LECTURE 2

Imposition of Boundary Conditions

In the preceding lecture I applied the method of Separation of Variables to the Heat Equation

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{2.1}$$

and thereby constructed: four 1-parameter sets of real-valued solutions

$$\phi_{k,1}(x) = e^{k^2 t} e^{\frac{kx}{a}} \qquad k \in \mathbb{R} \tag{2.2}$$

$$\phi_{k,2}(x) = e^{k^2 t} e^{\frac{-kx}{a}} \qquad k \in \mathbb{R} \tag{2.3}$$

$$\phi_{\lambda,1}(x) = e^{-k^2 t} \cos\left(\frac{kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{2.4}$$

$$\phi_{\lambda,2}(x) = e^{-k^2 t} \sin\left(\frac{-kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{2.5}$$

Let us now see if we can construct a solution of the PDE (2.1) that satisfies the following (Dirichlet) boundary conditions:

$$\phi\,(x, t = 0) = g(x) \tag{2.6}$$

$$\phi\,(x, t = +\infty) = 0 \tag{2.7}$$

$$\phi\,(x = 0, t) = 0 \tag{2.8}$$

$$\phi\,(x = L, t) = 0. \tag{2.9}$$

To construct such a solution, we first observe that the equation (2.1) is a homogeneous linear PDE. This fact implies that if ϕ_1 and ϕ_2 are solutions then so too is any linear combination of ϕ_1 and ϕ_2. Our method of solution will be to construct a particular linear combination of the solutions (2.2) - (2.5) of (2.1) that satisfies the boundary conditions (2.6) - (2.9).

The second boundary condition (2.7) requires ϕ to decay to 0 as $t \to \infty$. Noting that the positive exponential functions $e^{k^2 t}$ diverge as $t \to \infty$, we can immediately remove the solutions $\phi_{k,1}$ and $\phi_{k,2}$, $k \in \mathbb{R}$, from further consideration.

Similarly, the condition that ϕ vanish when $x = 0$ allows us to immediately disgard the solutions $\phi_{\lambda,1}$.

This leaves us with solutions constructed from linear combinations of

$$\phi_{\lambda,2} = e^{-\lambda^2 t} \sin\left(\frac{\lambda}{a} x\right) \quad , \quad \lambda \in \mathbb{R} \quad .$$

The last boundary condition, however requires that ϕ also vanish at $\chi = L$. To accomplish this, we require

$$\frac{\lambda}{a} = \frac{n\pi}{L},$$

because then the functions

$$e^{-\left(\frac{na\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right) \quad , \quad n \in \mathbb{N}$$

will vanish identically at both $x = 0$ and $x = L$.

Thus, after imposing (2.7), (2.8), and (2.9), we see that the most general solution of (2.1) and (2.6) - (2.9) that we can construct from linear combinations of the solutions (2.2) - (2.5) will have the form

$$\phi(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{na\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

for a suitable choice of coefficients c_n. To fix the c_n we have one more boundary condition to apply; equation (l-02.2a):

$$g(x) = \phi\left(x, t = 0\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) \quad .$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating between 0 and L we obtain

$$\begin{aligned}
\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\
&= \sum_{n=1}^{\infty} c_n \int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx \\
&= \sum_{n=1}^{\infty} c_n \left(\frac{L}{2}\delta_{mn}\right) \\
&= \frac{Lc_m}{2}
\end{aligned}$$

Here we have used the formula

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \frac{L}{2}\delta_{mn} \quad .$$

Thus, the coefficients c_n are given by the formula

$$c_n = \int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad .$$

We thus arrive at the following solution of (2.1) and (2.6) - (2.9):

$$\phi(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{na\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$c_n = \int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

