LECTURE 6

Laplace Transform Techniques

1. Review of Laplace Transform

THEOREM 6.1. *Let f be a function on the half line $[0,\infty)$ with the following properties*

1. $f, f', f'', \ldots, f^{(n-1)}$ *are continuous*
2. $f^{(n)}$ *is piecewise continuous*
3. *There exists constants K, a and M such that $|f(x)| \leq Ke^{at}$ for $t \geq M$.*

Then the Laplace transforms

$$\begin{aligned}
\mathcal{L}\left[f\right](s) &= \int_0^\infty e^{-st} f(t)dt \\
\mathcal{L}\left[f'\right](s) &= \int_0^\infty e^{-st} f'(t)dt \\
&\vdots \\
\mathcal{L}\left[f^{(n)}\right](s) &= \int_0^\infty e^{-st} f^{(n)}(t)dt
\end{aligned}$$

all exist for $s > a$, and moreover

$$\mathcal{L}\left[f^{(n)}\right](s) = s^n \mathcal{L}\left[f\right](s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \quad .$$

THEOREM 6.2. *If $F(s) = \mathcal{L}\left[f\right](s)$ and $G(s) = \mathcal{L}[g](s)$ both exist for $s > a \geq 0$, then*

$$\mathcal{L}^{-1}\left[F(s)G(s)\right] = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau \quad .$$

THEOREM 6.3. *If $F(s) = \mathcal{L}[f](s)$ then*

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)ds$$

Here the constant γ is chosen so that all the singularities of the integrand occur to the left of $Re(s) = \gamma$. For $t > 0$, the contour may be closed by an infinite semicircle in the left half plane.

EXAMPLE 6.4. The following example shows how the Laplace transform can be used to construct a solution of the boundary value problem

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{6.1}$$

$$\phi(x,0) = 0 \tag{6.2}$$

$$\phi(0,t) = f(t) \tag{6.3}$$

in the region $x > 0$, $t > 0$.

Multiplying (6.1) by e^{-st} and integrating along t from 0 to ∞ we get

$$\mathcal{L}\left[\frac{\partial\phi}{\partial t}\right](s) - a^2\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right] = 0$$

or

$$s\mathcal{L}\left[\phi\right] - \phi(x,0) - a^2\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right] = 0 \quad ,$$

which becomes

$$\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right] - \frac{s}{a^2}\mathcal{L}[\phi] = 0 \tag{6.4}$$

once we employ the initial condition $\phi(x,0) = 0$.

If $\phi(x,t)$ is sufficiently well behaved,

$$\mathcal{L}\left[\frac{\partial^2\phi}{\partial x^2}\right] = \frac{\partial^2}{\partial x^2}\mathcal{L}\left[\phi\right]$$

and (6.4) becomes

$$\frac{\partial^2\Phi}{\partial x^2}(x,s) - \frac{s}{a^2}\Phi(x,s) = 0 \quad .$$

Regarding this a second order linear differential equation with constant coefficients, we have as a general solution

$$\Phi(x,s) = A(s)e^{-\frac{\sqrt{s}}{a}x} + B(s)e^{\frac{\sqrt{s}}{a}x} \quad . \tag{6.5}$$

In order that our solution be a bounded function of t for all $x \geq 0$, we will demand that its Laplace transform go to zero as $s \to \infty$. This will require the coefficient $B(s) = 0$. To determine $A(s)$, we note that the boundary condition

$$\phi(0,t) = f(t)$$

has as its Laplace transform

$$\Phi(0,s) = \mathcal{L}[f](s) \quad . \tag{6.6}$$

On the other hand, when we evaluate (6.5) at $x = 0$, we have

$$\Phi(0,s) = A(s) \quad . \tag{6.7}$$

Comparing this with (6.4) with (6.5) we see that the coefficient $A(s)$ must be exactly $\mathcal{L}\left[f\right](s)$. Thus,

$$\Phi(x,s) = \mathcal{L}\left[f\right](s)e^{-\frac{\sqrt{s}}{a}x} \quad . \tag{6.8}$$

To find the solution $\phi(x,t)$ we now note that

$$e^{-\frac{\sqrt{s}}{a}x} = \mathcal{L}\left[\frac{x}{2\sqrt{\pi}at^{3/2}}e^{-\frac{x^2}{4a^2t}}\right] \tag{6.9}$$

and apply the Convolution Theorem to get

$$\begin{aligned} \phi(x,t) &= \mathcal{L}^{-1}\left[\mathcal{L}[f]\mathcal{L}\left[\frac{x}{2\sqrt{\pi}at^{3/2}}e^{-\frac{x^2}{4a^2t}}\right]\right] && (6.10)\\ &= \frac{x}{2a\sqrt{\pi}}\int_0^t f(t-\tau)\tau^{-3/2}e^{-\frac{x^2}{4a^2\tau}}d\tau && (6.11) \end{aligned}$$

