LECTURE 1

Introduction

1. Rough Classification of Partial Differential Equations

A partial differential equation is a equation relating a function ϕ of n variables $x_1, \ldots, x_n$, its partial derivatives, and the coordinates $x = (x_1, \ldots, x_n)$; i.e., an equation of the form

$$F\left[x, \phi, \partial_i \phi, \partial_i \partial_j \phi, \ldots, \partial_i \partial_j \cdots \partial_k \phi\right](x) = 0 \tag{1.1}$$

The order of the highest derivative appearing in (1.1) is the order of the partial differential equation (1.1). If the dependence of the functional F on ϕ and its partial derivatives is linear, then the PDE (1.1) is said to be linear (note, however, that a linear PDE is allowed to have a nonlinear dependence on x).

Example 1.1.

$$\left(\frac{\partial \phi}{\partial x_1}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2 + \phi = 3$$

is a nonlinear PDE of degree 1.

Example 1.2.

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \left(x_1^2 + x_2^2\right) \phi = 0$$

is a linear PDE of degree 2.

By a solution of the PDE (1.1) in a region $R \subset \mathbb{R}^n$, we mean an explicit function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$F\left[x, \Phi, \partial_i \Phi, \partial_i \partial_j \Phi, \ldots, \partial_i \partial_j \cdots \partial_k \Phi\right](x)$$

vanishes identically at each point $x \in R$. Note that if (1.1) has degree d then Φ must be of class C^d (i.e., Φ and each of partial derivatives up to order d must be continuous throughout R).

2. Three Fundamental Examples of 2^{nd} Order Linear PDEs:

2.1. Generic and Standard Forms of 2^{nd} Order Linear PDEs. The generic form of a second order linear PDE in two variables is

$$A(x,y)\frac{\partial^2 \phi}{\partial x^2} + B(x,y)\frac{\partial^2 \phi}{\partial x \partial y} + C(x,y)\frac{\partial^2 \phi}{\partial y^2} + D(x,y)\frac{\partial \phi}{\partial x} + E(x,y)\frac{\partial \phi}{\partial y} + F(x,y)\phi = G(x,y) \tag{1.2}$$

We shall see latter that by a suitable change of coordinates $x, y \to \xi(x,y), \eta(x,y)$ we can cast any PDE of the form (1.2) into one of the following three (standard) forms.

1. **Parabolic Equations:**

$$\frac{\partial^2 \Phi}{\partial \xi^2} + f_1(\xi,\eta)\frac{\partial \Phi}{\partial \xi} + f_2(\xi,\eta)\frac{\partial \Phi}{\partial \eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta) \tag{1.3}$$

2. **Elliptic Equations:**

$$\frac{\partial^2\Phi}{\partial\xi^2} + \frac{\partial^2\Phi}{\partial\eta^2} + f_1(\xi,\eta)\frac{\partial\Phi}{\partial\xi} + f_2(\xi,\eta)\frac{\partial\Phi}{\partial\eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta) \tag{1.4}$$

3. **Hyperbolic Equations:**

$$\frac{\partial^2\Phi}{\partial\xi\partial\eta} + f_1(\xi,\eta)\frac{\partial\Phi}{\partial\xi} + f_2(\xi,\eta)\frac{\partial\Phi}{\partial\eta} + f_3(\xi,\eta)\Phi = g(\xi,\eta) \tag{1.5}$$

Associated to each of these standard forms are prototypical examples, each of which, remarkably, corresponds to a fundamental PDE occuring in physical applications. During the next few weeks we shall discuss the solutions or each of these equations extensively.

2.2. The Heat Equation.

$$\frac{\partial\phi}{\partial t} - a^2\frac{\partial^2\phi}{\partial x^2} = 0 \tag{1.6}$$

This equation arises in studies of heat flow. For example, if a 1-dimensional wire is heated at one end, then the function $\phi(x,t)$ describing the temperature of the wire at position x and time t will satisfy (1.6). The heat equation is the prototypical example of a parabolic PDE.

2.3. Laplace's Equation.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1.7}$$

This equation arises in a variety of physical situations: the function $\phi(x,y)$ might be interpretable as the electric potential at a point (x,y) in the plane, or the steady state temperature of a point in the plane. Laplace's equation is the prototypical example of an elliptic PDE.

