sense. And so a more accurate definition of the delta function might be as follows: Let $T_{\delta_a(x-x_o)}$ be the distribution defined by

$$(7.12) \quad T_{\delta_a(x-x_o)}(f) = \int_{\mathbb{R}^n} f(x)\delta_a(x - x_o) d^n x$$

then the Dirac delta distribution $T_{\delta(x-x_o)}$ is defined as

$$T_{\delta(x-x_o)} = \lim_{a \rightarrow 0} T_{\delta_a(x-x_o)} .$$

Nevertheless, it is most common to regard the distributions, like the Dirac delta *function*, as some kind of *generalized function* (in fact, in the early literature, that was precisely what distributions were called), and

to represent their action on functions via (pseudo-) integral notation (e.g., the right hand side of (7.12). rather than functional notation (resp., the left hand side of (7.12)).

I should remark that like C^∞ functions, distributions can also be differentiated; more precisely, if $T \in \mathcal{D}'(\mathbb{R}^n)$ then $\partial_i T$ is the distribution defined by

$$(7.13) \quad \partial_i T(f) \equiv -T\left(\frac{\partial f}{\partial x^i}\right) \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^n) \quad .$$

In the more common (and more abusive) notation in which one represents a distribution as integration against some generalized function $g(x)$, the above definition takes the form

$$(7.14) \quad \int_{\mathbb{R}^n} \left(\frac{\partial g}{\partial x^i}(x)\right) f(x) d^n x \equiv - \int_{\mathbb{R}^n} g(x) \left(\frac{\partial f}{\partial x^i}\right) d^n x \quad ,$$

which can be (mis-) interpreted as the usual integration by parts formula (there is no boundary term since by definition $f \in C_0^\infty(\mathbb{R}^n)$ has only compact support).

I might also remark that it is possible to take the Laplace transform of the delta ‘function’: for the right hand side of

$$\mathcal{L}[\delta(x - x_o)](s) \equiv \int_{-\infty}^{+\infty} \delta(x - x_o) e^{-sx} dx = e^{-sx_o}$$

is perfectly well-defined.

3. Other Representations of the Delta Function

Consider the expressions

$$(7.15) \quad \delta(x) = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$

$$(7.16) \quad \delta(x) = \lim_{n \rightarrow \infty} \frac{n}{\pi} \frac{1}{1 + n^2 x^2}$$

$$(7.17) \quad \delta(x) = \lim_{n \rightarrow \infty} \frac{\sin(nx)}{\pi x}$$

$$(7.18) \quad \delta(x) = \lim_{n \rightarrow \infty} \int_{-n}^n e^{ixt} dt$$

Each of these expressions behaves like function $\delta(x)$ in (7.8) in the sense that

$$(7.19) \quad \delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and

$$(7.20) \quad \int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0)$$

if f is continuous and the limits in (7.18) are taken only after the integration over x has been carried out.

Here is yet another example of a representation of the delta function. Recall that the solution to

$$(7.21) \quad \frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

$$(7.22) \quad \phi(x, 0) = f(x)$$

was found (using Laplace transform techniques) to be

$$(7.23) \quad \phi(x, t) = \int_{-\infty}^{+\infty} f(\zeta) g(x - \zeta, t) d\zeta$$

where

$$(7.24) \quad g(x, t) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right) .$$

One can interpret the integral on the right hand side of (7.23) in the following way: $g(x - \zeta, t)$ represents the contribution to the total temperature at the point x and time t resulting from the propagation of the heat from the point ζ at time $t = 0$. Indeed, integral kernels like $g(x - \zeta, t)$ that allow one to convert boundary conditions directly to solutions are often called *propagators*.

Consider $g(x, t)$ as a 1-parameter family of functions:

$$\{g_t(x) := g(x, t) \mid t \in \mathbb{R}^+\} .$$

One can easily verify that

$$\int_{-\infty}^{+\infty} g_t(x) dx = 1 \quad , \quad \forall t \in \mathbb{R}^+ ,$$

$$\lim_{t \rightarrow 0} g(x, t) = \begin{cases} 0 & , \quad x \neq 0 \\ \infty & , \quad x = 0 \end{cases}$$

and that

$$\lim_{t \rightarrow 0} \int_{-\infty}^{+\infty} f(x) g(x, t) dx = f(0)$$

for any continuous function f . In other words,

$$\lim_{t \rightarrow 0} g(x, t) = \delta(x) .$$

This must be so since the original boundary condition on the solution (7.23) requires

$$f(x) = \phi(x, 0) = \lim_{t \rightarrow 0} \int_{-\infty}^{+\infty} f(\zeta) g(x - \zeta, t) d\zeta .$$

Here is yet another way of representing the delta function. Suppose $\{\beta_n(x)\}$ is a complete orthonormal set of functions on the interval (a, b) . Set

$$\delta(x - y) = \sum_{n=1}^{\infty} \beta_n(x) \beta_n(y) .$$

Then if

$$f(x) = \sum_{m=1}^{\infty} a_m \beta_m(x)$$

we have

$$(7.25) \quad \int_a^b f(x) \delta(x - y) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_a^b a_m \beta_m(x) \beta_n(x) \beta_n(y) dx$$

$$(7.26) \quad = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m \delta_{m,n} \beta_n(y)$$

$$(7.27) \quad = \sum_{m=1}^{\infty} a_m \beta_m(y)$$

$$(7.28) \quad = f(y) .$$

This representation will be very important latter on when we discuss Green's functions.

Homework: 2.6.2