$\square$

EXAMPLE 6.5. Let $\phi(x,t)$ satisfy the equation

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{6.12}$$

$$\phi(x,0) = f(x) \tag{6.13}$$

for $-\infty < x < +\infty$, $t > 0$. As before, the Laplace transform of the heat equation (with respect to the time variable t) is

$$s\Phi(x,s) - \phi(x,0) - a^2 \frac{\partial^2}{\partial x^2}\Phi(x,s) = 0$$

or

$$s\Phi(x,s) - f(x) - a^2 \frac{\partial^2 \Phi}{\partial x^2} = 0$$

or

$$\frac{\partial^2 \Phi}{\partial x^2} - \frac{s}{a^2}\Phi = -\frac{f(x)}{a^2} \quad . \tag{6.14}$$

The general solution to a ordinary differential equation of the form

$$y'' - p(x)y' + q(x)y = g(x)$$

is

$$y(x) = \left(A - \int^x \frac{y_2(\zeta)g(\zeta)}{W[y_1,y_2](\zeta)} d\zeta \right) y_1(x) + \left(B + \int^x \frac{y_1(\zeta)g(\zeta)}{W[y_1,y_2](\zeta)} d\zeta \right) y_1(x) \tag{6.15}$$

where $y_1(x)$ and $y_2(x)$ are any two linearly independent solutions of the corresponding homogeneous problem

$$y'' + p(x)y' + q(x)y = 0 \quad .$$

(See Boyce and DiPrima, Section 3.6.2.) In the case at hand, we can take

$$y_1(x) = e^{-\frac{\sqrt{s}}{a}x} \tag{6.16}$$

$$y_2(x) = e^{\frac{\sqrt{s}}{a}x} \tag{6.17}$$

$$g(x) = \frac{-f(x)}{a^2} \tag{6.18}$$

and

$$W[y_1,y_2](x) = y_1 y_2' - y_1' y_2 = \frac{2\sqrt{s}}{a}$$

so we have

$$\Phi(x,s) = \left(A + \frac{1}{2a\sqrt{s}} \int_0^x e^{\frac{\sqrt{s}}{a}\zeta} f(\zeta) d\zeta \right) e^{-\frac{\sqrt{s}}{a}x} \tag{6.19}$$

$$+ \left(B - \frac{}{2a\sqrt{s}} \int_0^x e^{-\frac{\sqrt{s}}{a}\zeta} f(\zeta) d\zeta \right) e^{\frac{\sqrt{s}}{a}x} \tag{6.20}$$

In order to avoid uncontrolled growth in $\Phi(x,s)$ as $x \to +\infty$ we require

$$B = \frac{1}{2a\sqrt{s}} \int_0^\infty e^{-\frac{\sqrt{s}}{a}\zeta} f(\zeta) d\zeta \tag{6.21}$$

and similarly to control the behavior of $\Phi(x,s)$ as $x \to -\infty$, we require

$$A = -\frac{1}{2a\sqrt{s}} \int_0^{-\infty} e^{\frac{\sqrt{s}}{a}\zeta} f(\zeta) d\zeta \tag{6.22}$$

So

$$\Phi(x,s) = \left(-\frac{1}{2a\sqrt{s}}\int_0^{-\infty} e^{\frac{\sqrt{s}}{a}\zeta} f(\zeta)d\zeta + \frac{1}{2a\sqrt{s}}\int_0^{x} e^{\frac{\sqrt{s}}{a}x'} f(\zeta)d\zeta\right) e^{-\frac{\sqrt{s}}{a}x} \tag{6.23}$$

$$+\left(\frac{1}{2a\sqrt{s}}\int_0^{\infty} e^{-\frac{\sqrt{s}}{a}\zeta} f(\zeta)d\zeta - \frac{1}{2a\sqrt{s}}\int_0^{x} e^{-\frac{\sqrt{s}}{a}\zeta} f(\zeta)d\zeta\right) e^{\frac{\sqrt{s}}{a}x} \tag{6.24}$$

$$\tag{6.25}$$

$$= \left(-\frac{1}{2a\sqrt{s}}\int_x^{-\infty} e^{\frac{\sqrt{s}}{a}\zeta} f(\zeta)d\zeta\right) e^{-\frac{\sqrt{s}}{a}x} + \left(\frac{1}{2a\sqrt{s}}\int_x^{\infty} e^{-\frac{\sqrt{s}}{a}\zeta} f(\zeta)d\zeta\right) e^{\frac{\sqrt{s}}{a}x} \tag{6.26}$$