1. Summary: Boundary Conditions and Series Expansions

A crucial step in our solution of

$$\begin{aligned}
&(2.10) & \frac{\partial\phi}{\partial t} - a^2\frac{\partial^2\phi}{\partial x^2} &= 0 \\
&(2.11) & \phi\left(x, t = 0\right) &= g(x) \\
&(2.12) & \phi\left(x, t = +\infty\right) &= 0 \\
&(2.13) & \phi\left(x = 0, t\right) &= 0 \\
&(2.14) & \phi\left(x = L, t\right) &= 0
\end{aligned}$$

was the representation of the solution as an infinite series

$$\phi(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{na\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) . \tag{2.15}$$

It should be noted, however, our introduction of this series expansion came after we used separation of variables to obtain four 1-parameter families of linearly independent solutions. We then imposed the boundary conditions to eliminate all but a countable 1-parameter set, and then applied the Superposition Principle (i.e., the fact that any linear combination of solutions of a linear PDE will also be a solution) justify the representation (2.15).

Suppose we had instead begun with the ansatz

$$\phi(x,t) = \sum_{n,m=0}^{\infty} a_{mn} x^m t^n , \tag{2.16}$$

which would surely be appropriate if we had in mind to look for C^∞ solutions, and plugged into the differential equation to try to find relations among the coefficients a_{mn}:

$$\sum_{n,m=0}^{\infty} n a_{mn} x^m t^{n-1} - a^2 \sum_{n,m=0}^{\infty} m(m-1) a_{mn} x^{m-2} t^n = 0. \tag{2.17}$$

One might hope to deduce from (2.17) suitable recursion relations for the coefficients a_{mn} that would fix these constants in terms of the initial conditions. However, the initial conditions are not so managable in this case. For example the condition

$$\phi\left(x, t = +\infty\right) = 0$$

does not yield any tangible information about the coefficients a_{mn}. The moral of this paragraph is although a series expansion like (2.16) might seem plausible from a purely formal point of view - in practice such an expansion will be useless unless the basis for the expansion allows for the boundary conditions to be implemented simply.

Even so, the underlying idea, that an arbitrary function (in particular, a solution to a PDE) can be representated as an (infinite) linear combination of a set of standard functions, is extremely important. This we shall develop next.

Homework:

1. (Problem 1.2.4 in text.) Let the temperature ϕ inside a solid sphere be a function of the radial distance r from the center and the time t. Show that the 3-dimensional heat equation

$$\frac{\partial \phi}{\partial t} - a^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

when transformed to spherical coordinates reduces to

$$\frac{\partial \phi}{\partial t} = a^2 \left(\frac{\partial^2 \phi}{\partial r^2} + 2\frac{\partial \phi}{\partial r}\right) .$$

Show also that a transformation of the form $\phi = r^\alpha \psi$, for a suitable choice of constant α reduces this equation to the form

$$\frac{\partial \phi}{\partial t} = C\frac{\partial^2 \phi}{\partial r^2}.$$

Discuss also the corresponding problem of 1-dimensional heat flow in a cylinder (consider here the transformation $\zeta = \ln |r|$).

2. (Problem 1.4.1 in text.) Using Separation of Variables, investigate solutions of of the Heat Equation

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \quad .$$

when the separation constant C is taken to be the square of a complex number. Which of these solutions relevant to the solution of (2.1) and (2.6) - (2.9)?

3. (Problem 1.4.2 in text.) Suppose we want to use the function

$$\Phi(x,t) = \sum_{n=1}^{N} c_n \exp\left[-n^2\pi^2 a^2 t/L^2\right] \sin\left(n\pi x/L\right)$$

where N is a chosen integer and the c_n are constants, to approximate solutions of the following problem: Find $\phi(x,t)$ satisfying the Heat Equation in the region $0 < x < L$, $0 < t$, with $0 = \phi(0,t) - \phi(L,t)$ and $\phi(x,0) = g(x)$. What would be a good way to determine the constants c_n. If we permit $N \to \infty$, what feature of the series would appear to ensure convergence for $t > 0$? (Hint: consider $\int_0^L \left[\phi(x,0) - \Phi(x,0)\right]^2 dx$.)