Math 4513
Solutions to Homework 10

1. Determine the order of the following multi-step methods.

(a)

$$x_n - x_{n-2} = 2hf_{n-1}$$

- This equation corresponds to a two step method with coefficients

$$\begin{aligned} a_2 &= 1 \quad , \quad a_1 = 0 \quad , \quad a_0 = -1 \\ b_2 &= 0 \quad , \quad b_1 = 2 \quad , \quad b_0 = 2 \end{aligned}$$

We have

$$\begin{aligned} d_0 &= \sum_{i=0}^{2} a_i = 1 + 0 - 1 = 0 \\ d_1 &= \sum_{i=0}^{2} [ia_i - b_i] = (2(1) - 0) + (0 - 2) + (0 - 0) = 0 \\ d_2 &= \sum_{i=0}^{2} \left[\frac{i^2}{2}a_i - ib_i\right] = (2 - 0) + (0 - 2) + (0 - 0) = 0 \\ d_3 &= \sum_{i=0}^{2} \left[\frac{i^3}{6}a_i - \frac{i^2}{2}b_i\right] = 0 - 1 + \frac{4}{3} = \frac{1}{3} \neq - \end{aligned}$$

so this method has order 2.

$\square$

(b)

$$x_n - x_{n-2} = h\left[\frac{7}{3}f_{n-1} - \frac{2}{3}f_{n-2} + \frac{1}{3}f_{n-3}\right]$$

- This equation corresponds to a three step method with coefficients

$$\begin{aligned} a_3 &= 1 \quad , \quad a_2 = 0 \quad , \quad a_1 = -1 \quad , \quad a_0 = 0 \\ b_3 &= 0 \quad , \quad b_2 = \frac{7}{3} \quad , \quad b_1 = -\frac{2}{3} \quad , \quad b_0 = \frac{1}{3} \end{aligned}$$

We have

$$\begin{aligned} d_0 &= \sum_{i=0}^{2} a_i = 1 + 0 - 1 + 0 = 0 \\ d_1 &= \sum_{i=0}^{2} [ia_i - b_i] = -\frac{1}{3} - \frac{1}{3} - \frac{7}{3} + 3 = 0 \\ d_2 &= \sum_{i=0}^{2} \left[\frac{i^2}{2}a_i - ib_i\right] = 0 + \frac{1}{6} - \frac{14}{3} + \frac{9}{2} = 0 \\ d_3 &= \sum_{i=0}^{2} \left[\frac{i^3}{6}a_i - \frac{i^2}{2}b_i\right] = 0 + \frac{1}{6} - \frac{14}{3} + \frac{9}{2} = 0 \\ d_4 &= \sum_{i=0}^{3} \left[\frac{i^4}{24}a_i - \frac{i^3}{6}b_i\right] = 0 + \frac{5}{72} - \frac{28}{9} + \frac{27}{8} = \frac{1}{3} \neq = \end{aligned}$$

so this method has order 3.

$\square$

(c)

$$x_n - x_{n-1} = h\left[\frac{3}{8}f_n + \frac{19}{24}f_{n-1} - \frac{5}{24}f_{n-2} + \frac{1}{24}f_{n-3}\right]$$

- This equation corresponds to a three step method with coefficients

$$\begin{aligned} a_3 &= 1 \quad , \quad a_2 = -1 \quad , \quad a_1 = 0 \quad , \quad a_0 = 0 \\ b_3 &= \frac{3}{8} \quad , \quad b_2 = \frac{19}{24} \quad , \quad b_1 = -\frac{5}{24} \quad , \quad b_0 = \frac{1}{24} \end{aligned}$$

We have

$$\begin{aligned} d_0 &= \sum_{i=0}^{2} a_i = 1 + 0 - 1 + 0 = 0 \\ d_1 &= \sum_{i=0}^{2} [ia_i - b_i] = -\frac{1}{24} + \frac{5}{24} - \frac{67}{24} + \frac{21}{8} = 0 \\ d_2 &= \sum_{i=0}^{2} \left[\frac{i^2}{2}a_i - ib_i\right] = 0 + \frac{5}{24} - \frac{43}{12} + \frac{27}{8} = 0 \\ d_3 &= \sum_{i=0}^{2} \left[\frac{i^3}{6}a_i - \frac{i^2}{2}b_i\right] = 0 + \frac{5}{48} - \frac{35}{12} + \frac{45}{16} = 0 \\ d_4 &= \sum_{i=0}^{3} \left[\frac{i^4}{24}a_i - \frac{i^3}{6}b_i\right] = 0 + \frac{5}{144} - \frac{31}{18} + \frac{27}{16} = 0 \\ d_5 &= \sum_{i=0}^{3} \left[\frac{i^5}{120}a_i - \frac{i^4}{24}b_i\right] = 0 + \frac{5}{576} - \frac{143}{180} + \frac{243}{320} = -\frac{19}{720} \neq 0 \end{aligned}$$

So this method has order 4..

$\square$

2. Consider the following intial value problem

$$\begin{aligned} \frac{dx}{dt} &= x^2 \\ x(1) &= s \end{aligned}$$

(a) At time $t = 1.5$ by how much does the solution at time corresponding to the initial condition $s = 1.0$ differ from the solution corresponding to the intial condtion $s = 1.01$.

- (Explicit Method.) This differential equation is separable:

$$\frac{dx}{x^2 + 1} = dt$$

Integrating both sides yields

$$\tan^{-1}(x) = t + C$$

Imposing the initial condition yields

$$\tan^{-1}(s) = 1 + C$$

so we have

$$C = \tan^{-1}(s) - 1$$

and so

$$\tan^{-1}(x) = t + \tan^{-1}(s) - 1$$

or

$$x = \tan\left(t - 1 + \tan^{-1}(s)\right)$$

When $s = 1$ we have

$$x(1.5, 1) = \tan\left(1.5 - 1 + \tan^{-1}(1.0)\right) = 3.408223437$$

and when $s = 1.01$ we have

$$x(1.5, 1.01) = \tan\left(1.5 - 1 + \tan^{-1}(1.0)\right) = 3.472072191$$

and

$$\Delta = 3.472072191 - 3.408223437 = 0.063849$$

(b) Assuming a numerical method of order 5, assuming $x < 4$ on $[1, 1.5]$ how small a step size should one use to ensure a soution at $t = 1.5$ that is accurate to 1 part in 10^4.

- According to the discusion of Lecture 24, the total error after n interations of order m method

$$\varepsilon_{tot} \approx h^m \frac{e^{\lambda t_n} - 1}{e^{\lambda h} - 1}$$

In the case at hand we have $m = 5$, $t_n = 0.5$, and since

$$f_x = 2x < 8 \quad , \quad t \in [1, 1.5]$$

we can safely choose $\lambda = 8$. Hence we need to solve

$$10^{-4} = \frac{\varepsilon}{x_n} = \frac{1}{4} h^5 \frac{e^{(.5)8} - 1}{e^{\lambda h} - 1} =$$

Using Maple we can solve this equation

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fsolve(10^(-4)=h^5(exp(4.0)-1)/(4*exp(4*h)-4),h);      ===>\ h = .2258021510
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$\square$