$$\tag{6.27}$$

$$= \int_{-\infty}^{x} \frac{1}{2a\sqrt{s}} e^{\frac{\sqrt{s}}{a}(\zeta - x)} f(\zeta)d\zeta + \int_x^{+\infty} \frac{1}{2a\sqrt{s}} e^{\frac{\sqrt{s}}{a}(x-\zeta)} f(\zeta)d\zeta \tag{6.28}$$

$$= \int_{-\infty}^{x} \frac{1}{2a\sqrt{s}} e^{-\frac{\sqrt{s}}{a}|\zeta - x|} f(\zeta)d\zeta + \int_x^{+\infty} \frac{1}{2a\sqrt{s}} e^{-\frac{\sqrt{s}}{a}|\zeta - x|} f(\zeta)d\zeta \tag{6.29}$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{s}} e^{-\frac{\sqrt{s}}{a}|\zeta - x|} f(\zeta)d\zeta \tag{6.30}$$

The third step comes form the observation that

$$e^{\frac{\sqrt{s}}{a}(\zeta - x)} = e^{-\frac{\sqrt{s}}{a}|\zeta - x|} \quad , \quad \forall\, \zeta \in (-\infty, x) \tag{6.31}$$

$$e^{\frac{\sqrt{s}}{a}(x - \zeta)} = e^{-\frac{\sqrt{s}}{a}|\zeta - x|} \quad , \quad \forall\, \zeta \in (x, +\infty) \tag{6.32}$$

Now we employ the identity that

$$\mathcal{L}\left[\frac{1}{2a\sqrt{\pi}}\frac{1}{\sqrt{t}} e^{-\frac{|\zeta - x|^2}{4a^2 t}}\right](s) = \frac{1}{2a\sqrt{s}} e^{-\frac{\sqrt{s}}{a}|\zeta - x|} \tag{6.33}$$

to write

$$\mathcal{L}\left[\phi(x,t)\right](s) \equiv \Phi(x,s) \tag{6.34}$$

$$= \int_{-\infty}^{+\infty} \mathcal{L}\left[\frac{1}{2a\sqrt{\pi}}\frac{1}{\sqrt{t}} e^{-\frac{|\zeta - x|^2}{4a^2 t}}\right](s) f(\zeta)d\zeta \tag{6.35}$$

$$= \mathcal{L}\left[\int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{|\zeta - x|^2}{4a^2 t}} f(\zeta)d\zeta\right](s) \tag{6.36}$$

Taking the inverse Laplace transform of both sides, we get

$$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{|\zeta - x|^2}{4a^2 t}} f(\zeta)d\zeta \quad . \tag{6.37}$$

2. The Method of Images

Consider now the problem

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{6.38}$$

$$\phi(x,0) = f(x) \quad , \quad 0 < x < \infty \tag{6.39}$$

$$\phi(0,t) = 0 \quad , \quad 0 < t < \infty \quad . \tag{6.40}$$

Note that in the statement of this problem we are given no information and asked no questions about the behavior of solutions as $x \to -\infty$. Yet in the preceding example, the assumption that $\Phi(x,s)$ was well behaved as $x \to \pm\infty$, was crucial to the determination of the constants of integration $A(s)$ and $B(s)$; which in turn allowed us to construct an explicit solution. Thus, the technique used in the preceding example can not be applied directly to the case at hand.

But a simple trick will suffice. We simply extend the domain of $f(x)$ to the whole real line by defining

$$f(-x) \equiv -f(x) \quad , \quad \forall\, x \in \mathbb{R}^{+} \quad .$$

The result of the preceding example then implies

$$\phi(x,t) = \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{|\zeta - x|^2}{4a^2 t}} f(\zeta) d\zeta \tag{6.41}$$

will satisfy the differential equation

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

and the boundary condition

$$\phi(x,0) = f(x) \quad .$$

We now verify that this solution also satisfies the boundary condition

$$\phi(0,t) = 0 \quad , \quad 0 < t < +\infty \quad .$$

Setting $x = 0$ in (6.41) yields

$$\phi(0,t) = \int_{-\infty}^{+\infty} \frac{1}{2a\sqrt{\pi t}} e^{-\frac{\zeta^2}{4a^2 t}} f(\zeta) d\zeta \quad . \tag{6.42}$$

Noting that the exponential function is a even function of ζ and that the function $f(\zeta)$ is defined to be an odd function of ζ, we conclude that the integrand in (6.42) is an odd function of ζ and so its integral between $-\infty$ and $+\infty$ vanishes. Thus,

$$\phi(0,t) = 0 \quad .$$

$\square$

Homework: 2.2.2, 2.2.3, 2.2.5