2.4. The Wave Equation.

$$\frac{\partial^2\phi}{\partial t^2} - a^2\frac{\partial^2\phi}{\partial x^2} = 0 \tag{1.8}$$

This equation governs the propagation of waves in a medium, such as the vibrations of a taunt string, pressure fluctuations in a compressible fluid, or electromagnetic waves. The wave equation is the prototypical example of a hyperbolic PDE. The coordinate transformation that casts (1.8) into the form (1.5) is

$$\begin{aligned} \xi &= x - ct \\ \eta &= x + ct \end{aligned}$$

3. Boundary Conditions

In stark constrast to the theory of ordinary differential equations where boundary conditions play a relatively innocuous role in the construction of solutions, the nature of the boundary conditions imposed on a partial differential equation can have a dramatic effect on whether or not the PDE/BVP (partial differential equation / boundary value problem) is solvable.

3.1. Cauchy Conditions. The specification of the function and its normal derivative along the boundary curve.

Cauchy boundary conditions are commonly applicable in dynamical situations (where the system is interpreted as evolving with respect to a time parameter t.

3.2. Dirichlet Conditions. The specification of the function on the boundary curve.

As an example of a PDE/BVP with Dirichlet boundary conditions, consider the problem of finding the equilibrium temperature distribution of a rectangular

sheet whose edges are maintained at some prescribed (but non-constant) temperature.

3.3. Neumann Conditions. The specification of the normal derivative of the function along the boundary curve.

As an example of a PDE/BVP with Neumann boundary conditions, consider the problem of determining the electric potential inside a superconducting cylinder.

4. Simple Solutions of the Heat Equation - Separation of Variables

In order to get a feel for the general nature of partial differential equations, we shall now look for simple solutions for the heat equation

$$\frac{\partial \phi}{\partial t} = a^2 \frac{\partial^2 \phi}{\partial x^2} \quad . \tag{1.9}$$

We shall construct solutions of this equation by presuming the existence of solutions of a particularly simple (but sufficiently general) form. Our initial assumptions will be justified by the fact that we obtain in this manner lots of solutions.

Let us then suppose that there exist solutions of (1.9) of the form

$$\phi(x,t) = F(x)G(t) \tag{1.10}$$

where F is a function of x alone and G is a function of t alone. Substituting this *ansatz* for ϕ into (1.9) yields

$$F(x)G'(t) = a^2 G(t)F''(x)$$

or

$$\frac{G'(t)}{G(t)} = a^2 \frac{F''(x)}{F(x)} \quad . \tag{1.11}$$

Now this equation should hold for all x and t. However, the left hand side depends only on t while the right hand side depends only on x. Consequently, if we vary t but keep x fixed, we must have $\frac{G'(t)}{G(t)}$ equal to the fixed number $a^2\frac{F''(x)}{F(x)}$. Thus, $\frac{G'(t)}{G(t)}$ equals some constant; say C. Similarly, by varying x and keeping t fixed we can conclude that $\frac{F''(x)}{F(x)}$ is a constant as well; say D. Equation (1.11) then becomes

$$C = a^2 D \quad .$$

Thus, when we presume the existence of solutions of the form (1.10), the diffusion equation (1.9) is equivalent to the following pair of ordinary differential equations

$$\frac{G'(t)}{G(t)} = C \tag{1.12}$$

$$\frac{F''(x)}{F(x)} = \frac{C}{a^2} \quad . \tag{1.13}$$

Therefore, if we can construct solutions G and F of the *ordinary differential equations* (1.12) and (1.13), then (1.10) will be a solution of the *partial differential equation* (1.9). Rewriting (1.12) and (1.13, respectively, as

$$G'(t) - CG(t) = 0 \tag{1.14}$$

$$F''(x) - \frac{C}{a^2}F(x) = 0 \tag{1.15}$$

We see that both of these ordinary differential equations are linear with constant coefficients. The general solution of (1.14) will be

$$G(t) = G_o e^{Ct}$$

and the general solution of (1.15) will have the form

$$F(x) = F_1 e^{\kappa x} + F_2 e^{-\kappa x} \qquad , \qquad \kappa = \sqrt{\frac{C}{a^2}} \quad .$$

Thus, any function of the general form

$$\begin{aligned} \phi_C(x,t) &= G_o e^{Ct}\left(F_1 e^{\kappa x} + F_2 e^{-\kappa x}\right) \\ &\approx c_1 e^{Ct+\kappa x} + c_2 e^{Ct-\kappa x} \end{aligned}$$

will be solutions of (1.9). Note that there are 3 undetermined parameters here; C, c_1 and c_2. For fixed values of $\kappa \neq 0$, we obtain a two dimensional space of solutions, since $\phi_{C,1}(x,t) = e^{Ct+\kappa x}$ and $\phi_{C,2}(x,t) = e^{Ct-\kappa x}$ are linearly independent. However, if $C' \neq C$, then the functions $\{\phi_{C,1}, \phi_{C,2}, \phi_{C',1}, \phi_{C',2}\}$ are all linearly independent.

