LECTURE 5

Solutions by Series Expansions, Cont'd

1. Review

In the preceding lecture we solved the initial value problem

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = w(x,t) \tag{5.1}$$
$$\phi(x,0) = f(x) \tag{5.2}$$
$$\phi(0,t) = 0 \tag{5.3}$$
$$\phi(L,t) = 0 \tag{5.4}$$

by presuming an expansion of the form

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\beta_n(x) \tag{5.5}$$

where the basis functions $\left\{\beta_n(x) = \sin\left(\frac{n\pi}{L}x\right)\right\}$ were chosen to be the complete set of orthogonal functions associated to the Sturm-Liouville problem

$$\beta'' - \lambda\beta = 0 \tag{5.6}$$
$$\beta(0) = 0 \tag{5.7}$$
$$\beta(L) = 0 \quad . \tag{5.8}$$

The major justification for this choice was the relative ease by which the resulting solution was constructed.

However, one could substitute another complete set of functions $\{\gamma_n(x)\}$ for the basis $\{\beta_n(x)\}$, and proceed as before. In this case, however, several difficulties will crop up. First of all, the boundry conditions

$$\begin{aligned} 0 &= \phi(0,t) = \sum_{n=1}^{\infty} a_n(t)\gamma_n(0) \\ 0 &= \phi(L,t) = \sum_{n=1}^{\infty} a_n(t)\gamma_n(L) \end{aligned}$$

will imply the coefficients $a_n(t)$ are now interrelated. One would also discover that the inconsistencies crop up when we assume that we can differentiate the resulting series term by term. In short, the choice (5.2) is made because it works; other choices for basis functions are to be avoided because without some simplifications the algorithm for constructing a series solution will fail.

2. Nonhomogeneous End Conditions

We will run into similar difficulties if we try to expand the solution of a non-homogeneous boundary value problem

$$\begin{aligned}
\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} &= w(x,t) && (5.9)\\
\phi(x,0) &= f(x) && (5.10)\\
\phi(0,t) &= g(t) && (5.11)\\
\phi(L,t) &= h(t) \quad . && (5.12)
\end{aligned}$$

Such problems can be avoided however by restating the problem in a slightly different manner. Define

$$\begin{aligned}
\psi(x,t) &= \phi(x,t) - \left[g(t) + \frac{x}{L}\left(h(t) - g(t)\right)\right] && (5.13)\\
\omega_1(x,t) &= w(x,t) - g'(t) - \frac{x}{L}\left(h'(t) - g'(t)\right) && (5.14)\\
f_1(x) &= f(x) - g(0) - \frac{x}{L}\left(h(0) - g(0)\right) && (5.15)
\end{aligned}$$

We then have

$$\begin{aligned}
\frac{\partial \psi}{\partial t} - a^2 \frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial t}\left(g + \frac{x}{L}(h-g)\right) - a^2 \frac{\partial^2 \phi}{\partial x^2} - a^2 \frac{\partial^2}{\partial x^2}\left(g + \frac{x}{L}(h-g)\right)\\
&= \frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} - g'(t) - \frac{x}{L}\left(h'(t) - g'(t)\right)\\
&= w(x,t) - g'(t) - \frac{x}{L}\left(h'(t) - g'(t)\right)\\
&= w_1(x,t)
\end{aligned}$$

$$\begin{aligned}
\psi(x,0) &= \phi(x,0) - \left[g(0) + \frac{x}{L}\left(h(0) - g(0)\right)\right]\\
&= f(x) - g(0) - \frac{x}{L}\left(h(0) - g(0)\right)\\
&= f_1(x)
\end{aligned}$$

$$\begin{aligned}
\psi(0,t) &= \phi(0,t) - \left[g(t) + \frac{0}{L}\left(h(t) - g(t)\right)\right]\\
&= g(t) - g(t)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\psi(L,t) &= \phi(L,t) - \left[g(t) + \frac{L}{L}\left(h(t) - g(t)\right)\right]\\
&= h(t) - g(t) - h(t) + g(t)\\
&= 0
\end{aligned}$$

Thus, the function $\psi(x,t)$ satisfies

$$\begin{aligned}
\frac{\partial \psi}{\partial t} - a^2 \frac{\partial^2 \psi}{\partial x^2} &= w_1(x,t) && (5.16)\\
\psi(x,0) &= f_1(x) && (5.17)\\
\psi(0,t) &= 0 && (5.18)\\
\psi(L,t) &= 0 && (5.19)
\end{aligned}$$

and so one can use the technique of the preceding lecture (for solving PDEs with homogeneous boundary conditions) to solve for $\psi(x,t)$. The solution of the original problem is then

$$\phi(x,t) = \psi(x,t) + \left[g(t) + \frac{x}{L}\left(h(t) - g(t)\right)\right] \tag{5.20}$$

Homework: 1.8.1