If we take the separation constant $C = k^2$, with k real, we obtain

$$\phi_k(x,t) = e^{k^2 t}\left(b_1 e^{\frac{k}{a}x} + b_2 e^{-\frac{k}{a}x}\right) \quad .$$

Varying c we thus obtain two 1-parameter families of linear independent solutions whose magnitudes grow exponentially in time:

$$\begin{aligned} \phi_{k,1}(x) = &\quad e^{k^2 t} e^{\frac{kx}{a}} \qquad k \in \mathbb{R} \\ \phi_{k,2}(x) = &\quad e^{k^2 t} e^{\frac{-kx}{a}} \qquad k \in \mathbb{R} \end{aligned}$$

If we take $C = -\lambda^2$, with λ real constant, we have

$$\kappa = \sqrt{\frac{-\lambda^2}{a^2}} = i\frac{\lambda}{a}$$

and so

$$\begin{aligned} \phi_{\lambda,1}(x,t) &= e^{-\lambda^2 t} e^{\frac{i\lambda}{a}x} \\ \phi_{\lambda,2}(x,t) &= e^{-\lambda^2 t} e^{\frac{-i\lambda}{a}x} \end{aligned}$$

and

$$\begin{aligned} \phi_\lambda(x,t) &= c_1 e^{-\lambda^2 t} e^{\frac{i\lambda}{a}x} + c_2 e^{-\lambda^2 t} e^{\frac{-i\lambda}{a}x} \\ &= e^{-\lambda^2 t}\left(c_1 \cos\left(\frac{\lambda x}{a}\right) + ic_1 \sin\left(\frac{\lambda x}{a}\right) + c_2 \cos\left(\frac{\lambda x}{a}\right) - ic_2 \sin\left(\frac{\lambda x}{a}\right)\right) \\ &= e^{-\lambda^2 t}\left(a_1 \cos\left(\frac{\lambda x}{a}\right) + a_2 \sin\left(\frac{\lambda x}{a}\right)\right) \end{aligned}$$

In the second step we have used Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

to replace the exponential functions $e^{\pm i\frac{\lambda}{a}x}$ by sine and cosine functions:

$$\begin{aligned} a_1 &= c_1 + c_2 \\ a_2 &= ic_1 - ic_2. \end{aligned}$$

Varying λ we obtain two more 1-parameter families of linearly independent solutions that decay exponentially as $t \to \infty$, and oscillate sinusoidally as one varies x.

$$\phi_{\lambda,1}(x) = \quad e^{-k^2 t}\cos\left(\tfrac{kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{1.16}$$

$$\phi_{\lambda,2}(x) = \quad e^{-k^2 t}\sin\left(\tfrac{-kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{1.17}$$

In summary, the method of separation of variables (i.e., the ansatz $\phi(x,t) = F(x)G(t)$) produces four 1-parameter sets of linearly independent, real-valued solutions

$$\phi_{k,1}(x) = e^{k^2 t} e^{\frac{kx}{a}} \qquad k \in \mathbb{R} \tag{1.18}$$

$$\phi_{k,2}(x) = e^{k^2 t} e^{\frac{-kx}{a}} \qquad k \in \mathbb{R} \tag{1.19}$$

$$\phi_{\lambda,1}(x) = e^{-k^2 t} \cos\left(\frac{kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{1.20}$$

$$\phi_{\lambda,2}(x) = e^{-k^2 t} \sin\left(\frac{-kx}{a}\right) \qquad \lambda \in \mathbb{R} \tag{1.21}$$

Given this plethora of linearly independent solutions, it is appropriate to ask under what additional conditions can we expect to find a unique solution. Clearly, specifying the value of ϕ at a single point will be insufficient. We shall see latter that in order to obtain a unique solution we will have to specify the values of ϕ and its partial derivatives at every point along some curve in order to completely determine a solution.